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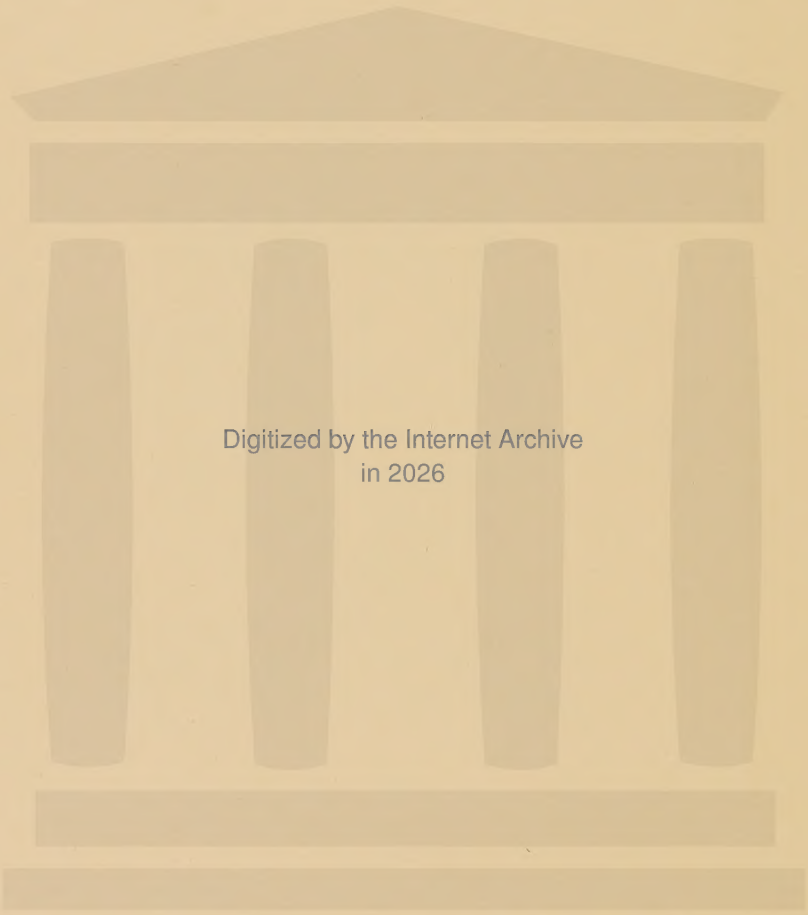
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OF

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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1911–JUNE, 1912.

Thursday, November 9th, 1911.

ANNUAL GENERAL MEETING.

Dr. H. F. BAKER, President, in the Chair.

Present twenty-four members.

Mr. J. M. Bottomley was elected a member.

Mr. J. E. Littlewood was admitted into the Society.

Professor Love presented the Treasurer's Report. On the motion of Lt.-Col. Cunningham, seconded by Mr. J. E. Littlewood, the Report was received.

Dr. J. G. Leathem was reappointed Auditor.

The President made a statement concerning changes in the mode of publication of the *Proceedings*. He quoted the following (see Vol. I, New Series, p. iv) :—

"With a view to the more rapid publication of *results*, the Council has adopted a plan for the publication of abstracts of the papers read at the meetings, of which plan the following is an outline :—The records of the proceedings of each meeting shall contain abstracts of the papers read at the meeting. Such extended records shall be published as soon as possible after the meetings to which they relate. The extended records shall afterwards be reprinted to form part of the Prefatory matter of a volume of *Proceedings*.

"It is proposed that the extended records should also contain abstracts of communications made at meetings but not at the time embodied in formal papers. By this plan an author may describe new methods or announce new results promptly, and yet gain time for more mature consideration."

On behalf of the Society the President presented the De Morgan Medal to Prof. H. Lamb.

The Council and Officers for the ensuing Session were elected as follows :—President, Dr. H. F. Baker ; Vice-Presidents, Mr. J. E. Campbell, Prof. A. E. H. Love ; Treasurer, Sir J. Larmor ; Secretaries, Mr. J. H. Grace, Dr. T. J. I'A. Bromwich ; other members of the Council, Mr. G. T. Bennett, Prof. W. Burnside, Mr. E. Cunningham, Mr. A. L. Dixon, Dr. L. N. G. Filon, Mr. J. H. Jeans, Mr. J. E. Littlewood, Prof. H. M. Macdonald, Major P. A. MacMahon, Dr. A. E. Western.

The following papers were communicated :—

*The Invariants of the Linear Partial Differential Equation of the Second Order in Two Independent Variables: Mr. J. E. Campbell.

†On Closed Linkages: Col. R. L. Hippisley.

*The System of Lines of a Cubic Surface: Mr. G. T. Bennett.

†On Invariants of a Canonical Substitution : Mr. H. Hilton.

†The Relations between Borel's and Cesàro's Methods of Summation: Messrs. G. H. Hardy and J. E. Littlewood.

*A Method of Establishing the 27-line Configuration on a Cubic Surface : Mr. W. P. Milne.

ABSTRACTS.

The Invariants of the Linear Partial Differential Equation of the Second Order: Mr. J. E. Campbell.

Let this equation be written in the form

$$az_{11}+2hz_{12}+bz_{22}+2gz_1+2fz+cz = 0, \qquad (1)$$

where the coefficients are functions of the independent variables, and where the suffixes denote that the functions, to which they are attached, are partial derivatives (1 with respect to x and 2 with respect to y) of the functions with the suffixes deleted.

By multiplying the equation (1) by a factor we can bring it to such a form that h^2-ab will be unity. We therefore take the equation in this standard form.

The equation (1) does not alter its form when multiplied by any

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* Printed in *Proceedings*, Ser. 2, Vol. 10.

† Printed in this volume.

function of the independent variables, nor when subjected to any transformation of the infinite group

$$x' = \phi(x, y), \quad y' = \psi(x, y), \quad z' = z\chi(x, y), \tag{2}$$

and we may suppose the multiplier chosen so as to keep the equation in standard form.

Let $2X = b(a_1+h_2-2g)-h(b_2+h_1-2f)$,

$2Y = a(b_2+h_1-2f)-h(a_1+h_2-2g)$,

$I = aX_1+h(X_2+Y_1)+bY_2+(a_1+h_2)X+(b_2+h_1)Y$
$+aX^2+2hXY+bY^2-c$,

$J = X_2-Y_1$,

$IC = a_{11}+2h_{12}+b_{22}+\frac{a_1b_2-a_2b_1}{2h}$,

$$I\Delta = \frac{\partial}{\partial x}\left(a\frac{\partial}{\partial x}+h\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\right).$$

It is proved that, if $Ia = J$,

$$\beta = \Delta(\log I)+C,$$

then a and β are absolutely unaltered by any of the operations in (2).

Let $ID(a, \beta) = (aa_1+ha_2)\beta_1+(ha_1+ba_2)\beta_2$,

$ID(a) = aa_1^2+2ha_1a_2+ba_2^2$,

$ID(\beta) = a\beta_1^2+2h\beta_1\beta_2+b\beta_2^2$,

then $\Delta(a)$, $\Delta(\beta)$, $D(a, \beta)$, $D(a)$, $D(\beta)$

will also be absolute invariants.

From any two invariants we can obtain others by these rules, and there are no invariants which cannot be thus obtained, so that all invariants are thus generated from the original two a and β.

From this result a method is obtained enabling us to decide whether any two assigned equations of form (1) are transformable into one another by an operation of group (2).

Some examples are given illustrating the application of the principle of invariants, and in particular it is shown under what conditions (1) will admit the transformations of a continuous group.

Closed Linkages: Col. R. L. Hippisley.

This paper is a continuation of the researches of Emch, and others, on the connection between three-bar motion and elliptic functions and poristic

polygons. In earlier work certain limitations were imposed on the relative lengths of the bars, but it is here shown that linkages of another sort can be constructed without observing any such restricting conditions. Tables are given and models have been constructed.

The System of Lines of a Cubic Surface: Mr. G. T. Bennett.

A diagram previously given (*Proc. London Math. Soc.*, Vol. 9, p. 349) to show the 135 intersections of the 27 lines of a cubic is here made the basis of a method of constructing such a configuration *ab initio*. The double six appears as an incident of the process.

Some properties are noticed of the plane figure, consisting of a triangle and an associated system of 32 lines, in which any one triple tangent plane is cut by the rest.

On Invariants of a Canonical Substitution: Mr. H. Hilton.

The number of homogeneous invariants of the second degree of a homogeneous linear substitution A is obtained. These invariants are actually obtained for the canonical form into which A may be transformed, and the invariants of A itself can be deduced therefrom.

The result is:—

"The number of independent quadratic invariants in the case of a substitution with invariant factors

$$(\lambda-\lambda_1)^a,\ (\lambda-\lambda_1^{-1})^a,\ (\lambda-\lambda_1)^b,\ (\lambda-\lambda_1^{-1})^b,\ (\lambda-\lambda_1)^c,\ (\lambda-\lambda_1^{-1})^c,$$
$$(\lambda-\lambda_1)^d,\ (\lambda-\lambda_1^{-1})^d,\ \ldots,$$

where $$a \geqslant b \geqslant c \geqslant d \geqslant \ldots,$$

and with invariant factors

$$(\lambda-1)^h,\ (\lambda-1)^k,\ (\lambda-1)^l,\ (\lambda-1)^m,\ \ldots,$$

where $$h \geqslant k \geqslant l \geqslant m \geqslant \ldots,$$

and invariant factors

$$(\lambda+1)^p,\ (\lambda+1)^q,\ (\lambda+1)^r,\ (\lambda+1)^s,\ \ldots,$$

where $$p \geqslant q \geqslant r \geqslant s \ldots,$$

is
$$I\left[\frac{h+1}{2}\right]+I\left[\frac{k+1}{2}\right]+I\left[\frac{l+1}{2}\right]+\ldots+2(k+2l+3m+\ldots)$$
$$+I\left[\frac{p+1}{2}\right]+I\left[\frac{q+1}{2}\right]+I\left[\frac{r+1}{2}\right]+\ldots$$
$$+2(q+2r+3s+\ldots)+\Sigma(a+3b+5c+7d+\ldots)$$

—where $I[t]$ denotes the integral part of t—the summation being taken over each distinct pair of reciprocal characteristic roots of the substitution."

A process is then given by means of which an orthogonal substitution A on the variables $x_1, x_2, \ldots, x_m$ can be transformed by means of an orthogonal substitution B into the form

$$\begin{aligned}
x'_1 = \pm x_1, \quad x'_2 = \pm x_2, \quad \ldots, \quad x'_h &= \pm x'_h, \\
x'_{h+1} &= \cos\theta_1 x_{h+1} - \sin\theta_1 x_{h+2}, \\
\pm x'_{h+2} &= \sin\theta_1 x_{h+1} + \cos\theta_1 x_{h+2}, \ \ldots, \\
x'_{h+2r-1} &= \cos\theta_r x_{h+2r-1} - \sin\theta_r x_{h+2r}, \\
\pm x'_{h+2r} &= \sin\theta_r x_{h+2r-1} + \cos\theta_r x_{h+2r}, \\
x'_t &= g_{t\,h+2r+1} x_{h+2r+1} + \ldots + g_{tm} x_m \quad (t = h+2r+1, \ldots, m),
\end{aligned}$$

where $$x'_t = g_{t\,h+2r+1} x_{h+2r+1} + \ldots + g_{tm} x_m \quad (t = h+2r+1, \ldots, m)$$

is an orthogonal substitution with no simple invariant factor.

If A is real, B is real, and $h+2r = m$.

A method is given by which a substitution A transformable into its inverse may be so transformed by means of a substitution of order 2, and the theory of invariants before developed is applied to express an orthogonal substitution as the product of two orthogonal substitutions of order 2 whose invariant factors are readily obtained.

The Relations between Borel's and Cesàro's Methods of Summation: Messrs. G. H. Hardy and J. E. Littlewood.

The principal object of this paper is to prove that *a series* Σa_n, *for which* $|a_n| < n^K$, *cannot be summable by Borel's method except when it is summable by one or other of Cesàro's means.* In particular, if $\lim a_n \sqrt{n} = 0$, the series must be convergent. It is almost certain that this condition may be replaced by the more general condition $|a_n \sqrt{n}| < K$, but this has not yet been proved.

A Method of Establishing the 27-line Configuration on a Cubic Surface: Mr. W. P. Milne.

This paper gives a method whereby the configuration of the 27 lines on the cubic surface becomes almost intuitive after a certain homographic correspondence has been established. Many of the elementary properties of the configuration become immediately evident. The correspondence is as follows:—If a, b, c be three coplanar lines of the configuration, and if

O be the point of contact of any one of the other four tangent planes drawn from c to the cubic surface, a homographic correspondence exists between the planes aP and bQ, where P and Q are the points of intersection with the surface of any chord drawn through O.

Thursday, December 14th, 1911.

Dr. H. F. BAKER, President, in the Chair.

Present thirteen members and two visitors.

Messrs. J. R. Airey, D. L. Board, A. Brown, J. C. Fields, S. Mukhopadhyaya were elected members.

The President read the Auditor's Report, and, on the motion of Lt.-Col. Cunningham, seconded by Mr. G. N. Watson, thanks were given to the Treasurer and the Auditor.

The following papers were communicated :—

*On the Pressure of Radiation on a Cylindrical Obstacle : Dr. J. W. Nicholson.

*On Hermitian Invariants of a Canonical Substitution : Mr. H. Hilton.

*On the Fundamental Lemma of the Calculus of Variations, and on some related Theorems : Prof. E. W. Hobson.

*On the Outer Isomorphisms of a Group : Prof. W. Burnside.

*Invariants of Linear Differential Equations with Applications to Ruled Surfaces in Five-Dimensional Space : Dr. E. B. Stouffer.

*A Method of Proving certain Theorems relating to Adjointness : Prof. J. C. Fields.

ABSTRACTS.

The Pressure of Radiation on a Cylindrical Obstacle : Dr. J. W. Nicholson.

The paper is a continuation of some work on the numerical value of the pressure which can be experienced by obstacles on which plane waves of light impinge. The case of a spherical obstacle has been discussed in

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* Printed in this volume.

the Monthly Notices of the R.A.S. (May 1910), and the present paper examines the case of the cylinder by a similar analysis. For a cylinder of large radius a, the pressure is given approximately in terms of the energy density E of incident radiation by the formula

$$p = \tfrac{8}{3}aE.$$

The paper tabulates this ratio for various radii of obstacle, and builds up the general solution of the problem from the two cases in which the electric and magnetic forces are respectively parallel to the axis of the cylinder.

During the discussion of the paper, a question was raised by Dr. Bromwich which suggested that some points needed a more complete consideration than the investigation had bestowed.

The author admits this necessity, and the work is to be revised in accordance with it.

On Hermitian Invariants of a Canonical Substitution : Mr. H. Hilton.

This paper is a continuation of a former paper in which were discussed the invariants $\Sigma e_{ij}x_i x_j$ of any substitution A,

$$x'_t = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \quad (t = 1, 2, \ldots, m).$$

Here the invariants of the type $\Sigma e_{ij}\bar{x}_i x_j$ are discussed, by actually obtaining these invariants for the canonical form into which A may be transformed.

The number of independent invariants of this type, and the number of fundamental independent Hermitian invariants, are given.

The number of positive and the number of negative signs in the reduced form of any fundamental Hermitian invariant are found to be equal, or to differ by unity.

A method of transforming a substitution with the invariant

$$x_1\bar{x}_1 + \ldots + x_k\bar{x}_k - x_{k+1}\bar{x}_{k+1} - \ldots - x_m\bar{x}_m$$

into a simpler form is obtained, similar to that previously given for orthogonal substitutions.

It is then proved that :—"If a substitution is transformable into the inverse of its conjugate, the transformation may be performed by means of a substitution identical with the inverse of its conjugate."

The paper concludes with a discussion of the effect of putting certain variables equal to zero in a fundamental quadratic invariant of a canonical substitution.

On the Fundamental Lemma of the Calculus of Variations, and on some related Theorems: Prof. E. W. Hobson.

The following theorems which are generalizations of theorems known in connection with the calculus of variations are established:—

(1) If $f(x)$ be a limited or unlimited function which is summable in the interval (x_0, x_1), and is such that $\int_{x_0}^{x_1} f(x) P(x) dx$ vanishes, provided $P(x)$ is any finite polynomial which, together with its first p derivatives, vanishes at the end-points of the interval, then $f(x)$ must vanish for every value of x, except at most those belonging to some set of which the measure is zero. Also $f(x)$ must vanish at any point at which it is continuous, and everywhere in any interval in which it is continuous. The number p may be zero, or have any assigned integral value.

(2) Let $f(x)$ be summable in the interval (x_0, x_1), and such that

$$\int_{x_0}^{x_1} f(x) Q(x) dx$$

vanishes for every finite polynomial which is such that

$$\int_{x_0}^{x_1} g(x) Q(x) dx = 0,$$

where $g(x)$ is another function summable in (x_0, x_1), and where $Q(x)$ also satisfies the condition that it, together with its first p derivatives, vanishes at the end-points of the interval. A determinate constant λ then exists, such that $f(x)+\lambda g(x)$ vanishes for all points in the interval (x_0, x_1), except at most those belonging to some set of which the measure is zero. In any interval in which $f(x)$ and $g(x)$ are continuous, $f(x)+\lambda g(x)$ vanishes without exception.

(3) If $f(x)$ be summable in the interval (x_0, x_1), and be such that $\int_{x_0}^{x_1} f(x) \frac{d^p}{dx^p} P(x) dx$ vanishes, provided $P(x)$ is any finite polynomial which, together with its first $p-1$ derivatives, vanishes at the end-points of the interval, then $f(x)$ must have the same values as some polynomial of degree $p-1$, for every value of x except possibly for those belonging to some set of measure zero. If $f(x)$ be continuous in any interval, then, through the whole of that interval, $f(x)$ has the same values as the polynomial. In case $p = 0$, the polynomial reduces to a constant.

(4) Let $q_0, q_1, \ldots, q_p$ be functions of x which are continuous and of limited total fluctuation in the interval (x_0, x_1), and let q_r be such that it

has a continuous differential coefficient of order r throughout the interval. Let $f(x)$ be a function, summable in (x_0, x_1), and such that

$$\int_{x_0}^{x_1} \left(q_0 y + q_1 \frac{dy}{dx} + q_2 \frac{d^2y}{dx^2} + \ldots + q_p \frac{d^p y}{dx^p}\right) f(x)\,dx$$

vanishes for every value of y that consists of a finite polynomial which, together with its first $p-1$ derivatives, vanishes at x_0 and x_1. Then, if $f(x)$ is continuous in the interval (x_0, x_1) it must be a solution of the equation adjoint to the differential equation

$$q_0 y + q_1 \frac{dy}{dx} + \ldots + q_p \frac{d^p y}{dx^p} = 0.$$

Moreover, if $f(x)$ is continuous only in some interval contained in (x_0, x_1), its value must satisfy the adjoint equation at all points in the interior of the interval of continuity.

On the Outer Isomorphisms of a Group: Prof. W. Burnside.

A hitherto undecided point with respect to the isomorphisms of a group is the question whether an outer isomorphism does or does not necessarily permute the conjugate sets. In this note an example is given of a group admitting outer isomorphisms which change every operation into a conjugate operation.

Invariants of Linear Differential Equations with Applications to Ruled Surfaces in Five-Dimensional Space: Dr. E. B. Stouffer.

Prof. Wilczynski has proved that the theory of non-developable ruled surfaces in ordinary space can be put into relation with the theory of a system of two linear homogeneous differential equations of the second order. Dr. Stouffer extends this motion to ruled surfaces in five-dimensional space, and shows that the projective differential properties of every surface of this kind which does not belong to a certain exceptional class can be obtained by studying a system of two linear homogeneous differential equations of the third order. The ruled surfaces to which this theory does not apply are those which have three consecutive generators in the same space of four dimensions. Complete systems of invariants and covariants of a system of two linear homogeneous differential equations of the third order are calculated, and some of the properties of ruled surfaces in five-dimensional space are discussed.

A Method of Proving certain Theorems relating to Adjointness: Prof. J. C. Fields.

This note deals with the simplification of the theory of algebraic functions. The fundamental algebraic equation being of order n in the dependent variable u, and of degree N in the two variables (u, z), it is proved—

(*a*) That if a rational function be adjoint for the value $z = a$, then, when put in reduced form, as an integral polynomial of order $n-1$ in u, the coefficient of u^{n-1} must be integral in regard to the element $z-a$.

(*b*) If this latter condition be satisfied for a rational function put into reduced form, and the function be adjoint relatively to all but one of the n branches corresponding to the value $z = a$, the function must also be adjoint relatively to the remaining branch.

(*c*) A reduced form which is adjoint for the value $z = \infty$ must be of degree less than N.

Thursday, January 11th, 1912.

Dr. H. BAKER, President, in the Chair.

Present sixteen members and two visitors.

Messrs. L. L. Conant and C. J. T. Sewell were elected members.

Dr. Bromwich, as Secretary, reported that the number of members at the beginning of last Session was 285. During the year three members had died, one had resigned, one name had been removed from the list, and twelve new members had been elected, so that the number of members at the beginning of the Session was 292. The Mathematical Society of Prague, the Royal Academy of Science and Letters of Denmark, Der Akademie der Wissenschaften in München, the South African Association for the Advancement of Science had been added to the list of those with which the Society exchanges publications.

The following papers were communicated:—

*A New Condition for the Truth of the Converse of Abel's Theorem relating to Power Series: Messrs. G. H. Hardy and J. E. Littlewood.

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* Printed in *Proceedings*, Ser. 2, Vol. 9.

On Mersenne's Numbers: Lt.-Col. A. Cunningham.

*Successions of Integrals and Fourier Series: Prof. W. H. Young.

*On Multiple Fourier Series: Prof. W. H. Young.

ABSTRACTS.

A New Condition for the Truth of the Converse of Abel's Theorem relating to Power Series: Messrs. G. H. Hardy and J. E. Littlewood.

Various sufficient conditions for the truth of the converse of Abel's theorem on the continuity of the series $\Sigma a_n x^n$ for $x = 1$ have been given by Tauber, Pringsheim, and Littlewood (*Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 434). Tauber's condition is $a_n = o(1/n)$, Littlewood's $a_n = O(1/n)$. The principal result of the present paper is

1. *A sufficient condition for the truth of the converse of Abel's theorem is that* $\Delta a_n \equiv a_n - a_{n+1} = O(a_n/n)$.

If $\Delta a_n = o(a_n/n)$ *the series* Σa_n *cannot converge, nor can Abel's limit exist: also* $s_n \sim n a_n$ *as* $n \to \infty$.

A typical series for which $\Delta a_n = O(a_n/n)$ is Σn^{-s}, where s is complex. If a_n is *real* the theorem is trivial, as a_n cannot change its sign. It follows from 1 that such a series cannot be summable by Cesàro's means, unless it is convergent. An alternative condition is that

$$n a_n = O\left(\frac{a_1 + 2a_2 + \ldots + n a_n}{n}\right).$$

Analogy suggests that there should be a similar theorem referring to Borel's method of summation, and related to the results given recently by the authors (*Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. v) as 1 is to the Tauber-Littlewood theorem. This theorem is

2. *A series for which* $\Delta a_n = o(a_n/\sqrt{n})$ *cannot be summable* (*B*) *unless it is convergent.*

Thus $$\Sigma n^{-b} e^{in^a} \quad (0 < a < \tfrac{1}{2})$$

is summable (B) if, and only if, it is convergent, *i.e.*, if $a + b > 1$.

On Mersenne's Numbers: Lt.-Col. A. Cunningham.

Lt.-Col. Cunningham announced the recent discovery by two† in-

* Printed in this volume.

† By Mr. R. E. Powers in America (in June, 1911), and M. H. Terry in France (in November, 1911).

dependent computers of a new Mersenne's *prime* $M_{89} = (2^{89}-1)$. This is particularly interesting in that—

1. It is not in Mersenne's original list of primes.

2. The exponent (q) is not of one of the forms $q = 2^k \mp 1$, $2^k \mp 3$, as yet known as giving $M_q = (2^q-1)$ prime.

3. It was affirmed *composite* by the late Ed. Lucas.

Successions of Integrals and Fourier Series: Prof. W. H. Young.

The paper consists of four sections. The first contains extensions of theorems given in a paper on semi-integrals* by the author. Thus it is shown that, *if* $\int_E f_n(x)\,dx$ *has no negative double limit as* E *approaches zero and* n *approaches infinity, then the succession of integrals* $\int_a^x f_n(x)\,dx$ *is bounded below and oscillates uniformly and homogeneously below. Moreover, if for any sub-set of values of the integer* n *the integrals converge to a unique limiting function, that function is an upper semi-integral and a lower semi-continuous function. If all the double limits of* $\int_E f_n(x)\,dx$ *are zero there is in every sub-succession of the integrals a sub-sequence of the integrals converging uniformly to an integral.* Here no assumption is made as to the summability of the upper and lower functions of the original succession.

Again, it is shown that, if *the same expression has no negative double limit, and the lower function* $l(x)$ *of the original succession is summable, and if, for some interval* (a, b),

$$\int_a^b l(x)\,dx = \text{lower limit}_{n\to\infty} \int_a^b f_n(x)\,dx,$$

then this equality holds for every interval or set of points contained in (a, b). Inequalities constituting generalisations of this equality are also given, as well as an example illustrating it.

Another kind of result given is a statement of certain conditions, required in the sequel, under which the triple limit of the integral $\int_E f_n(x)\,g_m(x)\,dx$ when E approaches zero and m and n approach infinity, exists and is zero.

---

* "On Semi-Integrals and Oscillating Successions of Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 286–324.

The second section contains a discussion of successions of integrals whose integrands are obtained by multiplying by one and the same function $g(x)$ a succession of functions $f_n(x)$, the limits of integration being finite or infinite. The theorems are usually given in the form of inequalities, constituting generalisations of theorems given by the author in a paper on "The Application of Expansions to Definite Integrals." In certain cases equalities are stated. Thus we have the theorem :—*If $f(x)$ is any function possessing a Harnack-Lebesgue integral, and the succession of functions $f_n(x)$ is bounded and consists of monotone descending functions of x, and is further such that the integrand of the integral which is the lower limit of the succession of integrals of the original succession is independent of the particular interval over which we integrate, viz., if*

$$\int_c^z L'(x)\,dx = \text{lower } \lim_{n\to\infty} \int_c^z f_n(x)\,dx,$$

for all values of c and z, then this equation remains true if we multiply the integrands of both sides by $g(x)$. In particular, if the succession of integrals $\int_a^x f_n(x)\,dx$ converges and $F(x)$ is the integral to which it necessarily converges uniformly, then

$$\int_c^z g(x)\,F'(x)\,dx = \operatorname*{Lt}_{n\to\infty} \int_c^z g(x) f_n(x)\,dx,$$

the convergence being also uniform.

Another result obtained is the following :—

If the sequence of Lebesgue integrals $\int s_n(x)\,dx$ converges boundedly to the Harnack-Lebesgue integral $\int f(x)\,dx$, and $g(x)$ is any function of bounded variation in the interval considered, then

$$\operatorname*{Lt}_{n\to\infty} \int s_n(x)\,g(x)\,dx = \int f(x)\,g(x)\,dx.$$

In the third section the theory of the succession of integrals whose integrands are the products of two successions is considered, and conditions are obtained that this should be a sequence, whose limiting function is independent of the mode in which the indices of the two factors of the integrand are chosen. To put the matter in another way, on the hypothesis that the integrals of the separate successions converge to $f(x)$ and $g(x)$, conditions are found that $\int f_n(x)\,g_m(x)\,dx$ should have as unique double limit $\int f(x)\,g(x)\,dx$.

The results obtained are capable of interpretation in the language of

integrals in a variety of ways. Thus they give us a number of sets of conditions that $\int_a^b f(x, t)\, g(y, t)\, dt$ should be a continuous function of (x, y).

In particular we have the following sets of sufficient conditions :—

(i) *$f(x, t)$ and $g(y, t)$ are both bounded, while $f(x, t)$ and the integral of $g(y, t)$ with respect to t are continuous functions of x and of y respectively.*

(ii) *$g(y, t)$ is bounded and its integral with respect to t is a continuous function of y, while $f(x, t)$ is positive, is a continuous function of x, and has an integral with respect to t which is a continuous function of x.*

(iii) *$g(y, t)$ is bounded and is a continuous function of y, while the integral of $f(x, t)$ with respect to t is a continuous function of x, and the integral of the $(1+p)$-th power of its modulus is, for some positive value of p, a bounded function of (x, t).*

(iv) *$g(y, t)$ is a function of t of bounded variation, whose total variation is a bounded function of (y, t), and whose integral with respect to t is a continuous function of y, while the integral of $f(x, t)$ with respect to t is a continuous function of (x, t).*

(v) *$g(y, t)$ is a monotone increasing function of t, and a bounded function of (y, t) and its integral with respect to t is a continuous function of y, while $f(x, t)$ has a Harnack-Lebesgue integral with respect to t, which is a continuous function of (x, t).*

(vi) *$g(y, t)$ is a continuous function of (y, t), while the integral of $f(x, t)$ with respect to t is a continuous function of x, and the integral of its modulus is bounded.*

(vii) *$g(y, t)$ and the $(1+1/p)$-th power of its modulus have integrals with respect to t which are continuous functions of y, while the integral of $f(x, t)$ with respect to t is a continuous function of x, and the integral of the $(1+p)$-th power of its modulus is bounded.*

(viii) *The integral of $g(y, t)$ with respect to t is a continuous function of y, and the integral of the $(1+1/q)$-th power of its modulus is bounded for some value of q less than p, while $f(x, t)$ is a continuous function of x, and the $(1+p)$-th power of its modulus has a bounded integral with respect to t.*

In the fourth section applications to Fourier series are given. Some additional preliminary theorems are needed. Among these may be noted

the theorem that, *if p be greater than unity, the p-th power of the n-th Cesàro partial summation of the Fourier series of the positive function $f(x)$ whose p-th power is summable, is never greater than the n-th Cesàro partial summation of the Fourier series of the p-th power of $f(x)$* ; and the theorem that, *if $f_n(x)$ denotes the n-th Cesàro partial summation of the Fourier series of a function $f(x)$ whose p-th power is summable, then $\int |f_n(x)|^p dx$ is a bounded function of n.*

The applications are of two kinds ; in the one kind known results* as to the circumstances under which the Fourier series of $f(x)$ may be integrated after multiplication by another function $g(x)$, are here re-obtained, by methods that are almost intuitive. Two distinct plans are adopted : the one turns on the consideration of the succession of integrals whose integrands are $g(x)f_n(x)$, where $f_n(x)$ is the ordinary, or the Cesàro, partial summation of the Fourier series of $f(x)$; in the other the original succession considered is not the succession of the partial summations of the Fourier series of $f(x)$, but the succession of functions $f(x+t)$ as t approaches zero, or any assigned value. In the former plan the new theorems in the theory of Fourier series, above stated, are required. In the latter, it is sufficient to employ the theorem that *if the integral $\int_a^b f(t)g(x+t)dt$ is a continuous function of x, it may be integrated under the sign of integration.* Each of the two plans gives all known results.

The second kind of application is to the determination of the circumstances under which, in evaluating a definite integral, of which the integrand is the product of two functions f and g, we may employ *simultaneously* the Fourier series of f and g, and so obtain the result as the sum of a double series, converging doubly, and containing only the Fourier constants of f and g. The results obtained have, as will be seen from their statement, a remarkable analogy with those obtained in the first mode of application. They are as follows :—

The process is allowable in the following cases :—

(i) *When $f(x)$ has bounded variation, $g(x)$ has a Harnack-Lebesgue integral, and the partial summations employed are the Cesàro ones.*

(ii) *When $f(x)$ is bounded and $g(x)$ is summable, and the summations are both Cesàro.*

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* Given in "On the Integration of Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 449–462, and "On a Class of Parametric Integrals, &c.," *Proc. Roy. Soc.*, A, Vol 85, pp. 401–414.

(iii) *When $f(x)$ has bounded variation and $g(x)$ is summable, and the summations are respectively ordinary and Cesàro.*

(iv) *When $f(x)$ and $g(x)$ have their squares summable, and the summations are one ordinary and one Cesàro.*

(v) *When $[f(x)]^{1+p}$ and $[g(x)]^{1+1/p}$ are summable, and the summations are both Cesàro ($p > 0$.)*

(vi) *When $f(x)$ and $g(x)$ have their squares summable, the summations are both ordinary, and one of the Fourier series converges, except at a set of content zero.*

On Multiple Fourier Series: Prof. W. H. Young.

The main object of the paper is the statement and proof of the following theorems:—

1. *If*

$$g(u, v) = \tfrac{1}{4} \operatorname{cosec} u \operatorname{cosec} v \int_0^u du \int_0^v [f(x+2u, y+2v)+f(x+2u, y-2v) + f(x-2u, y+2v)+f(x-2u, y-2v)]\, dv$$

is a continuous function of bounded variation with respect to (u, v) for all values of u and v in the rectangle $(0 \leqslant u \leqslant \pi, 0 \leqslant v \leqslant \pi)$, then the double Fourier series of $f(x, y)$ converges to the value $g(0, 0)$ at the point (x, y).

2. *If in the same rectangle $f(x, y)$ is a bounded function continuous with respect to x and*

$$g(u, v) = \tfrac{1}{4} \operatorname{cosec} v \int_0^v [f(x+u, y+2v)+f(x-u, y+2v)+f(x+u, y-2v) + f(x-u, y-2v)]\, dv$$

is a continuous function of (u, v), which has with respect to v a total variation which is bounded, then the double Fourier series of $f(x, y)$ converges doubly to $g(0, 0)$ when summed in the ordinary way with respect to y, and in the Cesàro way with respect to x.

It is also shewn how these results may be generalised so as to give theorems which hold for multiple Fourier series of functions of any number of variables, and it is indicated how other results corresponding to those given for a single variable in former papers by the author, in par-

ticular in the paper entitled "On the Convergence of a Fourier Series and of its Allied Series," may be obtained.

The method employed is similar to that in the paper just cited, but it involves considerations of a more subtle character. It is shewn first that functions of the type of the function $g(u, v)$ in (1) have converging double Fourier series, and that the Fourier constants $A_{m,n}$ of such a function satisfy the relations

$$mnA_{m,n} \to 0,$$

$$m \sum_{t=1}^{n} A_{m,t} \to 0, \qquad n \sum_{r=1}^{m} A_{r,n} \to 0,$$

when m and n increase without limit in any manner.

In the case of the function $g(u, v)$ in (2) the relation

$$\frac{n}{m+1} \sum_{r=1}^{m} \sum_{t=1}^{r} A_{t,n} \to 0$$

is proved.

From these relations and from the equations connecting the partial summations of the double Fourier series of $f(x, y)$ and of $g(u, v)$ at the point $u = 0$, $v = 0$, as well as from their generalisations, and others deduced from them, all the results in question follow.

It should be added that various preliminary theorems are proved, and that it is shewn that all multiple series of constant terms whose coefficients satisfy one or more of the relations above given, necessarily converge in the ordinary way, more or less completely, if they converge in any complete or partial Cesàro way of index not less than unity.

Thursday, February 8th, 1912.

Dr. H. F. BAKER, President, in the Chair.

Present seventeen members.

Messrs. F. M. Edwards and H. J. Priestley were elected members.

The President alluded to the loss the Society had sustained by the death of Mr. W. M. Coates.

The following papers were communicated :—

*On the Cases of Exception in Jacobi's Theorem concerning Double Residues : Prof. A. C. Dixon.

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* Printed in this volume.

*On Some Properties of Groups whose Orders are Powers of Primes: Prof. W. Burnside.

Some Results concerning Diophantine Approximations: Messrs. G. H. Hardy and J. E. Littlewood.

ABSTRACTS.

On the Cases of Exception in Jacobi's Theorem concerning Double Residues: Prof. A. C. Dixon.

In a former paper the author has shewn how to deduce the general properties of point-groups on curves from Jacobi's double residue theorem when the points are isolated; in the present paper the methods are extended to the more general case.

On Some Properties of Groups whose Orders are Powers of Primes: Prof. W. Burnside.

This paper is concerned chiefly with the maximum Abelian self-conjugate sub-groups of a group of prime-power order. In the case of an irreducible group of linear substitutions of prime-power order, it is shewn that the irreducible components of any maximum Abelian self-conjugate sub-group are all distinct; and from this conclusions are drawn as to the form of the group of linear substitutions. A process is also developed for constructing all the irreducible representations of any abstract group of prime-power order.

At the end of the paper it is shewn that no non-Abelian group with a cyclical central and no non-Abelian group the index of whose derived group is p^2 can be the derived group of a group of order p^n.

Some Results concerning Diophantine Approximations: Messrs. G. H. Hardy and J. E. Littlewood.

It is well known that if $(x) = n - [x]$, where $[x]$ is the integral part of x, and θ is an irrational number, then it is possible to find a sequence of positive integers $n_1, n_2, \ldots n_r, \ldots$, such that $(n_r\theta) \to \alpha$, where α is any number such that $0 \leqslant \alpha < 1$. Let $f(n)$ denote a positive increasing function of n, integral when n is integral, and let f_r denote the value of $f(n)$ for $n = n_r$. The result just stated suggests the following question, which seems to be of considerable interest:—*For what forms of f_n is it*

* Printed in this volume.

true that, for any value of a *such that* $0 \leqslant a < 1$, *a sequence* n_r *can be found such that* $(f_r, \theta) \to a$ *?*

It is easy to see that, when the increase of $f(n)$ is sufficiently rapid, the result suggested will not be generally true. Thus, if $f(n) = 2^n$, and θ is a number which, when expressed in the binary scale, shows at least k O's following upon every 1, it is plain that

$$(2^n\theta) < 2^{-1}+2^{-k-1}.$$

There is thus an "excluded interval" of values of a, the length of which can (by increasing k) be made as near to $\frac{1}{2}$ as we please. If $f(n) = 3^n$ we can obtain an excluded interval whose length is as near to $\frac{2}{3}$ as we please, and so on, while if $f(n) = n!$ it is (as is well known) possible to choose θ so that $(n!\,\theta)$ has a unique limit.

The first object of this investigation has been to prove the following theorem :—

1. *If* $f(n)$ *is a polynomial in* n, *with integral coefficients, and* $0 \leqslant a < 1$, *then a sequence can be found for which* $(f_r, \theta) \to a$.

The solution of this problem has suggested a number of others, some of which have already been solved. The following theorems will shew the kind of results that can be obtained :—

2. *A sequence can be found such that* $(n_r\theta)$, $(n_r^2\theta)$, ..., $(n_r^k\theta)$ *tend simultaneously to zero.*

3. *If* $\theta, \phi, \psi, \ldots$, *are linearly independent irrationals, and* $\alpha, \beta, \gamma, \ldots$, *lie between* 0 *and* 1, *a sequence can be found such that*

$$(n_r\theta) \to \alpha, \quad (n_r\phi) \to \beta, \quad (n_r\psi) \to \gamma, \quad \ldots,$$

$$(n_r^2\theta) \to 0, \quad (n_p^2\phi) \to 0, \quad (n_r^2\psi) \to 0, \quad \ldots.$$

4. *A sequence can be found such that* $(n_r\theta) \to \alpha$, $(n_r^2\theta) \to \beta$.

The authors have (with the kind assistance of Dr. W. W. Greg, Librarian of Trinity College) tabulated the values of $(n^2\theta)$, where $\theta = 1/\sqrt{10} = \cdot 31622776, \ldots$, for the first 300 values of n. The distribution of these values shews striking irregularities which suggest a variety of interesting questions. The answers to some of them would be likely to find applications in the theory of the Riemann ζ-function. That arithmetical theorems concerning approximation play an important part in the theory of Dirichlet's series in general, and the ζ-function in particular, has already been shewn in the most striking manner by Bohr.

Thursday, March 14th, 1912.

Mr. J. E. CAMPBELL, Vice-President, in the Chair.

Present also fourteen members.

Messrs. R. C. Archibald, E. B. Stouffer, J. B. Shaw, J. W. Nicholson, and E. O. Lovett were elected members.

The following papers were communicated :—

The Cubic Surface as a Degenerate Quartic : Mr. G. T. Bennett.

*On Differential Operators which Generate all Seminvariants and all Ternary Covariant Sources : Prof. E. B. Elliott.

Prof. W. H. Young made an informal communication on Goursat's Form of Cauchy's Theorem.

ABSTRACTS.

The Cubic Surface as a Degenerate Quartic : Mr. G. T. Bennett.

The surface treated is the general cubic surface represented in the form

$$(x+u)(y+u)(z+u)(t+u) = xyzt,$$

and gives fifteen lines and fifteen triple-tangent planes by inspection. The rest are dependent on a parameter determined by a quadratic and are tabulated. The form is closely related to the Cremona hexahedron and the transformation is given.

The reciprocal of the configuration of fifteen lines and fifteen planes is obtained barycentrically.

A metrical form of double-six is examined, with its lines lying on the faces of a parallelopiped. A special cubical form has twelvefold self-congruence.

On Differential Operators which Generate all Seminvariants and all Ternary Covariant Sources : Prof. E. B. Elliott.

If G is any gradient, *i.e.*, any rational, integral homogeneous isobaric function, in the unending series of letters a_0, a_1, a_2, ..., and if Ω denotes $a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \ldots$, then $G = \Omega F$, where F is another gradient

* Printed in this volume.

which can be found from G by direct differential operation. F may have to involve letters more advanced in the sequence $a_0, a_1, a_2, \ldots$ than G does.

A first result in this paper is the finding of an operator which derives an appropriate F from a given G, and has certain advantages over the operator generally used for the purpose.

It is also known that all seminvariants in the letters $a_0, a_1, a_2, \ldots$, can be obtained by direct operation on gradients. A new operator effective for this purpose is found.

After this it becomes easy to construct an operator for the generation of all *ternary* covariant sources from gradients in a double system of letters. Hitherto, I believe, such operators have been looked for in vain.

Thursday, April 11th, 1912.

Dr. H. F. BAKER, President, and temporarily Prof. A. E. H. LOVE, Vice-President, in the Chair.

Present also ten members.

The minutes of the last meeting were read and confirmed.

Messrs. A. M. Grundy and M. Jaffee were elected members.

The following papers were communicated :—

Mersenne's Numbers : Lt.-Col. A. Cunningham.

*Contributions to the Arithmetic Theory of Series : Messrs. G. H. Hardy and J. E. Littlewood.

*Some Transformations of Kummer's Surface ; and *On the Curves which lie on a Cubic Surface : Dr. H. F. Baker.

A Modification of Liouville's Theorem : Mr. G. N. Watson.

*A Theory of Binary Arithmetical Forms with Complex Integral Coefficients : Mr. G. B. Mathews.

An Application of the Theory of Integral Equations to the Equation $\nabla^2 u + k^2 u = 0$: Prof. H. S. Carslaw.

ABSTRACTS.

Mersenne's Numbers : Lt.-Col. A. Cunningham.

Lt.-Col. Cunningham announced the verification as composite of

* Printed in this volume.

another of the numbers affirmed by Mersenne to be composite, viz.,

$$2^{173}-1 \equiv 0 \pmod{730753},$$

thus leaving only 14 of those numbers affirmed composite still unverified, viz., those with exponents

101, 103, 107, 109, 137, 139, 149, 157, 167, 193, 199, 227, 229, 241.

And further reported that he had had all possible divisors of these 14 numbers tested up to 800000, without success. Every division was tried twice. The list of "trial divisors" had been collated with a list of the same prepared by M. A. Girardin of Nancy, France, Editor of *Sphinx-Œdipa*.

Contributions to the Arithmetic Theory of Series: Messrs. G. H. Hardy and J. E. Littlewood.

An abstract of the contents of part of this paper has already been printed under the title "A New Condition for the Truth of the Converse of Abel's Theorem" (*Records*, for January).

A number of general theorems are proved concerning the successive derivatives of a function of a real variable. The results referred to above, as well as others proved jointly or severally by the authors, are thus presented as parts of a general theory. The paper also contains a large number of new theorems concerning Dirichlet's series, multiplication of series, and other matters.

Some Transformations of Kummer's Surface: Dr. H. F. Baker.

This note examines algebraically a transformation of infinite period which has been stated to belong to a general Kummer surface. The conclusion reached is that the transformation is not of infinite period.

On the Curves which lie on a Cubic Surface: Dr. H. F. Baker.

The object of this paper is to explain a method of enumerating the algebraic curves lying on a general cubic surface, and incidentally to show that the formulation of the group of the lines of the surface as a linear group in six variables arises naturally in this way.

A Theory of Binary Arithmetical Forms with Complex Integral Coefficients: Mr. G. B. Mathews.

This paper is on the same topic as one by Hurwitz,* but the procedure

* *Acta Math.*, Vol. 25, pp. 231–290.

is different in several ways. A definition of reduced forms is given from geometrical considerations, and it leads at once to the finiteness of the number of reduced forms. Chain-fractions appear as an incident rather than a definition. Kronecker's restriction on the definition of proper equivalence is not required, and finally, in accordance with greater usage, binomial coefficients are omitted in the forms.

An Application of the Theory of Integral Equations to the Equation $\nabla^2 u + k^2 u = 0$: Prof. H. S. Carslaw.

The following applications of the theory of integral equations with a symmetrical kernel seem to have escaped notice.

Let $K(x, y, z; x_1, y_1, z_1)$ *be the solution of*

$$\nabla^2 u + k^2 u = 0,$$

with the following properties :—

It is to be finite and continuous, as also its first and second differential coefficients, in the interior of the closed surface S, except in the neighbourhood of the point (x_1, y_1, z_1), *where it is to be infinite as*

$$\frac{e^{-ikr}}{4\pi r},$$

when $r \to 0$.

At the surface of S, $\frac{du}{dn}$ *is to be zero.*

Adopting Kneser's notation for the kernel which he employs in "Some Applications of Integral Equations to the Theory of Heat,"* we denote this function by $K(0, 1)$. It is the velocity potential (omitting the time factor) for the case of a unit source in a medium filling the interior of S, the boundary being supposed rigid.

We shall call $K(0, 1)$ the Green's function of the problem.

Let ϕ *be any solution of*

$$\nabla^2 \phi + (\lambda + k^2)\, \phi = 0,$$

which is finite and continuous, as also its first and second differential coefficients, right through the interior of S, and satisfies the same boundary condition.

Then applying Green's theorem to the region between a small sphere

* Kneser, *Die Integralgleichungen und ihre Anwendung in der mathematischen Physik*, § 31 (Braunschweig, 1911).

Σ, whose centre is at (x_1, y_1, z_1), and the surface S, we have

$$\iiint (\phi\nabla^2 K - K\nabla^2\phi)\,dx\,dy\,dz$$
$$= \iint \left(\phi \frac{dK}{dn} - K\frac{d\phi}{dn}\right) dS - \iint \left(\phi \frac{dK}{dn} - K\frac{d\phi}{dn}\right) d\Sigma.$$

From the conditions imposed upon K and ϕ, it follows, on letting the radius of the sphere Σ tend to zero, that

$$\phi(x_1, y_1, z_1) = \lambda \iiint K(0, 1)\,\phi(x, y, z)\,dx\,dy\,dz.$$

And the integration can now be taken right through the region.

This result may be conveniently written in the form

$$\phi(1) = \lambda \iiint K(0, 1)\,\phi(0)\,dx\,dy\,dz.$$

Thus the Green's function, defined above, is the kernel of a homogeneous integral equation.

It is obvious from physical reasons that it is a symmetrical function. This can also be shown by a simple application of Green's theorem.

*The importance of this result depends upon the fact that a fundamental theorem in the theory of integral equations allows us, under certain conditions, to equate the sum of the series formed by the normalised orthogonal solutions of the equation to the kernel.**

I hope shortly to communicate to the Society a paper in which many important Green's functions are obtained in this way.

The theorem also holds for the corresponding Green's function in the two-dimensional case.

And the boundary condition $\frac{du}{dn} = 0$ may be replaced by $u = 0$, if desired.

We have thus a method of discussing various diffraction problems in sound waves and also in electric waves.

It has been shewn by Sommerfeld† that Green's theorem can be extended to the case of a Riemann's surface or space. The argument at the beginning of this communication will apply to such cases: and a modified Green's function will be introduced as kernel of the integral equation. In particular, the multiform solutions referred to in a former paper of mine in these *Proceedings*,‡ and others, can be obtained in this way.

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* Cf. Bôcher, "An Introduction to the Theory of Integral Equations" (*Cambridge Math. Tracts*, No. 10), p. 58. Kneser, *loc. cit.*, § 41.

† *Proc. London Math. Soc.*, Ser. 1, Vol. XXVIII, p. 403.

‡ *Proc. London Math Soc.*, Ser. 1, Vol. XXX, p. 121.

Thursday, May 9th, 1912.

Dr. H. F. BAKER, President, in the Chair.

Present twelve members and a visitor.

The minutes of the last meeting were read and confirmed.

Messrs. L. Isserlis, F. J. M. Stratton, and M. J. Couran, were elected members.

The following paper was communicated :—

*The Irreducibility of Legendre's Polynomials: J. B. Holt.

Prof. Love described a model made to exhibit the motion of the instantaneous axis of a rigid body rotating about its centre of mass.

In speaking of the same matter Prof. Lamb emphasised the distinction between the path of the instantaneous axis and that of the invariable line.

ABSTRACT.

The Irreducibility of Legendre's Polynomials: J. B. Holt.

The object of the paper is to shew that $P_n(x)$ (divided by x when n is odd) is irreducible when any of $n-1$, n, $n+1$, $n+2$, $\frac{n+1}{2}$, $\frac{n+2}{2}$ are primes: and again when n or $n-1$ is a power of 2. The method depends upon the following lemma.

Let $f(x)$ be a polynomial with integer coefficients, having no common factor. If the first r coefficients be divisible by a prime p, whilst the first is not divisible by p^2, $f(x)$ is either irreducible or contains an irreducible factor of the same type. The same holds with "last" for "first."

It is proved by putting $f(x)$ equal to the product of two polynomials, and equating coefficients.

Writing down the explicit form of $P_n(x)$ for n even and n odd, and integralising the coefficients by multiplication with a power of 2, it is shewn from the form of the coefficients that, p being an odd prime,

(1) If $E\left(\frac{n}{p}\right)$ be odd and $<p$: the last coefficient is divisible by p but not by p^2: the last $p+1$ coefficients are divisible by p when n is even: and the last $p-1$ when n is odd.

* Printed in this volume.

(2) If $E\left(\frac{2n}{p}\right)$ be odd and $<p$: the first coefficient is divisible by p but not by p^2: and if $r \equiv 2n+1 \bmod p$, and $r < p$, then the first r coefficients are divisible by p.

The lemma then shews at once that P_n is irreducible for the first four cases. For the fifth, where $n = 2p-1$, (1) and (2) shew that if P_n/x be reducible, it is the product of two irreducible factors of order $p-1$, and therefore cannot have one of greater order; but as there is a prime q between p and n, there is by (1) an irreducible factor of degree $q-1$ at least. Hence P_n/x is irreducible. The proof of the sixth case is similar.

For the case where n is a power of 2: start from

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x+y\cos\phi)^n\, d\phi \quad (y^2 = x^2-1)$$

$$= x^n + a_2 x^{n-2}y^2 + a_4 x^{n-4}y^4 + \ldots + a_n y^n.$$

It is shewn that $2^r a_r$ is an even integer for $r = 2, 4, \ldots, n$; and that $2^n a_n$ is not divisible by 4 when n is a power of 2.

Putting $n = 2m$, $z = \frac{4x^2}{x^2-1}$: the equation $P_n(x) = 0$ is equivalent to

$$z^m + 2^2 a_2 z^{m-1} + 2^4 a_4 z^{m-2} + \ldots + 2^n a_n = 0,$$

which is irreducible. From this it is shewn that $P_n(x)$ is irreducible.

An almost identical proof applies to the case $n-1 =$ a power of two.

Thursday, June 13th, 1912.

Dr. H. F. BAKER, President, in the Chair.

Present eleven members.

The minutes of the last meeting were read and confirmed.

Messrs. A. B. Grieve and L. J. Mordell were elected members.

The following papers were communicated:—

Some Properties of Symmetric and Orthogonal Substitutions: Mr. H. Hilton.

*Closed Orbits of Ejection and Related Periodic Orbits: Prof. F. R. Moulton.

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* Printed in this volume.

*On a Certain Series of Fourier: Prof. W. H. Young.

The Fourier Series of a Bounded Function: Prof. W. H. Young.

Some Properties of the Extended Zeta-Function: Mr. G. N. Watson.

*Curves of Contact of any Order on Algebraic Surfaces: Miss H. P. Hudson.

Dr. Bromwich made an informal communication "On a Solution of Laplace's Equation."

ABSTRACTS.

Some Properties of Symmetric and Orthogonal Substitutions: Mr. H. Hilton.

It is possible to transform a given symmetric substitution into the direct product of substitutions of the form

$$x_1' = ax_1 + x_2, \ \ldots, \ x_{s-1}' = ax_{s-1} + x_s, \ x_s' = ax_s;$$

and at the same time to transform every orthogonal substitution into one which has as invariant the sum of quadratic functions of the form

$$x_1x_s + x_2x_{s-1} + x_3x_{s-2} + \ldots + x_sx_1.$$

It is possible to transform a given orthogonal substitution into the direct product of substitutions of the form

$$\left.\begin{aligned} x_1' &= ax_1 \ \ + x_2, \ \ldots, \ x_{s-1}' = ax_{s-1} \ \ + x_s, \ x_s' = ax_s \\ y_1' &= a^{-1}y_1 + y_2, \ \ldots, \ y_{s-1} = a^{-1}y_{s-1} + y_s, \ y_s' = a^{-1}y_s \end{aligned}\right\},$$

where $a^2 \neq 1$, or else $a^2 = 1$, and s is even; and of substitutions of the form

$$X_1' = aX_1 + X_2, \ \ldots, \ X_{s-1}' = aX_{s-1} + X_s, \ X_s' = aX_s,$$

where $a^2 = 1$ and s is odd; and at the same time to transform every orthogonal substitution into one which has as invariant the sum of quadratic functions of the type $f_1(x, y)$ and $f_1(X, X)$ respectively; where $f_1(x, y)$ is the fundamental invariant of a canonical substitution defined in *Proc. London Math. Soc.*, Ser. 2, Vol. 10 (1911), p. 426.

These results are employed to find the orthogonal substitutions permutable with a given symmetric or orthogonal substitution.

Properties of quasi-Hermitian and quasi-unitary substitutions are given similar to those proved for symmetric and orthogonal substitutions.

* Printed in this volume.

Closed Orbits of Ejection and related Periodic Orbits: Prof. F. R. Moulton.

In this paper it is proved that, in the restricted problem of three bodies with one finite mass sufficiently small, orbits of ejection from the larger finite mass exist such that they are also orbits of collision; that is, they are geometrically closed. The direction of ejection is either toward or from the smaller finite mass, and the period from ejection to collision is an arbitrary multiple of the period of revolution of the finite masses. The closed orbits of ejection are not periodic either physically or in the analytic sense, but they are the limits of two families of periodic orbits.

It is proved that periodic orbits exist, when one of the finite masses is sufficiently small, which pass near the larger finite mass an arbitrary number of times before being closed. When there is but a single close approach to the finite mass, the limits of the periodic orbits are the closed orbits of ejection.

It is proved that, when one of the finite masses is sufficiently small, it is possible to determine the direction of motion of the infinitesimal body so it shall pass through an arbitrary point with arbitrary speed, and be either an orbit of ejection or of collision with the larger finite mass.

On a Certain Series of Fourier: Prof. W. H. Young.

The series considered is $\sum_{n=1} \frac{\cos nx}{n}$. The importance of the discussion of such special series is first pointed out. It is then shewn that the chasm function of the series has the value $+\infty$ at the origin, so that the divergence there is uniform in the sense previously employed by the author. It appears further that the ratio of the sum-function to the n-th partial summation has the unique double limit unity. The method employed involves in part the consideration of the maxima and minima of the partial summations. As an example of the use of the results arrived at, application is made to shew that, denoting by $a_1, a_2, \ldots$, the coefficients of the cosine terms in the Fourier series of a function $f(x)$, which is such that $|f(x)| \log |f(x)|$ is summable, then $\sum_{n=1} a_n/n$ is a convergent series. For this purpose a certain inequality is employed, viz.,

$$ab \leqslant a \log a + e^{b-1}, \quad (0 \leqslant a).$$

Two proofs of this inequality are appended, one immediate and the other by means of a limiting process from a known inequality.

The Fourier Series of a Bounded Function: Prof. W. H. Young.

In this paper a number of theorems are proved dealing with the convergence of the Fourier series of bounded functions and of their allied series, when the coefficients have been multiplied by certain convergence factors. Beside the general class of bounded functions, two special classes of bounded functions are considered, viz., the class of those that are simply discontinuous and the class of those which in addition possess bounded variation. It is shown that the Fourier series of the functions of the last-named class so narrowly escape being the Fourier series of integrals that almost any monotone succession of constants, with zero as limit, which occurs naturally, will, when employed as convergence factors, change the series into the Fourier series of an integral. A scarcely less wide class of constants will even change the allied series of such a Fourier series into the Fourier series of an integral. It follows, in particular, that, if a_n and b_n be, as usual, the typical cosine and sine constants of a function of bounded variation, the series $\Sigma n q_n a_n b_n$ will certainly converge if q_n is of the form $1/(1_r n)^p$, $(0 < p)$. If it is known that a bounded function $f(x)$ has all its discontinuities of the first kind, it is found that the series $\Sigma a_n/\log n$ necessarily converges, and an expression is found for its sum. This requires an examination of the series $\Sigma \cos nx/\log n$, and it is shown that, though the corresponding sine series is, as is known, not a Fourier series, converging as it does to a function which has an infinity at the origin of the order $1/x \log x$, the cosine series has a sum which is at least as summable as $1/x(\log x)^2$, though it is not as summable as $1/x(1_1 x)^2 1_2 x \ldots (1_n x)^{1+p}$ $(0 < p)$. If all that is known of the function $f(x)$ is that it is bounded, we can only assert that the series $\Sigma a_n/\log n$ converges when summed in the Cesàro manner, index unity. We have, however, another series of theorems relating to the general class of bounded functions, viz., the series whose general term is $a_n q_n$ converges if q_n is such that $q_n \log n$ has the unique limit zero, while the $q_1, q_2, \ldots$ is a monotone descending sequence whose decrements form another sequence of the same kind, while the series whose general term is $b_n q_n$ converges if $q_1, q_2 \ldots$ be any monotone descending sequence with zero as limit, which is such that $\Sigma q_n/n$ converges.

Some Properties of the Extended Zeta-Function: Mr. G. N. Watson.

The behaviour of $\xi(s, a) = \sum_{n=0}^{\infty} (a+n)^{-s}$ (where $s = \sigma + it$), when $|t|$ is large, presents many resemblances to the behaviour of the ordinary zeta-

function $\xi(s) = \xi(s, 1)$. The results I have obtained are as follows:

$$\xi(s, a) = O(e^{t \arg a}) + O(t^{\tau(\sigma)} \log t),$$

where* $\tau(\sigma) = 0, \quad \sigma \geqslant 1,$

$$= \tfrac{1}{2}(1-\sigma), \quad 0 \leqslant \sigma \leqslant 1,$$

$$= \tfrac{1}{2} - \sigma, \qquad \sigma \leqslant 0.$$

The $\log t$ can be omitted except when $0 \leqslant \sigma \leqslant 1$.

The work is of the same nature as that given in Landau's *Primzahlen*, § 228, when $\sigma \geqslant \frac{1}{2}$, but, since there is no relation corresponding to Riemann's relation

$$2^{1-s} \Gamma(s) \xi(s) \cos(\tfrac{1}{2} s\pi) = \pi^s \xi(1-s)$$

(except that the relation of Hurwitz holds when $0 < a \leqslant 1$), we cannot deduce the behaviour of $\xi(s, a)$, when $\sigma < \frac{1}{2}$, from its behaviour when $\sigma \geqslant \frac{1}{2}$. It is accordingly necessary to employ some elaborate transformations of contour integrals to obtain the required result.

Curves of Contact of any Order on Algebraic Surfaces: Miss H. P. Hudson.

The object of this paper is to find formulæ for the postulation and equivalence for a family of algebraic surfaces of contact of any order i with a given surface along a given curve C. These formulæ are required in the theory of Cremona transformations. The results are

$$\text{Postulation} = iP_1 - N$$

$$\text{Equivalence} = iE_1 - 2N$$

where P_1, E_1 are the postulation and equivalence of C as a simple curve, and N depends on the nature of C and the singularities of the given surface. If $N \geqslant 0$, the curve of contact can be regarded as the limit of i distinct curves of simple intersection meeting one another in N points; but this is not so in the general case. Similar formulæ are found for a multiple curve.

A Solution of Laplace's Equation: Dr. T. J. I'A. Bromwich.

In potential problems which have an axis of symmetry, it has usually been assumed that there are two, and only two, solutions of Laplace's

* The function $\tau(\sigma)$ is that defined by Landau in connection with the ordinary zeta-function.

equation, which are available near the origin: namely,

$$r^n P_n(\mu), \qquad r^n Q_n(\mu),$$

where $\mu = z/r$, and P_n is Legendre's polynomial of degree n, while

$$Q_n(\mu) = \tfrac{1}{2} P_n \log\left(\frac{1+\mu}{1-\mu}\right) - \left(\frac{2n-1}{1.n} P_{n-1} + \frac{2n-5}{3(n-1)} P_{n-3} + \ldots\right).$$

Apart from the generalisations introduced by supposing n not to be an integer, I have found no reference to other solutions which have an axis of symmetry; but it appears that, if

$$F_n(r, z) = r^n \{P_n \log(r+z) - 2\phi(\mu)\},$$

where $$\phi(\mu) = \frac{2n-1}{1.2n} P_{n-1} - \frac{2n-3}{2(2n-1)} P_{n-2} + \frac{2n-5}{3(2n-2)} P_{n-3} - \ldots,$$

then $F_n(r, z)$ is a solution of Laplace's equation, which is obviously symmetrical about the axis of z. Here, as throughout this note, n is taken to be a positive integer, and there are n terms in the series of polynomials $\phi(\mu)$.

It will be found that we can write

$$r^n Q_n(\mu) = \tfrac{1}{2} \{F_n(r, z) - (-1)^n F_n(r, -z)\},$$

allowing for the changes of sign introduced by changing the sign of z (and so of μ).

It may be noticed that $F_n(r, z)$ has no singularity on the *positive* half of the axis of z. Thus, in order to prove that a solution of Laplace's equation coincides with $r^n P_n$, it is *not* sufficient to prove that the solution is symmetrical and is finite when we write $\mu = 1$.

The solution corresponding to $F_n(r, z)$ which tends to zero when r tends to infinity is found by inversion as usual, and is

$$\frac{1}{r} F_n\left(\frac{1}{r}, \frac{z}{r^2}\right) = \frac{1}{r^{n+1}} \left\{P_n \log\left(\frac{r+z}{r^2}\right) - 2\phi(\mu)\right\},$$

a solution which (as was pointed out to me by Prof. Hobson) can be written in the symbolic form

$$\frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n} \left\{\frac{1}{r} \log\left(\frac{r+z}{r^2}\right) + \frac{A}{r}\right\}.$$

The function $F_n(r, z)$, when n is even, presents itself in the calculation of the potential of a circular disc whose surface density is proportional to r^{n-1}. To take the simplest case, suppose that the surface density is kr,

and it will be found that on the axis of the disc the potential is equal to

$$\pi k a^2 \left(1+\tfrac{1}{2}\frac{r^2}{a^2}-\frac{1.1}{2.4}\frac{r^4}{a^4}+\frac{1.1.3}{2.4.6}\frac{r^6}{a^6}-\ldots\right)$$

$$-\pi k r^2 \left\{\log\left(\frac{2a}{r}\right)+\tfrac{1}{2}\frac{r^2}{2a^2}-\frac{1.3}{2.4}\frac{r^4}{4a^4}+\frac{1.3.5}{2.4.6}\frac{r^6}{6a^6}-\ldots\right\} \quad (r<a).$$

To obtain the general formula for the potential, the terms in r^n are to be replaced by $r^n P_n$, except when $n=2$; but for those in r^2 we have to use the formulæ

$$\pi k r^2\left[P_2\left\{\log\left(\frac{r+z}{4a}\right)+\tfrac{5}{3}\right\}-\tfrac{3}{2}\mu+\tfrac{1}{3}\right] \quad (z>0),$$

or

$$\pi k r^2\left[P_2\left\{\log\left(\frac{r-z}{4a}\right)+\tfrac{5}{3}\right\}+\tfrac{3}{2}\mu+\tfrac{1}{3}\right] \quad (z<0),$$

which are respectively of the forms

$$\pi k\left\{F_2(r,z)+Ar^2P_2\right\}, \qquad \pi k\left\{F_2(r,-z)+Ar^2P_2\right\}.$$

LIBRARY

Presents.

BETWEEN June, 1911, and November, 1912, the following presents were made to the Library, from their respective Authors or Publishers :—

Bachelier, L.—"Les probabilités à plusieurs variables," 1910.
Burnside, W.—"Theory of Groups of Finite Order," 2nd ed., 1911.
del Corral, José Isaac.—"Nuevos Métodos para resolver Ecuaciones numéricas," 1912.
Cramer, A.—"Pubertät und Schule," 1911.
de Donder, Th.—"Quelques Remarques sur le Multiplicateur de Jacobi et de Multiplicateur généralisé, 1911.
———— "Sur les Invariants Intégraux Relatifs et leurs Applications à la Physique Mathématique," 1911.
Dugan, Raymond Smith.—"The Algol-System R. T. Persei," 1911.
Forsyth, A. R.—"Lectures on the Differential Geometry of Curves and Surfaces," 1912.
Harris, Frank.—"Gravitation," 1912.
Kelvin, Lord.—"Mathematical and Physical Papers," vol. 6, 1912.
Lebon, E.—"Gaston Darboux, Biographie," 1910.
Martin, Artemas.—"Algebraic Solution of the celebrated 'Three Point' Problem," 1910.
Nielsen, Niel.—"Beretnung om den andea Skandinaviski Matematiski Kongres i Kjøbenhavn, 1911," 1912.
Poincaré, Henri.—"The Future of Mathematics," 1910.
Prym, Fr., und Rost, Georg.—"Theorie der Prym'schen Functionen enter Ordnung," 1911.
Raman, C. V.—"Experimental Investigations on the Maintenance of Vibrations," 1912.
Rahman Khan, Habibur.—"Water Wireless Telegraphy," 1911.
Schaller, J. Georg.—"Beweis der Richtigkeit des Grossen Fermatschen Satzes," 1912.
See, T. J. J.—"Researches on the Evolution of the Stellar Systems," vol. 2, 1910.
Stone, Ormond.—"Simon Newcomb," 1910.
"Signatures in the First Journal Book and the Charter Book of the Royal Society," 1912.
"Record of the Royal Society," 3rd ed., 1912.
"The Newhaven Mathematical Colloquium," Lectures by E. H. Moore, E. J. Wilczynski, and M. Mason, 1910.
"Abhandlungen über den Mathematischen Unterricht in Deutschland veranlasst durch die Internationale Mathematische Unterrichtskommission," herausgegeben von F. Klein, bd. 1, hefte 1–4 ; bd. 2, hefte 1–7 ; bd. 3, hefte 1–7 ; bd. 4, hefte 1, 4, 6, 7 ; bd. 5, hefte 1–3 ; 1909–12.
"Berichte und Mittheilungen veranlasst durch die Internationale Unterrichtskommission," nos. 4–6 ; 1910–11.
"Schriften des Deutschen Ausschusses für den Mathematischen und Naturwissenschaftlichen Unterricht," hefte 1–3, 5–14 ; 1909–12.

Calcutta : Indian Engineering, vol. 49, nos. 16–26 ; vol. 50, 1911 ; vol. 51 ; vol. 52, nos. 1–14, 1912.
Cape Town : South African Journal of Science, vol. 6, nos. 2–12 ; vol. 7 ; vol. 8 ; vol. 9, nos. 1–3, 1911–12.
Coimbra : Accademia Polyt. de Porto, Ann. Scientificos, vol. 7, nos. 1, 2, 1912.

Edinburgh: Math. Notes, no. 6, 1910; no. 7, 1911.
Kansas: Univ. Science Bulletin, vol. 5, nos. 12–21, 1911; vol. 6, no. 1, 1912.
London: Educational Times, vol. 64, nos. 602–608, 1911; vol. 65, nos. 609–619, 1912.
London: Educational Times, Math. Questions and Solutions, new series, vols. 19, 20, 1911; vol. 21, 1912.
London: Mathematical Gazette, vol. 6, nos. 92–100, 1911–2.
London: Nautical Almanac for 1914.
Madras: Indian Mathematical Society, Journal, vol. 3, 1911; vol. 4, nos. 1–5, 1912.
Madrid: Rivista de la Sociedad Matematica Española, año 1, núm 1.
Manchester: Municipal School of Technology, Journal, vol. 4, 1910; vol. 5, 1912.
Paris: L'Enseignement Math., ann. 13, nos. 3–6, 1911; ann. 14, nos. 1–5, 1912.
Sendai: Tôhoku Mathematical Journal, vol. 1, 1911; vol. 2, pts. 1 and 2, 1912.
Tokyo: Sūgaku-Butsurigaku, vol. 6, nos. 3–19, 1911–12.
Warsaw: Wiadomości Matem., tom. 15, 1911; tom. 16, zeszyt 1–4, 1912.
Washington: Journal of the Academy of Sciences, vol. 1, nos. 1–4, 1912.

Exchanges.

Between June, 1911, and November, 1912, the following exchanges were received:—

American Journal of Mathematics, vol. 32, nos. 3, 4, 1910; vol. 33, 1911; vol. 34, no. 1, 1912.
American Mathematical Society, Transactions, vol. 12, nos. 3, 4, 1911; vol. 13, nos. 1, 2, 1912.
American Mathematical Society, Bulletin, vol. 17, nos. 7–10, 1911; vol. 18, 1911–12; vol. 19, no. 1, 1912.
American Philosophical Society, Proceedings, vol. 50, 1911; vol. 51, nos. 203–205, 1912.
Amsterdam: Nieuw Archiev, deel 9, stuk 4, 1911; deel 10, stuk 1, 1912.
Amsterdam: Revue Semestrielle, tome 19, pt. 2, 1911; tome 20, 1912.
Amsterdam: Wiskundige Opgaven, deel 11, stuk 1–3, 1911.
Amsterdam: Wiskundige Opgaven, Register, 1875–1910, 1911.
Belgique: Académie Royale des Sciences, Bulletin, 1911, nos. 3–12; 1912, nos. 1–7.
Berlin: Jahrbuch über die Fortschritte, bd. 39, heft 3, 1911; bd. 40, hete 1–3, 1911–12.
Berlin: Journal für die Mathematik, bd. 140, 1911; bd. 141, 1912.
Berlin: Sitzungsberichte der K. Preuss. Akademie, 1911; 1912, nos. 1–38.
Bologna: R. Accademia, Memorie, ser. 6, tom. 7, and supplement, 1910, tom. 8, 1911.
Bologna: R. Accademia, Rendiconti, new series, vol. 14, 1910; vol. 15, 1911.
Bordeaux: Société des Sciences, Bulletin, Commission Météorologique, Gironde, 1910, 1911.
Bordeaux: Société des Sciences, Procès-Verbaux, 1910–11.
Boston: American Academy of Arts and Sciences, vol. 46, nos. 18–25, 1911; vol. 47, nos. 1–21, 1912; vol. 48, nos. 1–3, 1912.
Cambridge Philosophical Society, Proceedings, vol. 16, pt. 2, 1911–12.
Cambridge Philosophical Society, Transactions, vol. 21, pts. 15–18, 1911; vol. 22, pt. 1, 1912.
Cambridge, Mass.: Annals of Mathematics, vol. 12, nos. 2, 3, 1911; vol. 13, 1912; vol, 14, no. 1, 1912.
Canadian Institute, Transactions, no. 20, 1910; no. 21, 1912.
Catania: Accademia Gioenia, Atti, ser. 5, vol. 4, 1911.
Catania: Accademia Gioenia, Bollettino, ser. 2, fasc. 16–19, 1911; fasc. 20–23, 1912.
Dublin: Royal Irish Academy, Proceedings, ser. A, vol. 28, nos. 2, 3, 1910; vol. 29, 1911.
Edinburgh: Mathematical Society, Proceedings, vol. 28, 1910; vol. 29, 1911.
Edinburgh: Royal Society, Proceedings, vol. 30, nos. 4–7, 1910; vol. 31, nos. 1–5, 1911–12.
Erlangen: Physik.-medicin. Societät, Sitzungsberichte, bd. 42, 1910.
France: Société Mathématique, Bulletin, tome 39, fasc. 2, 3, 1911; tome 40, 1912.
Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1911; hefte 2–6, 1912.

Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Geschaft. Mitth., 1911; 1912.
Holland: Archives Néerlandaises, ser. A, tome 1, liv. 1, 2, 1911; ser. B, tome 1, liv. 1–4, 1911–12.
København: Kgl. Danske Vidensk. Selsk., Forhandlinger, Oversigt, 1911, nos. 2–6; 1912, nos. 1, 2.
København: Kgl. Danske Vidensk. Selsk., Skrifter, ser. 7, tome 6, nos. 7, 8; tome 8, no. 6, 1911–12.
Leipzig: Beiblätter zu den Annalen der Physik, bd. 35, hefte 9–24, 1911; bd. 36, hefte 1–22, 1912.
Leipzig: K. Sächsische Gesell., Math. Klasse, Berichte, 1910, nos. 6, 7; 1911; 1912, 1, 2.
Leipzig: K. Sächsische Gesell., Math. Klasse, Abhandlungen, bd. 32, nos. 3–5, 1911.
Livorno: Periodico di Matematica, anno 27, 1911–12; anno 28, fasc. 1, 1912.
Livorno: Periodico di Matematica, Supplemento, anno 14, fasc. 7–9, 1911; anno 15, 1912.
Lombardo: Reale Instituto, Memorie, vol. 21, fasc. 5, 1911.
Lombardo: Reale Instituto, Rendiconti, vol. 43, fasc. 17–12, 1910; vol. 44, 1911; vol. 45, fasc. 1–15, 1912.
London: Institute of Actuaries, Journal, vol. 44, pts. 2–4, 1910; vol. 45, 1911; vol. 46, pt. 1, 1912.
London: Institution of Naval Architects, Transactions, vol. 51, 1911; vol. 52, 1912.
London: National Physical Lab., Collected Researches, vol. 7, 1911; vol. 8, 1912.
London: National Physical Lab., Reports, 1910, 1911.
London: Nature, vols. 87, 88, 89, and 90, nos. 2236–2246, 1911–12.
London: Physical Society, Proceedings, vol. 23, pts. 4, 5, 1911; vol. 24, 1912.
London: Royal Society, Proceedings, Series A, nos. 577–597, 1911–12.
London: Royal Society, Transactions, Series A, vol. 211, 1912.
Manchester Literary and Philosophical Society, Memoirs, vol. 54, pts. 2, 3, 1910; vol. 55, 1911.
Marseille: Annales de la Faculté des Sciences, tome 18, 1909; tome 19, 1910; tome 20, 1911; 20 suppl., 1912.
Milan: Annali di Matematica, tome 43, fasc. 17–20, 1910; tome 44, fasc. 1–14, 1911.
Modena: Regia Accademia, Memorie, vol. 9, 1910.
München: K. Bayer Akad. d. Wissen., Math.-Phys. Kl., Abhandlungen, bd. 25, abh. 5–10, 1911–12.
München: K. Bayer Akad. d. Wissen. Math.-Phys. Kl., Sitzungsberichte, 1910, abh. 10–15; 1911; 1912, abh. 1, 2.
Napoli: Accademia delle Scienze, Rendiconti, vol. 17, 1911; vol. 18, fasc. 1–9, 1912.
Odessa: Société des Naturalistes, tome 34–36, 1909–11.
Palermo: Rendiconti del Circolo Matematico, tome 31, fasc. 3; tome 32, 1911; tome 33; tome 34, fasc. 1, 2, 1912.
Paris: Bulletin des Sciences Mathématiques, May–Dec., 1911; Jan.–Sept., 1912.
Paris: Journal de l'Ecole Polyt., cah. 15, 1911; cah. 16, 1912.
Pisa: R. Scuola Normale Superiore, Annali, vol. 12, 1912.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 20, sem. 1, fasc. 7–12, sem. 2, 1911; vol. 21, sem. 1, 2, 1912.
Roma: Reale Accademia dei Lincei, Rendiconti delle Sedute Solenni, vol. 2, pp. 503–634, 1911–12.
Stockholm: Acta Mathematica, bd. 34, pt. 4, 1911; bd. 35, 1911–12; bd. 36, pts. 1, 2, 1912.
Torino: R. Accademia delle Scienze, Atti, vol. 46, 1911; vol. 47, 1912.
Torino: R. Accademia delle Scienze, Memorie, vol. 59, 1909.
United States Naval Observatory, Publications, ser. 2, vols. 6, 7, 1911.
Upsala: Reg. Soc. Scientiarum, Nova Acta, ser. 4, vol. 2, fasc. 2, 1912.
Venezia: Atti del R. Istituto, tome 17, disp. 6–10; tome 18, 19, 20, disp. 1–8, 1908–11.
Venezia: Atti del R. Istituto, Osservazioni Meteorologiche e Geodinamische, 1907, 1908.
Warsaw: Prace Matematyczno-Fizyczne, tome 21, 1910; tome 22, 1911.
Wien: Monatshefte für Mathematik, jahr. 22, viertel. 2–4, 1911; jahr. 23, 1912.
Zurich: Vierteljahrsschrift, hefte 3, 4, 1910; 1911; hefte 1, 2, 1912.

International Catalogue of Scientific Literature.

In the year May, 1911, to November, 1912 (inclusive), the following exchanges were sent in the first instance to Dr. Bromwich to be indexed for the International Catalogue of Scientific Literature :—

"Proceedings of the Edinburgh Mathematical Society," Vol. XXIX, 1911.
"Transactions of the Institution of Naval Architects," London, 1911.
"Journal of the Institute of Actuaries," Vol. XLV, Pts. 2–5, and Vol. XLVI, Pt. 1, 1911–12.
"Proceedings of the Manchester Literary and Philosophical Society," Vol. LV, Pts. 2, 3, 1910–11.
"Proceedings of the Royal Society of Edinburgh," Vol. XXXI, Pts. 3–5, 1912.
"Proceedings of the Royal Irish Academy," Vol. XXIX, Section A, Nos. 1–4, 1911.

The following were also sent for the purposes of the Catalogue :—

"Mathematical Gazette," Nos. 92–100 ; London, 1911–12.
"Educational Times," Nos. 600–619 ; London, 1911–12.

OBITUARY NOTICES

W. M. COATES.

William Montgomery Coates was born at Midleton, co. Cork, on June 2nd, 1857, and, after a distinguished career at Trinity College, Dublin, entered Queens' College, Cambridge, in 1884. He became a fellow of that foundation soon after taking his degree as Third Wrangler in 1886, and was afterwards Bursar, Assistant Tutor, and Mathematical Lecturer of the College.

In the mathematical world it is as a teacher that Coates will be remembered. He was one of the most successful of recent times in Cambridge. His breezy cheerful businesslike manner made his college lectures a real pleasure to listen to, whilst his patience with private pupils was astonishing; he seemed to be able to work pretty nearly all day and yet was always willing to do more. It was characteristic of him that, having some terms free after his first illness, he spent them in attending lectures with a zest that a man less than half his age might have envied. After that illness he seemed outwardly as strong and cheerful as ever, but he may not have really been so, and a second illness ended in his death, on January 16th, 1912.

Like many other Irishmen, Coates was keenly interested in geometry, and taught it in most inspiring fashion, but, although his opinion of new work was sought by his friends, he never wrote papers himself. He was always too much occupied with teaching and college business for that.

His nature was kindly and lovable. He held strong views and expressed himself vigorously, but was scrupulously fair and courteous to opponents, whose goodwill he always retained.

JULES HENRI POINCARÉ.

JULES HENRI POINCARÉ was born on the 29th of April, 1854, at Nancy, where his father was a physician held in great respect among the citizens. He received his school education at the Lycée of his native city, but this was interrupted by the war of 1870–71, during which he acted as secretary to his father in his attendance upon the sick and wounded. It is recorded that he was so eager for news of the progress of the war that at this time he learnt German in order to read the German newspapers. At school, although he early showed a special aptitude for mathematics, and it was always expected that he would become a mathematician, he nevertheless outclassed the other boys of his age in all the subjects of instruction. He proceeded to Paris in 1873, and was admitted "first" to the École Polytechnique, proceeding in 1875 to the École des Mines, and becoming finally qualified as a mining engineer in 1879, the same year in which he was created Docteur ès Sciences mathématiques in the University of Paris, and was appointed to give instruction in Analysis in the Faculté des Sciences at Caen. In 1881 he returned to Paris to take up an appointment as Maître de Conférences d'Analyse in the Faculté des Sciences of the University. In 1885, he began to give instruction in Mechanics in the same institution, becoming Professor of Mathematical Physics in 1886 and Professor of Mathematical Astronomy and Celestial Mechanics in 1896, a title he continued to hold till the time of his death on July the 17th, 1912. From 1883 to 1897 he acted as Répétiteur d'Analyse, and from 1904 to 1908 as Professor of Astronomy, at the École Polytechnique. In 1902 he was appointed Professor of Theoretical Electricity at the École des Postes et des Télégraphes in Paris. During the last few years of his life he occupied so many positions of dignity and eminence, and his pen was so constantly in request for articles, prefaces, addresses, and reports, that it is surprising that he could continue to find time for research work.

Poincaré's original investigations cover a wider range in Pure Mathematics, Theoretical Physics, Celestial Mechanics, and the Philosophy of the exact sciences than those of any other mathematician of the age. A mere catalogue of his writings would occupy more space than would be suitable in this Notice, and it will be desirable to select for mention and description some of the more important. His earliest writing, occupied with functions that are defined by differential equations, is dated 1878,

and among his first pieces of research work was an essay submitted in competition for the Grand Prix offered in 1880 by the Académie des Sciences for an improvement in the theory of differential equations. His great discoveries in this theory began to be published in a series of notes inserted in the *Comptes Rendus* for 1881. Here he first propounded his construction of Fuchsian and Kleinian groups and the corresponding automorphic functions. A brief general account of this theory was published in the *Mathematische Annalen*, Bd. 19, 1892, and the developed theory was expounded as a coherent whole in a series of memoirs published in the *Acta Mathematica*, tt. 1–5, 1882–84. The elliptic functions of a complex argument are uniform functions which have the property of taking the same value at corresponding points in a doubly infinite set of parallelograms. These form a lattice covering the whole plane of the complex argument, the corresponding points being determined by a linear substitution which is equivalent to the result of two compounded simple translations, parallel to the sides of one of the parallelograms, through distances which are numerical multiples of the lengths of the sides. Poincaré's idea was to substitute for one of these parallelograms a curvilinear polygon, formed in general by arcs of circles, and for the others a set of related polygons, the whole set covering the plane, or a prescribed portion of the plane, and corresponding points being determined by the linear substitutions belonging to an infinite discontinuous group. His automorphic functions, or as he called them "Fuchsian" functions, are uniform functions which have the property of taking the same value at the corresponding points of all the polygons. He showed how expressions for such functions could be constructed, and how by means of them it is possible to construct the integrals of any linear differential equation with algebraic coefficients. He also showed that by means of the same functions a pair of variables connected by any algebraic equation can simultaneously be expressed as uniform functions of a single parameter. The great generality and power of these researches won for him instant recognition as an analyst of the first rank.

While this work was still in course of publication, Poincaré's attention was attracted to questions of Mechanics, in particular to the integration of the differential equations of the problem of three bodies, and he began to write on this subject as early as 1882. It may be surmised that it was in the process of assimilating the work of others on analytical dynamics that he became acquainted with Thomson and Tait's "Natural Philosophy," some passages in which, dealing with the equilibrium of rotating masses of fluid, appear to have set him thinking about the theory of stability and to have given rise to the memoir "Sur l'équilibre d'une masse fluide

animée d'un mouvement de rotation," published in the *Acta Mathematica*, t. 7, 1885. In this memoir he introduced the important conceptions of "bifurcation" and "exchange of stabilities," explained their bearing upon the distinction previously emphasized by Thomson and Tait between ordinary and secular stability, and applied them to the problem of determining the conditions under which spheroidal and ellipsoidal figures of equilibrium of rotating masses of fluid become unstable. He further demonstrated the existence of a previously unknown pear-shaped figure, and suggested the possibility that the progressive deformation of this figure with increasing angular velocity might result in the breaking up of the rotating body into two detached masses. The bearing of these discoveries upon cosmic speculation as to the history of the Earth-Moon system and the origin of double stars is obvious. The fundamental conception involved in this theory is that of the dependence of the equilibrium configuration upon a variable parameter; in the special case the parameter is the angular velocity of the rotating body. As the parameter varies the quantities which determine the configuration, for example, the ellipticity of a rotating spheroid, also vary; and the totality of the configurations that can be specified by the same quantities and the parameter form what Poincaré calls a "linear series" of configurations. There may be several linear series, and when the parameter takes one or other of certain special values the same configuration may belong to two series. This possibility is expressed in the notion of bifiurcation, and Poincaré proved that if, as the parameter increases, a configuration is reached at which there is bifurcation, and if for values of the parameter smaller than the critical value the configurations of one series are stable, then for greater values they become unstable, and the property of stability passes to the configurations of the other series. Isolated problems which can now be recognized as exemplifying this theory had been solved before, particularly in the theory of elasticity, but the conceptions were for the first time formulated effectively in the memoir cited above, and they have since proved fruitful in many departments of mathematical physics.

Poincaré himself proceeded to develop the application of these conceptions to celestial mechanics. We have seen that he had begun to write on the problem of three bodies before the publication of the memoir on figures of equilibrium of rotating fluid. His further work on the subject issued in the publication in 1890 of a memoir "Sur le problème des trois corps et les équations de la dynamique" (*Acta Mathematica*, t. 13), for which he was awarded a prize founded by King Oskar II of Sweden and Norway. The methods and results of this memoir were afterwards amplified and presented as part of a systematic treatise in the work entitled

Les Méthodes nouvelles de la Mécanique céleste (t. I, 1892; t. II, 1894; t. III, 1899). G. W. Hill, in 1877, had proved the existence of a particular periodic solution of what is known as the restricted problem of three bodies, taking account of that inequality which in the lunar theory is called the "variation." Poincaré showed that Hill's solution is only one of a quadruple infinity of periodic solutions of the general problem, and that such solutions fall into families in respect of which the notions of bifurcation and exchange of stabilities find an application. He proved, further, that any motion of the bodies, if it is not exactly periodic, must differ from some one or other of those which can be expressed by periodic solutions by a very small deviation, so that a knowledge of the approximate periodic solution would suffice for the calculation of the actual motion during a very long period of time. It may happen that one of the bodies approaches closer and closer to a periodic orbit, traces a path which lies very near to this orbit during a great number of revolutions, but ultimately moves far away from it. Such solutions are called "asymptotic." The memoir of 1890 contains also the first introduction of the analytical conception of an integral invariant, an important but depressing theorem as to the non-existence of any uniform integrals other than those of energy, momentum, and moment of momentum, and a disconcerting but well supported statement to the effect that the series traditionally used by astronomers in the lunar and planetary theories are divergent. Of this memoir and treatise Sir George Darwin remarked, "It is probable that for half a century to come it will be the mine from which humbler investigators will excavate their materials."

Divergent series had no terrors for Poincaré, nor did he share Abel's opinion that such series are an invention of the devil. He knew that in many instances they had been employed with success for purposes of approximate numerical computation, especially in connection with irregular integrals of linear differential equations, and one of his memoirs (*Acta Mathematica*, t. 8, 1886) is devoted to the construction of a general theory of what are now called "asymptotic expansions" and their applications to the expression of such integrals. In another paper dating from the same year (*Bulletin de la Soc. Math. de France*, t. 14) he formulated a theory of the convergence of an infinite determinant, a novel type of infinite expression which had been introduced very shortly before by G. W. Hill in his researches on the lunar theory. In fact, Poincaré never gave up working at analysis, making important contributions among other things to the theory of functions of several variables, and the discussion of related questions concerning *analysis situs* or the connectivity of algebraic loci in space of any number of dimensions. One of his memoirs on this subject

was communicated by him to this Society, at the meeting held in June, 1900, when he had occasion to be in London to take part in the conferences which led to the establishment of the International Catalogue of Scientific Literature. In another, perhaps the last that he wrote, he utilised the notions of *analysis situs* to extend his previous work on periodic solutions of the problem of three bodies (*Rendiconti d. Circ. mat. d. Palermo*, t. 33, fasc. 3, 1912). Although this memoir is unfinished, it affords a very remarkable example of Poincaré's mode of working, and the unexpected use that he could make of geometrical imagery.

Poincaré's activity in almost all departments of mathematical physics is well shown by the number of treatises which he published in the form of lectures delivered at the Sorbonne. The latest of these, the *Leçons de Mécanique céleste*, t. 3, *Théorie des Marées*, published in 1910, and the *Leçons sur les Hypothèses cosmogoniques*, published in 1911, deal with astronomical subjects, the latter containing a very interesting discussion of Laplace's nebular hypothesis; but the earlier volumes are occupied with Optics, Electricity, Capillarity, Vortex Motion, Elasticity, Conduction of Heat, Thermodynamics, Theory of the Potential, and Figures of Equilibrium of rotating fluid. Of these the *Électricité et Optique* in its two editions attracted the most attention, and had much influence in spreading Maxwell's ideas abroad in Europe. Besides all this work of expounding physical theories, always with a touch, or more than a touch, of novelty, Poincaré made important original contributions to theoretical Optics in two memoirs on the diffraction of light at a straight edge (*Acta Mathematica*, t. 16, 1892, and t. 20, 1897), but his chief work in physical subjects may be held to lie in the discussion of theorems concerning the existence of the functions which satisfy the partial differential equations of mathematical physics, and also satisfy prescribed conditions at assigned boundaries. The most famous of these is the one known under the name of Dirichlet's principle, to the effect that there exists a function which, with its differential coefficients of the first order, is finite and continuous at all points of the region of space bounded by a given surface, which satisfies Laplace's equation at all points within the region, and which takes at every point on the surface a prescribed value. One method of proof is to construct the function as the limit of a series of functions which all satisfy the differential equation, but do not satisfy the boundary condition exactly, only with continually improving approximation. This was the method developed by C. Neumann, and it seemed to be impossible to prove that the approximation really improves without limit unless the surface is everywhere convex. Poincaré overcame the difficulty by constructing a series of functions which do satisfy the boundary condition, but

do not satisfy the equation. Such functions are the potentials of distributions of matter through the region, and the successive functions are invented to have the property that the density of the matter diminishes without limit. This is the method of sweeping (*balayage*), the matter being always, as it were, swept out of a portion of the region and distributed over the boundary of that portion. This method was propounded in a memoir published in the *American Journal of Mathematics*, Vol. 12, 1890, and afterwards expounded in the course of lectures entitled "Théorie du Potentiel Newtonien," 1899. Other famous memoirs by Poincaré on the partial differential equations of mathematical physics appeared in the *Rendiconti d. Circ. mat. d. Palermo*, t. 8, 1894, and in the *Acta Mathematica*, t. 20, 1897. The first of these contains a method for arriving at a series of quantities analogous to the frequencies of a vibrating system, and forms a step towards the theory of integral equations afterwards developed by Fredholm and others. At the end of the second are given suggestions, put forward as probable but not as proved, for the construction of functions, called by the author "fundamental," belonging to any given surface, and designed to play the same part in solutions of problems of potential theory related to the surface as spherical harmonics play in the solution of similar problems related to a sphere. Stekloff and others have developed these suggestions with some modifications so as to obtain a more complete theory. Poincaré seems to have been much interested in the theory of integral equations, and among his writings on this subject may be mentioned the first three of the lectures which he gave at Göttingen, in 1909, at the invitation of the University ("Sechs Vorträge über ausgewählte Gegenstände aus der reinen Mathematik und mathematischen Physik," Leipzig, 1910).

If Poincaré's original contributions to theoretical physics are concerned rather with the discussion of the equations than with the construction of comprehensive theories of phenomena, as a critic and expositor of such theories he was unrivalled. Particular reference may be made to the brilliant tract of 80 pages, entitled *La théorie de Maxwell et les oscillations Hertziennes*, which he contributed to the *Scientia* series in 1899. This tract was one of the earliest of his popular writings, but the most celebrated of them is the volume entitled *La Science et l'Hypothèse*, which has been translated into six languages, and has had a circulation of 20,000 copies. His philosophical writings, of which this book is the best known example, seem to have begun in 1887 with a discussion of the foundations of Geometry, but rapidly extended to a review of the foundations of physical science. The particular notion that is associated with his name is that of the possibility of alternative laws of nature enunciated in the Introduction

to the *Électricité et Optique*, in the words—" Si donc un phénomène comporte une explication mécanique complète, il en comportera une infinité d'autres qui rendront également bien compte de toutes les particularités révélées par l'expérience." The idea that the laws of nature which we arrive at gradually, building on the foundations laid by our predecessors, may constitute merely one of an infinite number of conceivable rational schemes for coordinating and interpreting experience, may be regarded as a development of the views of Kirchhoff and Mach, but the special form which it received at the hands of Poincaré is all his own, and the wonderfully simple and vivid style of his writings secured for his ideas a wider circle of readers than falls to the lot of most mathematicians and philosophers.

Poincaré was honoured during his lifetime. In his own country he was elected a member of the Académie des Sciences in 1887, subsequently receiving every honour that can be conferred upon a French man of science by his countrymen. The Royal Society of Sciences of Göttingen was quick to realise that a new star had risen in the mathematical firmament and elected him a foreign member in 1883. Afterwards very many foreign academies and scientific societies did the same, and he was the recipient of many medals and other distinctions. In England the Cambridge Philosophical Society elected him an Honorary Member in 1890, and our own Society and the Manchester Literary and Philosophical Society followed in 1892. In 1894 he became a Foreign Member of the Royal Society and an Associate of the Royal Astronomical Society. He received the gold medal of the Royal Astronomical Society in 1900, and in 1901 was the first recipient of the recently founded Sylvester medal of the Royal Society. Both Cambridge and Oxford conferred Honorary Degrees upon him. A few months before his death he gave, by invitation of the University of London, a series of lectures which were attended by many English mathematicians. Several accounts of his life and writings have been published, the most complete being that by E. Lebon in the series entitled *Savants du Jour* (2nd ed., 1912). There can be no doubt that France was proud of him, and moreover curious about him. This curiosity led to his giving a most interesting account of the way in which he made discoveries in mathematics. He held that mathematical reasoning does not consist in piling syllogism on syllogism, but essentially in the order in which the syllogisms are placed; and he described how, in circumstances in which he might be occupied with anything rather than mathematics, the combination of ideas which would prove to be effective in the development of some theory would suddenly flash into his mind with a conviction of absolute certainty. He had a very retentive memory,

especially for anything he heard ; and it is said of him that he associated symbols with sounds, his mind acting rather by audition than by visualising. Calculators have often been visualisers, but Poincaré distrusted his powers of calculation and preferred to work by a combination of analytical and geometrical conceptions, often expressed in extremely simple terms, towards necessary general conclusions. The record of his life shows that he was not one of those who sit by the roadside waiting for inspiration. He was always at work, ever acquiring fresh knowledge by assimilating the work of others, and constantly giving verbal expression to the form in which the acquired knowledge presented itself to his mind and the relation in which it stood to the things that he had known before. It is given to few men to dream even one such dream as the invention of automorphic functions, the formulation of the geometrical criterion of stability, or the discovery of families of periodic orbits. There was needed for these a mind stored with knowledge of groups and functions and series and differential equations, of general theorems concerning the equations of dynamics, of harmonic analysis, of non-Euclidean geometry and geometry of many dimensions, of *analysis situs*. All this knowledge and much besides he could bring to bear upon any matter to which it could be applied. But there was another requisite—the indefinable thing that we call genius. His right is recognized now, and it is not likely that future generations will revise the judgment, to rank among the greatest mathematicians of all time.

A. E. H. L.

PAPERS

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THE RELATIONS BETWEEN BOREL'S AND CESÀRO'S METHODS OF SUMMATION

By G. H. Hardy *and* J. E. Littlewood.

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I.

1. The results obtained in this paper are developments of an idea that has been prominent in much recent work on the theory of divergent series.

Consider the series

$$a_0+a_1+a_2+\dots . \tag{1}$$

This series may converge, or may possess a "sum" according to one or other of a large variety of definitions of what is meant by the "sum" of a series: Hölder's, Cesàro's, or Riesz's definitions by mean values, Euler's definition as the limit of a power series, Borel's exponential definition, and so forth. To each of these definitions correspond certain limits of applicability. Thus Hölder's and Cesàro's definitions can never be successful unless

$$n^{-k}a_n \to 0$$

for some value of k; Euler's definition requires that $\Sigma a_n x^n$ should converge for $|x| < 1$; Borel's that

$$\Sigma \frac{a_n x^n}{n!}$$

should be an integral function of x. Roughly, we may say that any method of summation will fail if the series to which it is applied is *too divergent.** Or, in other words, to any method corresponds a certain upper limit of its power, the specification of which is a problem generally not difficult and often uninteresting.

It is only recently that it has been observed that the range of applicability of all methods of summation is limited from below, so to say, as well as from above. Methods fail, not only when the series to which it is attempted to apply them is *too divergent*, but also if it is *too nearly convergent:* not only is their *power* limited, but also their *delicacy*. The theorems which express this latter fact are more subtle than those which express the "limitation from above," and take a rather different form: they assert that, if a series is too nearly convergent, it cannot be summable unless it is *actually* convergent.

The theorems of this character which correspond to Cesàro's (or Hölder's) and to Euler's methods were discussed by us in two recent papers in these *Proceedings.*† It follows from a well known theorem of Tauber‡ that a series for which

$$na_n \to 0 \tag{2}$$

cannot be summable by any of these methods unless it is convergent. In these papers we showed that this condition may be replaced by the more general condition

$$|na_n| < K. \tag{2'}$$

Similar results hold for Riesz's more general methods: these results will also be found in the papers referred to.

The primary object of this paper was to establish the analogue of Tauber's theorem for Borel's exponential method of summation. But this problem has led us on to a number of others, some of which are discussed here, while to others we hope to return later. There is one important point with regard to which our results are not complete. We

* In these general explanations (as in the phrase "Theory of Divergent Series") it is convenient to use *divergent* as meaning simply non-convergent. In detailed work it is essential to distinguish between divergent and oscillatory series.

† Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 301; Littlewood, *ibid.*, Vol. 9, p. 434. The results of the first of these papers may be deduced as corollaries from those of the second: they have been extended, in a somewhat different direction, by Landau (*Prac Matematyczno-Fizycznych*, t. 21, p. 97), who shows that, when a_n is real, it is enough to suppose $na_n < K$ or $na_n > -K$, and makes an interesting application of the result to the Theory of Prime Numbers.

‡ If $\Sigma a_n x^n \to s$, as $x \to 1$, and $na_n \to 0$ as $n \to \infty$, then Σa_n is convergent (to sum s). For a proof see Littlewood, *l.c. supra*, and Bromwich, *Infinite Series*, p. 251.

prove (Theorem 1 below) that a series for which

$$\sqrt{n} \, . \, a_n \to 0 \tag{3}$$

cannot be summable by Borel's method, unless it is convergent. It can hardly be doubted that this result (the analogue of Tauber's theorem) is susceptible of the same generalisation: that is to say, that (3) may be replaced by

$$| \sqrt{n} \, . \, a_n | < K. \tag{3'}$$

But this we have not at present succeeded in proving; and the difficulties attendant on the generalisation of Tauber's theorem suggest forcibly that the proof may not be at all easy to find.*

We shall use the symbols

$$K, \ \epsilon, \ \delta, \ O, \ o, \ \ldots,$$

in special senses for an explanation of which we must refer elsewhere.†

II.

2. Borel gave two definitions of the sum of the series

$$a_0+a_1+a_2+\ldots. \tag{1}$$

According to his first definition the sum is

$$s = \lim_{x \to \infty} e^{-x} \Sigma \frac{s_n x^n}{n!}, \tag{2}$$

where

$$s_n = a_0+a_1+\ldots+a_n.$$

According to the second it is

$$\int_0^\infty e^{-x} a(x) \, dx, \tag{2a}$$

where

$$a(x) = \Sigma \frac{a_n x^n}{n!}. \tag{2b}$$

These definitions are not exactly equivalent, the second being slightly

* In the abstract of this paper which appeared in the "Notes and Corrections" to Vol. 9 of the *Proceedings*, we implied that we were in possession of a proof. We have since discovered that our belief that we had found one was mistaken.

† See Hardy, "Orders of Infinity," *Camb. Math. Tracts*, No. 12. The only symbol whose use is not explained there is the small o, introduced by Landau (*Handbuch der Lehre von der Verteilung der Primzahlen*, p. 61). We write

$$f = o(\varphi),$$

if ϕ is positive and $f/\phi \to 0$.

more general.* They are certainly equivalent whenever $a_n \to 0$. For the present we shall adopt the first definition as fundamental; if (2) holds, we shall say that the series (1) is *summable* (*B*) to sum s.

3. Our first object is to prove the following theorem:—

THEOREM 1.—*If* Σa_n *is summable* (*B*), *and*

$$\sqrt{n} \, . \, a_n \to 0,$$

then Σa_n *is convergent.*

This theorem, however, is only one of a hierarchy of theorems connecting Borel's with Cesàro's methods of summation. To establish these we shall require a number of lemmas.

We shall write

$$\begin{aligned} s_n &= a_0+a_1+\ldots+a_n, \\ s_n^1 &= s_0+s_1+\ldots+s_n, \\ s_n^2 &= s_0^1+s_1^1+\ldots+s_n^1, \\ &\ldots \quad \ldots \quad \ldots, \end{aligned}$$

so that Σa_n is summable (*Ck*) if

$$k! \, s_n^k / n^k \to s.\dagger$$

LEMMA 1.—*If* $c_n \sim An^\alpha$ *as* $n \to \infty$, *then*

$$e^{-x} \Sigma c_n \frac{x^n}{n!} \sim Ax^\alpha,$$

as $x \to \infty$.

This is, of course, well known.

LEMMA 2.—*We have identically*

$$e^{-x} \sum_0^\infty s_n^k \frac{x^{n+k}}{(n+k)!} = \left(\int_0^x dx\right)^k \left(e^{-x} \sum_0^\infty s_n \frac{x^n}{n!}\right).$$

We use the identity

$$s_n^k = s_n+\begin{bmatrix} k \\ 1 \end{bmatrix} s_{n-1}+\begin{bmatrix} k \\ 2 \end{bmatrix} s_{n-2}+\ldots+\begin{bmatrix} k \\ n \end{bmatrix} s_0,$$

where

$$\begin{bmatrix} p \\ q \end{bmatrix} = \frac{p\,(p+1)\,\ldots\,(p+q-1)}{q!}.$$

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* Bromwich, *Infinite Series*, p. 297.

† We need hardly point out that s_n^k does not stand for a power of s_n.

The coefficient of s_ν on the left-hand side is

$$e^{-x}\left\{\frac{x^{\nu+k}}{(\nu+k)!}+\begin{bmatrix}k\\1\end{bmatrix}\frac{x^{\nu+k+1}}{(\nu+k+1)!}+\begin{bmatrix}k\\2\end{bmatrix}\frac{x^{\nu+k+2}}{(\nu+k+2)!}+\cdots\right\},$$

and on the right-hand side is

$$\left(\int_0^x dx\right)^k\left(e^{-x}\frac{x^\nu}{\nu!}\right).$$

That these two expressions are identical may be verified immediately by induction.

LEMMA 3.—*If* Σa_n *is summable* (B) *to sum* s, *then*

$$e^{-x}\sum^{\infty} s_n^k\frac{x^n}{(n+k)!}\to\frac{s}{k!}.$$

This is an immediate corollary from Lemma 2.

4. Now suppose Σa_n summable (B), so that

$$e^{-x}\Sigma s_n^k\frac{x^n}{(n+k)!}\to\frac{s}{k!} \tag{3}$$

for any value of k. And let us suppose that there is a number α such that

$$s_n^{k-1}=o(n^\alpha).* \tag{4}$$

Then, if for shortness we write $s_n^k=t_n$, we have

$$\Delta t_n\equiv t_n-t_{n+1}=-s_{n+1}^{k-1}=o(n^\alpha). \tag{5}$$

If, in (3), we put $x=m$, we have

$$S_1+S_2\to s/k!, \tag{6}$$

where $S_1=e^{-m}t_m\Sigma\frac{m^n}{(n+k)!}$, $S_2=e^{-m}\Sigma(t_n-t_m)\frac{m^n}{(n+k)!}$.

By Lemma 1,

$$S_1=m^{-k}t_m(1+\epsilon_m). \tag{7}$$

* If $k=0$, $s_n^{k-1}=a_n$.

We proceed to discuss S_2, which we write in the form

$$(8) \qquad \dots e^{-m}\left(\sum_{0}^{(1-H)m} + \sum_{(1-H)m}^{(1+H)m} + \sum_{(1+H)m}^{\infty}\right) = S_3+S_4+S_5,$$

say. Here H is a positive constant less than unity, and $\sum_{\mu}^{\nu}$ denotes a summation extended to all integral values n such that $\alpha < n < \beta$: it is convenient to suppose H *irrational*, so that $(1-H)m$ and $(1+H)m$ cannot be integral, but the limits of summation may be left indefinite to the extent of a term or two without any effect on the argument.

To obtain upper limits for S_3, S_4, and S_5, we use the inequalities

$$(9) \qquad \begin{cases} |t_n-t_m| < Km^K & (in\ S_3), \\ |t_n-t_m| \leqslant \epsilon_m m^a |n-m| & (in\ S_4), \\ |t_n-t_m| < Kn^K & (in\ S_5), \end{cases}$$

which are immediate consequences of inequalities already established.

Let $u_n = m^n/n!$. Then we find at once, by an elementary application of Stirling's theorem, that

$$e^{-m}u_\mu < e^{-Km}, \qquad e^{-m}u_\nu < e^{-Km},$$

where u_μ and u_ν are the last u which occurs in S_3 and the first which occurs in S_5.

Hence
$$\begin{aligned} |S_3| &< Km^K e^{-m} \sum_{0}^{(1-H)m} n^{-k}u_n \\ &< m^K e^{-m}(u_\mu+u_{\mu-1}+\dots+u_0) \\ &< m^K e^{-Km}\{1+(1-H)+(1-H)^2+\dots\} \\ &< e^{-Km}. \end{aligned} \qquad (10)$$

Similarly,
$$\begin{aligned} |S_5| &< Ke^{-m} \sum_{(1+H)m}^{\infty} n^{K-k}u_n \\ &< e^{-m} \sum_{(1+H)m}^{\infty} n^K u_n \\ &< m^K e^{-Km}\left\{1+\frac{2^K}{1+H}+\frac{3^K}{(1+H)^2}+\dots\right\} \\ &< e^{-Km}. \end{aligned} \qquad (11)$$

Finally we have to consider S_4, which we write in the form

$$S_4 = \sum_{n>m} + \sum_{m>n} = S_6+S_7.$$

We have $$|S_6| = \left| e^{-m} \sum_{m}^{(1+H)m} (t_n - t_m) \frac{m^n}{(n+k)!} \right|$$

$$< Km^{-k} e^{-m} \sum_{m}^{(1+H)m} |t_n - t_m| \frac{m^n}{n!}$$

$$< m^{a-k} e^{-m} \epsilon_m \sum_{1}^{Hm} r \frac{m^{m+r}}{(m+r)!} \quad *$$

$$< m^{a-k} e^{-m} \frac{m^m}{m!} \epsilon_m \Sigma \frac{rm^r}{(m+1)(m+2)\dots(m+r)}$$

$$< m^{a-k-\frac{1}{2}} \epsilon_m \Sigma \frac{r}{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\dots\left(1+\frac{r}{m}\right)}$$

Now, for $1 \leqslant r < Hm$, we have

$$1 \Big/ \left(1+\frac{r}{m}\right) = \exp\left(-\frac{r}{m} + \frac{r^2}{2m^2} - \dots\right) < e^{-r/2m},$$

$$r \Big/ \left\{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\dots\left(1+\frac{r}{m}\right)\right\} < r \exp\left(-\frac{1}{2m} - \frac{2}{2m} - \dots - \frac{r}{2m}\right)$$

$$< re^{-r^2/4m}.$$

Hence $$|S_6| < m^{a-k-\frac{1}{2}} \epsilon_m \sum_{0}^{\infty} re^{-r^2/4m}$$

$$< m^{a-k-\frac{1}{2}} \epsilon_m \int_0^\infty xe^{-x^2/4m} dx \dagger$$

$$< m^{a-k+\frac{1}{2}} \epsilon_m$$

(12) $$= o(m^{a-k+\frac{1}{2}}).$$

Again, $$|S_7| = \left| e^{-m} \sum_{(1-H)m}^{m} (t_n - t_m) \frac{m^n}{(n+k)!} \right|$$

$$< Km^{-k} e^{-m} \sum_{(1-H)m}^{m} |t_n - t_m| \frac{m^n}{n!}$$

$$< m^{a-k} e^{-m} \epsilon_m \sum_{1}^{Hm} r \frac{m^{m-r}}{(m-r)!}$$

$$< m^{a-k-\frac{1}{2}} \epsilon_m \Sigma r \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{r-1}{m}\right).$$

In this case we use the inequality

$$1 - \frac{r}{m} < e^{-r/m},$$

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* Using (5).

† When m is large, the integral is of order m. The difference between the integral and the series is less than a constant multiple of the maximum of $xe^{-x^2/4m}$, which is of order $\sqrt{m}$.

and the argument proceeds as before ; so that

$$|S_7| = o(m^{\alpha-k+\frac{1}{2}}). \tag{13}$$

From (4), (7), (10), (11), (12), and (13), we deduce

$$s_m^k(1+\epsilon_m)/m^k + o(m^{\alpha-k+\frac{1}{2}}) \to s/k!. \tag{14}$$

From this relation we can deduce Theorem 1 and the other theorems referred to in § 3.

5. Suppose first $\alpha = k-\frac{1}{2}$. Then

$$s_m^k/m^k \to s/k!. \tag{15}$$

Hence we obtain

THEOREM 2.—*If* Σa_n *is summable* (B), *and*

$$s_n^{k-1} = o(n^{k-\frac{1}{2}}),$$

then Σa_n *is summable* (Ck).

This reduces to Theorem 1 when $k = 0$ (when s_n^{-1} must be interpreted as meaning a_n).*

Next, suppose $\alpha > k-\frac{1}{2}$. Then we obtain

THEOREM 3.—*If* Σa_n *is summable* (B) *and*

$$s_n^{k-1} = o(n^{\alpha})$$

where $\alpha > k-\frac{1}{2}$, *then* $s_n^k = o(n^{\alpha+\frac{1}{2}})$.†

THEOREM 4.—*If* Σa_n *is summable* (B), *and*

$$s_n^{k-1} = o(n^k),$$

then Σa_n *is summable* $(C, k+1)$.

For, by Theorem 3, we have $s_m = o(n^{k+\frac{1}{2}})$, and Theorem 4 accordingly follows from Theorem 2. The special case in which $k = 0$ is particularly interesting and deserves a separate statement.

THEOREM 4*a*.—*If* Σa_n *is summable* (B), *and* $a_n \to 0$, *then* Σa_n *is summable* $(C1)$.

By a repeated application of the argument which led to Theorem 4, we deduce

* The theorem was stated for the particular cases in which $k = 0, 1$ in the Abstract of this paper referred to above.

† That $s_n^k = o(n^{\alpha+1})$ is trivial. The point of the theorem lies in the reduction of $\alpha+1$ to $\alpha+\frac{1}{2}$ as a result of the hypothesis of Borel summability.

THEOREM 5.—*If* Σa_n *is summable* (B), *and*

$$s_n^{k-1} = o(n^{k+\frac{1}{2}(r-1)}),$$

then Σa_n *is summable* $(C, k+r)$.

6. If $\Sigma a_n x^n$ is convergent for $|x| < 1$, and the function $f(x)$ represented by the sum of the series is regular for $x = 1$, the series Σa_n is summable (B). It therefore follows from Theorems 1 and 4a that

(i) *If* $f(x)$ *is regular for* $x=1$, *and* $\sqrt{n}.a_n \to 0$, *then* Σa_n *is convergent.*

(ii) *If* $f(x)$ *is regular for* $x=1$, *and* $a_n \to 0$, *then* Σa_n *is summable* $(C1)$.

Each of these corollaries of our theorems is included in Fatou's theorem* that, *if* $f(x)$ *is regular for* $x = 1$, *and* $a_n \to 0$, *then* Σa_n *is convergent.* But we have, of course, assumed much less than regularity for $x = 1$.

If Σa_n is summable by Cesàro's means, or, more generally, if Abel's limit exists, we can only infer convergence if

$$a_n = O(1/n).$$

To assume that Σa_n is summable (B) is to assume *more* than that it is summable (C)† or by Abel's limit, but less than that $f(x)$ is regular. To this corresponds the fact that $\sqrt{n}.a_n \to 0$ asserts less than $a_n = O(1/n)$, but more than $a_n \to 0$.

7. The results of § 5 may be represented conveniently by means of a diagram.

If $a_n =$	$s_n =$	$s_n^1 =$	$s_n^2 =$	...	*then the series is*
$o(n^{-\frac{1}{2}})$					*convergent*
$o(1)$	$o(n^{\frac{1}{2}})$				*summable* $(C1)$
$o(n^{\frac{1}{2}})$	$o(n)$	$o(n^{\frac{3}{2}})$			,, $(C2)$
$o(n)$	$o(n^{\frac{3}{2}})$	$o(n^2)$	$o(n^{\frac{5}{2}})$		,, $(C3)$
...	...	...	...	...	...

* Fatou, *Thèse* (Paris, 1906) and *Acta Mathematica*, t. 30, p. 389. A simpler proof, and a series of important generalisations, have been given by Riesz, *Crelle's Journal*, Bd. 140, S. 89, and *Comptes Rendus*, Nov. 22, 1909; see also Landau, *Prac Matematyczno-Fizycznych*, t. 21, p. 151.

† See Theorem 6 below for a precise statement.

This diagram at once suggests that there should be further theorems corresponding to the spaces which we have not filled in, such as :—

(*a*) if $s_n = o(1)$, and Σa_n is summable (*B*), then Σa_n is convergent ;

(*b*) if $s_n^1 = o(\sqrt{n})$, and Σa_n is summable (*B*), then Σa_n is convergent ;

and so on. These theorems are all trivial or false. Thus (*a*) is obviously trivial: the same is true of all the theorems which correspond to the vacant spaces which have two sides in common with those filled in in the diagram.

On the other hand, (*b*) is false. For, if $s_n^1 = o(\sqrt{n})$, we have

$$(s_0+s_1+\ldots+s_n)/(n+1) = o(1/\sqrt{n}) ;$$

and so we can *deduce* from the condition $s_n^1 = o(\sqrt{n})$ that Σa_n is summable (*B*) to sum 0.* Hence the theorem suggested would show that $s_n^1 = o(\sqrt{n})$ by itself implies convergence to zero, and this is untrue, as s_n may well be of the form

$$\epsilon_n \sqrt{n} - \epsilon_{n-1} \sqrt{(n-1)}$$

without tending to zero.

A very interesting general conclusion may be drawn from the theorems comprised in our diagram, viz.,

THEOREM 6.—*If* Σa_n *is summable* (*B*), *and*

$$a_n = o(n^k)$$

for some value of k, *then* Σa_n *is summable by Cesàro's means of sufficiently high order.*

In the language of § 1, we may express this by saying that *Borel's method, although more powerful than Cesàro's, is never more delicate, and often less so.*

A particular case of Theorem 6 deserves special notice. It is well

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* If

$$\frac{s_0 + s_1 + \ldots + s_n}{n+1} = s + o\left(\frac{1}{\sqrt{n}}\right),$$

then Σa_n is summable (*B*) to sum s. See Hardy, *Quarterly Journal*, Vol. 35, p. 40 ; Bromwich, *Infinite Series*, pp. 319–322. It may be shown more generally (cf. Bromwich, *l.c.*) that

$$k!\, s_n^k / n^k = s + o(1/\sqrt{n})$$

implies the same conclusion : we have thus a general condition which enables us to infer Borel from Cesàro summability. For some examples of series summable by Cesàro's, but not by Borel's, method, see Hardy, *l. c. supra* and *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 290, and §§ 10, 11 below.

known that a power series is summable (B) at any regular point on its circle of convergence. It therefore follows that, *if* $f(x)$ *is regular for* $x = 1$, *and* $a_n = o(n^k)$, *then* Σa_n *is summable* (C). This result has been found by Riesz.*

8. It is natural to enquire whether the preceding results may be extended to non-integral orders of Cesàro summation.† The necessary analysis is not difficult, but, as its conclusions are obvious generalisations of those already established, we shall be content to sketch the argument very shortly.

In the first place, Lemma 1 of § 3 is quite independent of any assumption as to the arithmetic nature of a.

Secondly, Lemma 2 may be replaced by the equation

$$e^{-x} \sum_0^\infty s_n^k \frac{x^{n+k}}{\Gamma(n+k+1)} = \frac{1}{\Gamma(k)} \int_0^x (x-t)^{k-1} e^{-t} \left(\sum_0^\infty s_n \frac{t^n}{n!} \right) dt,$$

where

$$s_n^k = s_n + \begin{bmatrix} k \\ 1 \end{bmatrix} s_{n-1} + \ldots + \begin{bmatrix} k \\ n \end{bmatrix} s_0,$$

$$\begin{bmatrix} p \\ q \end{bmatrix} = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q+1)}.$$

This equation may be shown to hold for any positive value of k. From it may be deduced the analogue of Lemma 3, viz., that

$$e^{-x} \sum_0^\infty s_n^k \frac{x^n}{(n+k)!} \to \frac{s}{\Gamma(k+1)},$$

if Σa_n is summable (B). We then deduce equation (14) of § 4,‡ precisely as in that section. We thus obtain Theorems 2, 3, 4, 5, freed from the restriction that k is an integer. The effect of this is to replace each set of theorems, corresponding to a set of spaces lying on a line parallel to the principal diagonal of the diagram of § 7, by a *continuous* series of theorems.

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* *L.c. supra* (p. 9). Riesz assumes more than we do, and so obtains a more precise result: in fact, he establishes summability (Ck), whereas all that can be deduced from our hypothesis is summability ($C, 2k+1$).

† For the theory of such methods of summation, see Chapman, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 369; Hardy and Chapman, *Quarterly Journal*, Vol. 42, p. 181; and various writings of Bohr, Knopp, and Riesz quoted in the latter paper. A later note by Riesz appeared in the *Comptes Rendus*, June 12th, 1911.

‡ Of course with $\Gamma(k+1)$ for $k!$.

9. So far we have confined ourselves to Borel's definition (2). The question remains whether our results remain valid when this definition is replaced by the integral definition expressed by (2*a*) and (2*b*).

If $a_n \to 0$, the two definitions are certainly equivalent. For the necessary and sufficient condition for equivalence is that

$$e^{-x} a(x) \to 0.$$

It is, however, not difficult to see that all our results still hold when Borel's integral definition is adopted.

For, if $$a_0+a_1+a_2+\dots$$

is summable by the integral definition, then

$$0+a_0+a_1+a_2+\dots$$

is summable by the definition (2).* Moreover, if the first series satisfies one of our conditions

$$a_n = o(n^{-\frac{1}{2}}), \quad a_n = o(1). \quad s_n = o(n^{\frac{1}{2}}), \quad \dots,$$

the second satisfies a corresponding condition, and is accordingly summable by the appropriate one of Cesàro's means. But the two series are completely equivalent in regard to the application of Cesàro's method. Thus all our results apply as well to one of Borel's definitions as to the other.

III.

10. A good deal of light may be thrown on the foregoing theorems by the study of a particular series, viz., the series

(15) $$\Sigma n^{-b} e^{in^a},$$

where a and b are real and $0 < a < 1$.†

This series is convergent if $a+b > 1$, summable ($C1$) if $2a+b > 1$, summable ($C2$) if $3a+b > 1$, and so on.‡ If $b = 1-a$, it is finitely oscillating. In this case, if $a < \frac{1}{2}$, we have

$$n^{-b} e^{in^a} = o(n^{-\frac{1}{2}});$$

* Hardy, *Quarterly Journal*, Vol. 35, p. 34.

† We might equally well consider the more general series

$$\Sigma n^{-b} e^{i\lambda n^a}.$$

‡ Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 142.

and so, by Theorem 1, the series cannot be summable (B). We are thus led to expect, in regard to the summability (B) of the series, different results according as $a > \frac{1}{2}$ or $a < \frac{1}{2}$. We shall, in fact, prove that *the series* (15) *is summable* (B) *for all values of* b *if* $a > \frac{1}{2}$, *but is never summable* (B) *when* $a < \frac{1}{2}$, *except when it is convergent.*

The proof of this result is tedious rather than difficult, and we shall content ourselves with sketching its main features.

In the first place, we can easily verify that, if

$$a_n = n^{-b} e^{in^a},$$

then
$$s_n = -(i/a) n^{1-a-b} e^{in^a} + \sum_{(\nu)} K n^\nu e^{in^a} + C + o(1),$$

where the summation extends over a finite number of values of ν, all less than $1-a-b$, and C is a constant arising from the application of the Euler-Maclaurin sum-formula.

We can now prove that, *if* $a > \frac{1}{2}$,

$$e^{-x} \sum_0^\infty n^\nu e^{in^a} \frac{x^n}{n!} \to 0 \tag{16}$$

as $x \to \infty$, *for all values of* ν. It will then follow that the series (15) is summable to sum C. Let

$$m = [x].$$

Then it is easy to see that we may replace the left-hand side of (16) by

$$e^{-x} \sum_{-\mu}^{\mu} (m+r)^\nu e^{i(m+r)^a} \frac{x^{m+r}}{(m+r)!}, \tag{17}$$

where
$$\mu \sim m^{\frac{1}{2}+\delta}.$$

We then show, by a straightforward but tedious process of approximation, that, if we write $x = m+f$, and keep f constant as $m \to \infty$, we can write (17) in the form

$$K m^{\nu-\frac{1}{2}} e^{im^a} \sum_{-\mu}^{\mu} [e^{iam^{a-1}r - (r^2/2m)} \{1 + \Sigma K r^A m^B + \Sigma O(r^\alpha m^\beta)\}], \tag{18}$$

where the number of terms in each sum is finite, A is integral, and

$$\nu + \tfrac{1}{2}a + \beta < 0$$

for each pair of indices α, β.

Next, it is easy to see that

$$Km^{\nu-\frac{1}{2}}e^{im^a}\sum_{-\mu}^{\mu}\{e^{iam^{a-1}r-(r^2/2m)}O(r^\alpha m^\beta)\} = O(m^{\nu+\beta-\frac{1}{2}})\int_{-\infty}^{\infty}e^{-r^2/2m}|r|^\alpha dr$$
$$= O(m^{\nu+\frac{1}{2}\alpha+\beta}) = o(1).$$

Thus these terms may be neglected, and everything is reduced to showing that a number of terms of the type

$$m^p\sum_{-\mu}^{\mu}r^q e^{iam^{a-1}r-(r^2/2m)}, \tag{19}$$

where q is an integer, tend to zero. It is easily proved that the limits of summation may be replaced by ∞ and $-\infty$.

First suppose $q = 0$. Then

$$\sum_{-\infty}^{\infty}e^{iam^{a-1}r-r^2/2m} = 1+2\sum_{1}^{\infty}e^{-r^2/2m}\cos r\theta = \vartheta_3(v,\tau),^*$$

where $\theta = am^{a-1}, \quad v = \theta/2\pi, \quad \tau = i/2m\pi.$

Now† $$\vartheta_3(v,\tau) = \sqrt{\left(\frac{i}{\tau}\right)}e^{-\pi iv^2/\tau}\vartheta_3\left(\frac{v}{\tau}, -\frac{1}{\tau}\right),$$

$$\vartheta_3\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) = 1+2\sum_{1}^{\infty}e^{-2mr^2\pi^2}\cosh 2mr\theta\pi = 1+o(1).$$

Hence $$\vartheta_3(v,\tau)\sim\sqrt{(2m\pi)}\,e^{-\frac{1}{2}m\theta^2} = \sqrt{(2m\pi)}\,e^{-\frac{1}{2}a^2m^{2a-1}},$$

which tends to zero more rapidly than any power of m.

When q is not zero the argument is a little more complicated, but in essence the same: in this case we use the q-th derivative of the theta-function with respect to v.

Thus the left-hand side of (16) tends to zero if $x = m+f$, where $0\leqslant f<1$, and $m\to\infty$. Moreover it does so uniformly with respect to f, and so our proof is completed.

11. When $0<a<\frac{1}{2}$, the discussion is somewhat similar, but rather simpler. The essential difference lies in the fact that we can choose δ so that $iam^{a-1}r$ is small throughout a range of values of r of magnitude $m^{\frac{1}{2}+\delta}$, so that, in approximating to $e^{i(m+r)^a}$, we can use e^{im^a} instead of the

* Tannery and Molk, *Fonctions Elliptiques*, t. II, p. 252.
† *Ibid.*, p. 263.

more accurate approximation

$$e^{im^a+ia m^{a-1}r}.$$

The result is that the dominant term of our final result assumes the form

$$-\frac{i}{a\sqrt{(2\pi)}}\, m^{\frac{1}{2}-a-b}\, e^{im^a} \sum_{-\mu}^{\mu} e^{-r^2/2m} \sim -(i/a)\, m^{1-a-b}\, e^{im^a},$$

and the series (15) is summable if and only if $a+b>1$, *i.e.*, if and only if it is convergent.

12. If $a=\frac{1}{2}+\delta$, we require $b>\frac{1}{2}-\delta$ for convergence. Hence we can find a series of the type (15), summable (B), but not convergent, and such that $a_n = O(n^{-\frac{1}{2}+\delta})$. This affords a formal proof that the index $\frac{1}{2}$ of Theorem 1 cannot be replaced by any lower index. We can show similarly, by means of the series considered in §§ 10, 11, that the indices of the powers of n, which occur in Theorems 2–5, are as small as they can possibly be.

A much more difficult question remains. *Is Theorem* 1 *true if the condition* $\sqrt{n}.a_n \to 0$ *is replaced by* $|\sqrt{n}.a_n| < K$, *and can similar changes be made in the other theorems?* It has been proved recently* that a similar extension may be given to Tauber's converse of Abel's theorem, and it is natural to suppose that the extension is possible here also.

It is interesting to consider for a moment what light is thrown on this question by the series of §§ 10, 11. The crucial case is that in which the series oscillates finitely; *i.e.*, when

$$a_n = n^{a-1} e^{in^a}.$$

Then $\quad a_n = o(n^{-\frac{1}{2}}) \quad (a<\tfrac{1}{2}), \qquad a_n = O(n^{-\frac{1}{2}}) \quad (a=\tfrac{1}{2}).$

In the first case the series is certainly not summable, by Theorem 1 or by the results of §§ 10, 11. And the question of interest is *whether*

$$\Sigma \frac{e^{i\sqrt{n}}}{\sqrt{n}}$$

is summable (B).†

The answer to this question is (as analogy would lead us to expect) in

* Littlewood, *l.c.*, p. 434.

† If not, Theorem 1 shows that $\Sigma n^{-b} e^{\sqrt{n}}$ is never summable unless convergent.

the negative. In fact,

$$s_n = C - 2ie^{i\sqrt{n}} + o(1),$$

and it may be shown that

$$e^{-x} \Sigma e^{i\sqrt{n}} \frac{x^n}{n!} \sim e^{-(1/8)+i\sqrt{x}}.*$$

Thus the evidence obtained from this series points to the truth of the suggested generalisations. But, as we stated in § 1, we have not been able to find a satisfactory proof of them.

13. Theorem 1 has an interesting application to the problem of the multiplication of series. It is easy to prove that *if* Σa_n *and* Σb_n *are summable* (*B*), *and*

$$a_n = O(1/n), \qquad b_n = O(1/n),$$

then the product series Σc_n, *formed in accordance with Cauchy's rule, is also summable* (*B*). But it is evident that

$$c_n = O(\log n/n) = o(1/\sqrt{n}),$$

and therefore, by Theorem 1, Σc_n is *convergent*. We thus obtain a simple proof of a known theorem.†

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* If $\alpha < \frac{1}{2}$,

$$e^{-x} \Sigma e^{in^\alpha} \frac{x^n}{n!} \sim e^{ix^\alpha},$$

while, if $\alpha > \frac{1}{2}$, the left-hand side tends exponentially to zero (see §§ 10, 11). In the critical case we obtain a result resembling the first, but differing owing to the presence of the factor $e^{-i/8}$. The correspondence between the oscillations of the original series and of Borel's integral is not so precise in this case as it is shown to be, when $\alpha < \frac{1}{2}$, by the formula at the end of § 11. For an illustration of the corresponding phenomenon in connection with Tauber's theorem, see Littlewood, *l.c. supra*, p. 436: in the formulæ there given a constant term $\zeta(1+\alpha i)$ is omitted, but the insertion of this term does not affect the argument.

† Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410. We have obtained a number of further results on this subject, to which we hope to return shortly. In particular we have proved that any convergent series for which $a_n = O(1/n)$—and therefore any such series summable by any of Cesàro's means—is summable $(C, -1+\delta)$ for any positive δ. We are thus able, by the use of Mr. Chapman's negative indices of summability, to deduce the multiplication theorem referred to above from the theorem that a series of this type cannot be summable (*C*) without being convergent.

ON THE FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS, AND ON SOME RELATED THEOREMS

By E. W. HOBSON.

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THE fundamental lemma of the calculus of variations states that, if $f(x)$ be a function which is continuous in the interval (x_0, x_1), and be such that $\int_{x_0}^{x_1} \eta(x) f(x)\, dx$ vanishes for all functions $\eta(x)$ which satisfy certain prescribed conditions, then $f(x)$ must have the value zero throughout the interval (x_0, x_1). This lemma is required for the establishment of Euler's equation as the condition that the first variation of an integral

$$\int_{x_0}^{x_1} F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right) dx$$

should vanish.

In the form in which this lemma was first established by Du-Bois-Reymond, the function $\eta(x)$ is prescribed to belong to the class of all those functions which vanish at x_0 and x_1 and which have a continuous derivative in the interval (x_0, x_1). The same writer further shewed that the theorem is still valid in case the function $\eta(x)$ is restricted to belong to the class of either (1) all functions which vanish at x_0, x_1 and which have continuous derivatives of the first p orders in the interval, or (2) all functions which have continuous derivatives of all orders. It has further been shewn by H. A. Schwarz that the theorem holds in case the functions $\eta(x)$ are restricted to the class of functions that are regular in the interval, *i.e.*, such that in the neighbourhood of any point a in the interval they are representable by a power-series in powers of $x-a$. It is clear that the more restricted the class of functions to which $\eta(x)$ is assumed to belong, that may be found sufficient for the establishment of the theorem, the greater is the degree of generality of that theorem. Moreover, if the function $f(x)$ need not be assumed to be continuous in the whole interval (x_0, x_1) the generality of the theorem will be still further increased. In the present communication it is shewn to be sufficient to assume that the function $\eta(x)$ is restricted to be any finite polynomial which, together with

its first p derivatives, vanishes at the ends of the intervals, p being any fixed integer (including zero). Moreover, it is shewn that if no assumption be made as to $f(x)$, except that is summable in the interval, in the sense that, whether it be limited in the interval or not, it has a Lebesgue integral, the theorem still holds in the sense that $f(x)$ must be zero at every point of the interval, with the possible exception of points belonging to an exceptional set with the measure zero. Moreover, $f(x)$ is zero at every point at which it is continuous. When the theorem thus generalized is applied for the purposes of the calculus of variations it is seen that, in order to establish Euler's differential equation satisfied by the extremals, it is sufficient to take weak variations δy of the restricted type which consists only of finite polynomials, which, together with a prescribed number of their derivatives, vanish at the extremities of the interval.

A theorem* which is of use in the theory of isoperimetric problems, and which is due to Du-Bois-Reymond, states that if $f(x)$, $g(x)$ are continuous in the interval (x_0, x_1), and such that $\int_{x_0}^{x_1} f(x)\psi(x)dx$ vanishes for all functions $\psi(x)$ which vanish at x_0, x_1, have continuous differential coefficients in the interval (x_0, x_1), and are such that $\int_{x_0}^{x_1} g(x)\psi(x)dx$ vanishes, then a constant λ exists such that $f(x)+\lambda g(x)$ vanishes throughout the interval (x_0, x_1). This theorem is here generalized by assuming that $f(x)$, $g(x)$ are any summable functions, and that $\psi(x)$ is restricted to be a finite polynomial of the same type as in the former theorem. It is then shewn that a constant λ exists, such that $f(x)+\lambda g(x)$ vanishes at all points except those of an exceptional set with measure zero, and that it vanishes at every point at which $f(x)$, $g(x)$ are continuous.

Another known theorem which has an important application in the Calculus of Variations is that, if $\int_{x_0}^{x_1} \frac{d\eta(x)}{dx} f(x)dx$ vanishes for all functions $\eta(x)$ which have continuous differential coefficients in the interval (x_0, x_1), where $f(x)$ is a continuous function, then $f(x)$ must be constant in the interval. This was also established by Du-Bois-Reymond, and has been extended to the case in which $f(x)$, though limited, has a finite set of discontinuities in the interval.

This theorem is here extended to the case in which $f(x)$ is assumed only to be summable in (x_0, x_1), and in which $\eta(x)$ is restricted to belong to the class of those finite polynomials which vanish at x_0, x_1. It is shewn that $f(x)$ has a constant value at all points which do not belong to

* See Bolza's *Vorlesungen über Variationsrechnung*, p. 462.

some set of measure zero, and that it has this constant value at all points of any interval contained in (x_0, x_1) in which it is continuous. The theorem is proved for the more general case of the integral

$$\int_{x_0}^{x_1} \frac{d^p \eta(x)}{dx^p} f(x)\, dx,$$

in which a polynomial of degree $p-1$ takes the place of the constant for the special case $p = 1$.

Lastly, the case of the vanishing of the more general expression,

$$\int_{x_0}^{x_1} \left\{ q_0 y + q_1 \frac{dy}{dx} + \ldots + q_p \frac{d^p y}{dx^p} \right\} f(x)\, dx,$$

is considered, where $q_0, q_1, \ldots, q_p$ are functions of x satisfying certain conditions.

It is shewn that, if y be restricted to belong to the class of those finite polynomials which, together with their first $p-1$ derivatives, vanish at x_0 and x_1, then, in any interval of continuity of $f(x)$, that function must be a solution of the differential equation which is adjoint to the equation

$$q_0 y + q_1 \frac{dy}{dx} + \ldots + q_p \frac{d^p y}{dx^p} = 0.$$

1. Let $f(x)$ be a function which is summable (whether limited or not) in the interval (x_0, x_1). Let it be assumed that for every function $\eta(x)$ which consists of a finite polynomial such that the function and its first p differential coefficients vanish for the values x_0, x_1 of x, the integral $\int_{x_0}^{x_1} \eta(x) f(x)\, dx$ has the value zero.*

It is clear that $\eta(x)$ is of the form $(x-x_0)^{p+1}(x_1-x)^{p+1} P_r(x)$, where $P_r(x)$ is any finite polynomial. If we write

$$\phi(x) = (x-x_0)^{p+1}(x_1-x)^{p+1} f(x),$$

the above assumption is equivalent to the one that $\int_{x_0}^{x_1} P_r(x)\, \phi(x)\, dx$ vanishes for every polynomial $P_r(x)$.

Let ξ, ξ' denote any two numbers such that $x_0 < \xi < \xi' < x_1$, and let ϵ be an arbitrarily chosen positive number, less than both $\xi - x_0$ and $x_1 - \xi'$. Let $\chi(x)$ denote that function, continuous in the interval (x_0, x_1),

* It can be shewn that, if $f(x)$ is summable in the interval (x_0, x_1), then the product of $f(x)$ by any limited summable function is also summable in the same interval. The functions $\eta(x)$ fall into this class.

which has the values

$$\chi(x) = 0, \qquad \text{for} \quad x_0 \leqslant x \leqslant \xi - \epsilon,$$

$$\chi(x) = \frac{1}{\epsilon}(x - \xi + \epsilon), \qquad \text{for} \quad \xi - \epsilon \leqslant x < \xi,$$

$$\chi(x) = 1, \qquad \text{for} \quad \xi \leqslant x \leqslant \xi',$$

$$\chi(x) = \frac{1}{\epsilon}(\xi' + \epsilon - x), \qquad \text{for} \quad \xi' \leqslant x \leqslant \xi' + \epsilon,$$

$$\chi(x) = 0, \qquad \text{for} \quad \xi' + \epsilon \leqslant x \leqslant x_1.$$

Thus $\chi(x)$ is represented graphically by the thickened line in the figure.

In accordance with the well known theorem of Weierstrass, a finite polynomial $P_r(x)$ can be so determined that, if ζ be an arbitrarily chosen positive number, the condition

$$|\chi(x) - P_r(x)| < \zeta$$

holds for every value of x in the interval (x_0, x_1).

We have

$$\int_{x_0}^{x_1} \chi(x)\,\phi(x)\,dx = \int_{x_0}^{x_1} \phi(x)\,\{\chi(x) - P_r(x)\}\,dx + \int_{x_0}^{x_1} \phi(x)\,P_r(x)\,dx.$$

Now
$$\int_{x_0}^{x_1} \chi(x)\,\phi(x)\,dx = \int_{\xi}^{\xi'} \phi(x)\,dx + \int_{\xi-\epsilon}^{\xi} \frac{1}{\epsilon}(x - \xi + \epsilon)\,\phi(x)\,dx + \int_{\xi'}^{\xi'+\epsilon} \frac{1}{\epsilon}(\xi' + \epsilon - x)\,\phi(x)\,dx,$$

and the second and third integrals on the right-hand side of this equation are less in absolute value than $\int_{\xi-\epsilon}^{\xi} |\phi(x)|\,dx$, $\int_{\xi'}^{\xi'+\epsilon} |\phi(x)|\,dx$; and thus, in accordance with a well known property of Lebesgue integrals, they converge to zero as ϵ converges to zero.

Also
$$\left| \int_{x_0}^{x_1} \phi(x)\,\{\chi(x) - P_r(x)\}\,dx \right| < \zeta \int_{x_0}^{x_1} |\phi(x)|\,dx.$$

By hypothesis $$\int_{x_0}^{x_1} \phi(x)\, P_r(x)\, dx = 0;$$

it then follows that $\left| \int_{\xi}^{\xi'} \phi(x)\, dx \right|$ has a value which can be made as small as we please by taking ϵ and ζ small enough. Therefore

$$\int_{\xi}^{\xi'} \phi(x)\, dx = 0,$$

where ξ, ξ' are any numbers such that $x_0 < \xi < \xi' < x_1$. Since the integral is a continuous function of either the upper or the lower limit, it follows that the integral vanishes for all pairs of values of ξ and ξ' such that $x_0 \leqslant \xi < \xi' \leqslant x_1$. It is a known theorem * that, if the integral of $\phi(x)$ vanishes when taken through any sub-interval whatever of (x_0, x_1), then $\phi(x)$ vanishes for every value of x, with the possible exception of those values belonging to a set with measure zero.

Since $f(x)$ vanishes for any value of x for which $\phi(x)$ vanishes, except for $x = x_0$, $x = x_1$, we see that $f(x)$ must vanish for all values of x except at most those belonging to some set with measure zero.

The fundamental lemma of the Calculus of Variations has now been established in the following generalized form:—

If $f(x)$ be a function which is summable in the interval (x_0, x_1), and is such that $\int_{x_0}^{x_1} f(x)\, P(x)\, dx$ vanishes, provided $P(x)$ is any finite polynomial which, together with its first p differential coefficients, vanishes at the end-points of the interval, then $f(x)$ must vanish for every value of x except those belonging to some set of which the measure is zero. Also $f(x)$ must vanish at any point at which it is continuous. In particular, if

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* This theorem was first given by Vitali; see *Rend. di Palermo*, Vol. xx, p. 136.

It may easily be proved as follows:—Let E be that set of points in the interval (x_0, x_1), at each of which $\phi(x) > 0$; then all the points of E can be enclosed in intervals of a set Δ, such that $m(\Delta) - m(E)$ is arbitrarily small; where $m(E)$, $m(\Delta)$ denote the measures of E and Δ respectively. The integral of $\phi(x)$ through Δ vanishes, by hypothesis, and thus

$$\int_{(E)} \phi(x)\, dx + \int_{(\Delta - E)} \phi(x)\, dx = 0.$$

The latter integral may be made arbitrarily small by choosing $m(\Delta) - m(E)$ small enough; therefore

$$\int_{(E)} \phi(x)\, dx = 0.$$

It follows that the measure of that set of points of E at which $\phi(x) > 0$ must be zero. Similarly it is seen that the measure of the set of points in (x_0, x_1) at each of which $\phi(x) < 0$ must be zero. Thus the theorem is established.

$f(x)$ is continuous in an interval (α, β) which may be the whole or a part of (x_0, x_1), it must vanish everywhere in (α, β).

It is clear that, instead of $P(x)$, we may take the functions

$$(x-x_0)^{p+1}(x_1-x)^{p+1}x^k,$$

where k has the values 0, 1, 2, 3, ...

It will be observed that the proof here given establishes the theorem that, if $\phi(x)$ be any summable function such that $\int_{x_0}^{x_1}\phi(x)x^k dx$ vanishes for all positive integral values of k, including zero, then $\phi(x)$ vanishes in the interval (x_0, x_1), except possibly at points belonging to some set of measure zero; and that in particular $\phi(x)$ vanishes at any point at which it is continuous. The particular case of this theorem which arises when $\phi(x)$ is continuous throughout the interval has been proved by Landau,* who gives references to an earlier proof by Lerch.

2. Let $f(x)$, $g(x)$ be functions which are summable in the interval (0, 1), and let us suppose that $\int_0^1 f(x)\,Q(x)\,dx$ vanishes for every finite polynomial such that $\int_0^1 g(x)\,Q(x)\,dx$ vanishes, and such also that $Q(x)$ and its first p differential coefficients all vanish for $x=0$ and $x=1$. The number p may have any fixed integral value, including zero.

Let A and B be so chosen that

$$\int_0^1 g(x)\,x^{p+1}(1-x)^{p+1}(Ax^n+Bx^m)dx = 0,$$

where n and m are positive integers. When the two integrals

$$\int_0^1 g(x)\,x^{p+1}(1-x)^{p+1}x^n\,dx, \qquad \int_0^1 g(x)\,x^{p+1}(1-x)^{p+1}x^m\,dx$$

have values which are both different from zero, which we suppose to be the case, the assumed condition determines uniquely the ratio A/B.

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* See the *Rend. del Circ. Mat. di Palermo*, Vol. XXV, p. 343. A referee has called attention to the fact that the theorem is not valid for the case of an indefinitely great interval. Thus Stieltjes has pointed out (see Borel's *Théorie des séries divergentes*, p. 68) that if

$$\phi(x) = e^{-x^{\frac{1}{2}}}\sin(x^{\frac{1}{2}}),$$

then
$$\int_0^\infty \phi(x)\,x^k\,dx = 4\int_0^\infty e^{-y}\sin y\,.\,y^{4k+3}\,dy = \frac{1}{2^{2k}}\Gamma(4k+4)\sin(k+1)\pi,$$

which vanishes when $k = 0, 1, 2, 3, \ldots$.

We have then, from the original assumption,

$$A\int_0^1 f(x)\,x^{v+1}(1-x)^{p+1}x^n\,dx+B\int_0^1 f(x)\,x^{p+1}(1-x)^{p+1}x^m\,dx=0.$$

From the two equations, we see that

$$\int_0^1 x^{p+1}(1-x)^{p+1}\{f(x)+\lambda g(x)\}\,x^n\,dx=0,$$

where λ has the value

$$-\int_0^1 x^{m+p+1}(1-x)^{p+1}f(x)\,dx\Big/\int_0^1 x^{m+p+1}(1-x)^{p+1}\,g(x)\,dx,$$

and thus, if m has any fixed value, λ is a determinate constant.

The integer m may be so chosen that $\int_0^1 g(x)x^{p+m+1}(1-x)^{p+1}dx$ is not zero, for if this integral were zero for every value 0, 1, 2, 3, ... of m, it would follow from the theorem of § 1 that $g(x)$ vanishes everywhere except at points of some set of measure zero, and we may provisionally suppose $g(x)$ to be such that this possibility is excluded. Moreover, there must be an indefinitely great number of values of m for which the integral does not vanish; for, if it vanished for all values of m which exceed some fixed value m_1, we should see that $x^{m_1}g(x)$, and therefore $g(x)$, would vanish for all values of x except those of some set of measure zero. Further, for any value of n for which

$$\int_0^1 x^{p+1}(1-x)^{p+1}\,g(x)\,x^n\,dx=0,$$

we should also have $\int_0^1 x^{p+1}(1-x)^{p+1}f(x)\,x^n\,dx=0,$

in virtue of the original assumption. It has now been shewn that a constant λ exists, such that

$$\int_0^1 x^{p+1}(1-x)^{p+1}\{f(x)+\lambda g(x)\}\,x^n\,dx=0,$$

for $n=0,\ 1,\ 2,\ 3,\ \ldots.$

It follows from the theorem of § 1, that $f(x)+\lambda g(x)$ is zero at all points of the interval (0, 1) with the exception of those belonging to some set with measure zero. Also in any interval (α, β) in which $f(x)$, $g(x)$ are continuous, $f(x)+\lambda g(x)$ vanishes in the interval.

In case $g(x)$ vanishes everywhere in (0, 1) with the exception of points of some set with measure zero, the integral

$$\int_0^1 x^{p+1}(1-x)^{p+1}\,g(x)\,x^n\,dx$$

would vanish for every value of n; and therefore

$$\int_0^1 x^{p+1}(1-x)^{p+1} f(x)\, x^n dx$$

would also vanish for every value of n. It would then follow that $f(x)$ vanishes everywhere except at points of some exceptional set of measure zero; in this case we shall have $\lambda = 0$. It is clear that the limits of the integral can be taken to be x_0, x_1, instead of 0, 1.

The following theorem has now been established:—

Let $f(x)$ be a function summable in the interval (x_0, x_1), and such that $\int_{x_0}^{x_1} f(x)\, Q(x)\, dx$ vanishes for every finite polynomial $Q(x)$ which is such that

$$\int_{x_0}^{x_1} g(x)\, Q(x)\, dx = 0,$$

where $g(x)$ is another function summable in (x_0, x_1), and where $Q(x)$ also satisfies the condition that it and its first p derivatives vanish for $x = x_0, x_1$. A determinate constant λ then exists, such that $f(x)+\lambda g(x)$ vanishes for all points in the interval (x_0, x_1) except at most those belonging to some exceptional set which has the measure zero. In any interval (α, β), which may be a part or the whole of (x_0, x_1), and in which $f(x)$, $g(x)$ are continuous, $f(x)+\lambda g(x)$ vanishes without exception.

3. Let us assume that $f(x)$ is a function defined for the interval (0, 1), summable in that interval, and such that

$$\int_0^1 \frac{d^p \eta(x)}{dx^p} f(x)\, dx$$

vanishes for every value of $\eta(x)$ which is a finite polynomial such that it and its first $p-1$ differential coefficients all vanish at the end-points of the interval. Let $\eta(x) = x^p(1-x)^p x^k$, where k is a positive integer; we have then

$$0 = \int_0^1 f(x) \frac{d^p}{dx^p}\left[x^{p+k}(1-x)^p\right] dx$$

$$= \int_0^1 f(x) \Big[(p+k)(p+k-1)\dots(k+1)x^k - p(p+k+1)(p+k)\dots(k+2)x^{k+1} + \frac{p(p-1)}{2!}(p+k+2)(p+k+1)\dots(k+3)\, x^{k+2} - \dots\Big] dx.$$

Let $$u_k = (p+k)(p+k-1)\dots(k+1)\int_0^1 x^k f(x)\, dx;$$

the above equation becomes then

$$u_k - pu_{k+1} + \frac{p(p-1)}{2!} u_{k+2} - \ldots = 0;$$

which is a linear difference equation that can be written in the form

$$(1-E)^p u_k = 0,$$

where E is the usual operator, such that

$$Eu_k = u_{k+1}.$$

The general solution of this difference equation is

$$u_k = A_0 + A_1 k + A_2 k^2 + \ldots + A_{p-1} k^{p-1};$$

where $A_0, A_1, \ldots, A_{p-1}$ are constants independent of the value of k. It follows that

$$\int_0^1 f(x) x^k dx = \frac{a_1}{k+1} + \frac{a_2}{k+2} + \ldots + \frac{a_p}{k+p};$$

where $a_1, a_2, \ldots, a_p$ are independent of k. Hence

$$\int_0^1 x^k [f(x) - a_1 - a_2 x - a_3 x^2 - \ldots - a_p x^{p-1}] dx = 0.$$

Since the function $f(x) - a_1 - a_2 x - a_3 x^2 - \ldots - a_p x^{p-1}$ when multiplied by any finite polynomial and integrated through (0, 1) is such that the integral vanishes, it follows from the theorem in § 1 that this function vanishes for all points of the interval (0, 1) except possibly those belonging to a set of which the measure is zero. It is clear that, by a linear transformation, the limits of the integral can be taken to have any values x_0, x_1 instead of 0 and 1; thus we have established the following theorem:—

If $f(x)$ be a function which is summable in the interval (x_0, x_1) and is such that $\int_{x_0}^{x_1} f(x) \frac{d^p}{dx^p} P(x) dx$ vanishes provided $P(x)$ is any finite polynomial which, together with its first $p-1$ differential coefficients, vanishes at the end-points of the interval (x_0, x_1), then $f(x)$ must have the same values as some polynomial of degree $p-1$ for every value of x except possibly for those belonging to a set of measure zero. If in an interval which is a part of, or the whole of, the interval (x_0, x_1) the function $f(x)$ is continuous, then through the whole of such interval $f(x)$ must have the same values as the polynomial.

In the case $p = 1$, the theorem states that if

$$\int_{x_0}^{x_1} f(x) \frac{d}{dx} P(x)\, dx = 0,$$

for all polynomials $P(x)$ which vanish at x_0, x_1, then $f(x)$ must have a constant value at all points of the interval, with the possible exception of those belonging to a set of measure zero. Moreover, the function has this constant value at every point where it is continuous.

4. Let $q_0, q_1, q_2, \ldots q_p$ be functions of x which are continuous and of limited total fluctuation in the interval (0, 1). Moreover, let it be assumed that q_p has a continuous differential coefficient of order p, q_{p-1}, a continuous differential coefficient of order $p-1$, and so on. Let it be assumed that a summable function $f(x)$ is such that

$$\int_0^1 \left\{ q_0 y + q_1 \frac{dy}{dx} + q_2 \frac{d^2y}{dx^2} + \ldots + q_p \frac{d^p y}{dx^p} \right\} f(x)\, dx$$

vanishes for every value of y that is represented by a finite polynomial which, together with its first $p-1$ derivatives, vanishes at the end-points 0, 1 of the interval. The method of integration by parts being applicable to Lebesgue integrals, we find, by successive employment of the process, that

$$\int_0^1 q_{p-r} f(x) \frac{d^{p-r} y}{dx^{p-r}}\, dx = (-1)^r \int_0^1 \left\{ \frac{d^p y}{dx^p} \int_0^x dx \int_0^x dx \ldots \int_0^x q_{p-r} f(x)\, dx \right\} dx.$$

Hence, when y is any polynomial having the above property, we see that

$$\int_0^1 \left[q_p f(x) - \int_0^x q_{p-1} f(x)\, dx + \int_0^x dx \int_0^x q_{p-2} f(x)\, dx - \ldots \right] \frac{d^p y}{dx^p} dx$$

vanishes. From the theorem established in § 3, we see that

$$q_p f(x) - \int_0^x q_{p-1} f(x)\, dx + \int_0^x dx \int_0^x q_{p-2} f(x)\, dx - \ldots$$

must, at every point x of the interval (0, 1), with the possible exception of the points of a set of measure zero, have the value of some polynomial $\alpha_0 + \alpha_1 x + \ldots + \alpha_p x^{p-1}$, of degree $p-1$.

Remembering that an integral $\int_0^x U dx$ has a differential coefficient with respect to its upper limit x, equal to U, at any point at which U is continuous, we see that at any interior point of an interval (α, β) con-

tained in (0, 1), such that $f(x)$ is continuous in (α, β), $f(x)$ has a continuous differential coefficient; since throughout this interval we have

$$q_p f(x) = a_0 + a_1 x + \ldots + a_p x^{p-1} + \int_0^x q_{p-1} f(x)\,dx - \int_0^x dx \int_0^x q_{p-2} f(x)\,dx + \ldots .$$

Thus, throughout this interval,

$$\frac{d}{dx}\{q_p f(x)\} = a_1 + 2a_2 x + \ldots + (p-1)a_p x^{p-2} + q_{p-1} f(x) - \int_0^x q_{p-2} f(x)\,dx + \ldots ;$$

it now follows that $q_p f(x)$ has a continuous second differential coefficient through the interior of (α, β), and thus that this holds also for $f(x)$. Proceeding in this way we see that, in the interior of the interval (α, β), the first p differential coefficients of $f(x)$ all exist and are continuous, and also that $f(x)$ there satisfies the condition

$$\frac{d^p}{dx^p}\{q_p f(x)\} - \frac{d^{p-1}}{dx^{p-1}}\{q_{p-1} f(x)\} + \frac{d^{p-2}}{dx^{p-2}}\{q_{p-2} f(x)\} - \ldots = 0.$$

This is the differential equation known as the one adjoint * to the differential equation

$$q_0 y + q_1 \frac{dy}{dx} + q_2 \frac{d^2 y}{dx^2} + \ldots + q_p \frac{d^p y}{dx^p} = 0.$$

In case $f(x)$ is continuous in the whole interval (0, 1) it must be a function which everywhere satisfies this adjoint equation.

The following theorem has now been established:—

Let $q_0, q_1, \ldots q_p$ be functions of x which are continuous and of limited total fluctuation in the interval (x_0, x_1), and let q_r be such that it has a continuous differential coefficient of order r throughout the interval. Let $f(x)$ be a function summable in the interval (x_0, x_1) and such that

$$\int_{x_0}^{x_1} \left\{ q_0 y + q_1 \frac{dy}{dx} + q_2 \frac{d^2 y}{dx^2} + \ldots + q_p \frac{d^p y}{dx^p} \right\} f(x)\,dx$$

vanishes for every value of y that consists of a finite polynomial which, together with its first $p-1$ differential coefficients, vanishes at x_0 and x_1. Then, if $f(x)$ be continuous in the interval (x_0, x_1), it must be a solution of

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* See Forsyth's *Theory of Differential Equations*, Part III.

the equation which is adjoint to the equation

$$q_0 y + q_1 \frac{dy}{dx} + \ldots + q_p \frac{d^p y}{dx^p} = 0.$$

Moreover, if $f(x)$ is continuous only in some interval contained in (x_0, x_1), its value must satisfy the adjoint equation at all points in the interior of the interval of continuity.

It will be observed that, as in the former cases, no assumption has been made *a priori* as to the existence of differential coefficients of $f(x)$, which has been assumed only to be summable, whether limited or not, in the sense that it is assumed to be such as to possess a Lebesgue integral.

CLOSED LINKAGES

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The linkages described in this paper consist of a series of identical three-bar linkages revolving about the same pair of pivots, and formed by repeatedly rotating one of the triangles into which a diagonal divides the quadrilateral about that diagonal, alternate diagonals being used in succession. If, after a given number of such operations, a vertex coincides with its original position, the linkage is said to "close."

The subject has been discussed in an article by Arnold Emch,* who has pointed out the connection between the problem of closing such linkages and the corresponding problem of Poristic Polygons. The treatment is based upon the statement by Darboux † that such a chain of linkages could be formed, and that, if the required coincidence occurred at all, it would occur whatever was the initial position of the quadrilateral. It follows, therefore, that such a compound linkage is poristic. It possesses one degree of freedom, although it fulfils the conditions of a rigid framework.

Darboux has defined three quantities,

$$p_{12} = a+b-c-d, \quad p_{13} = a-b+c-d, \quad p_{14} = a-b-c+d,$$

where a, b, c, and d are the sides of the quadrilateral; and has limited his investigations to the case where p_{12} and p_{13} are positive and p_{14} is negative.‡ Emch has made the same restriction. In the later case the reason is evident; for it is only with this restriction that closed linkages of the sort he had in view can be constructed. But these three quantities may each be either positive or negative; and the object of the present

* *Annals of Mathematics*, Series 2, Vol. 1 (1900).

† *Bulletin des Sciences Mathématiques*, 2nd Series, Tome iii, 1879, pp. 120, 121.

‡ Darboux, *op. cit.*, pp. 111, 116.

paper is to remove the restriction and to show that closed linkages of another sort can be constructed whatever the values of p_{12}, p_{13}, and p_{14} may be, and that they likewise lead to poristic polygons whose circumscribing and inscribed circles may be partly or wholly external to one another.

Let us consider the side d as fixed. The nature of the motion of the linkage as regards the rotation or oscillation of the various links, and the characteristics of the curve traced out by any point rigidly connected to the opposite side b, will depend upon the relative lengths of the links. The case taken by Darboux and by Emch is the one which allows all three of the moving links a, b, c to revolve continuously through complete circles.

If all the permutations of the four quantities a, b, c, and d be written out, there will result twenty-four ways in which the order of magnitude of these links can be varied. In each of these there is one critical value of the ratios which permits the three moving links to lie altogether on the line d, namely, when the sum of the extremes is equal to the sum of the means. The locus of the carried point has then an extra double point in addition to the three ordinary nodes on the focal circle and to the two triple points at infinity. The curve becomes unicursal, and the elliptic functions by which the variables can be expressed degenerate. The locus is unipartite or bipartite according as one of these sums is greater or less than the other. There are, therefore, forty-eight different species of linkage giving rise to different kinds of curves having special characteristics.

In applying elliptic functions to three-bar motion we have to deal with the following quantities :

$$\alpha_1 = (b+c)^2, \quad \alpha_2 = (a+c)^2, \quad \alpha_3 = (a+b)^2,$$

$$\beta_1 = (a+d)^2, \quad \beta_2 = (b+d)^2, \quad \beta_3 = (c+d)^2,$$

$$\gamma_1 = (b-c)^2, \quad \gamma_2 = (a-c)^2, \quad \gamma_3 = (a-b)^2,$$

$$\delta_1 = (a-d)^2, \quad \delta_2 = (b-d)^2, \quad \delta_3 = (c-d)^2.$$

It is with the relative values of α, β, γ, and δ that we are concerned. Now, the forty-eight different species of linkage can be grouped into eight classes, each of six species. All the species in any one class exhibit the same peculiarities in the rotation or oscillation of the links, in the number of branches in the locus of the carried point, and in the relative magnitudes of the twelve quantities $\alpha, \beta, \gamma, \delta, \ldots$. The following table gives an analysis of them :—

Class	Relative Magnitudes	Relative Magnitudes	Relative Magnitudes	a passes OO'		c passes OO'		Links which rotate	Number of Branches
				externally	internally	externally	internally		
1	$\alpha_1\ \beta_1\ \gamma_1\ \delta_1$	$\alpha_2\ \beta_2\ \gamma_2\ \delta_2$	$\beta_3\ \alpha_3\ \delta_3\ \gamma_3$	yes	no	no	yes	—	1
2	$\alpha_1\ \beta_1\ \gamma_1\ \delta_1$	$\beta_2\ \alpha_2\ \delta_2\ \gamma_2$	$\alpha_3\ \beta_3\ \gamma_3\ \delta_3$	yes	no	yes	no	—	1
3	$\alpha_1\ \beta_1\ \delta_1\ \gamma_1$	$\alpha_2\ \beta_2\ \delta_2\ \gamma_2$	$\alpha_3\ \beta_3\ \delta_3\ \gamma_3$	yes	yes	yes	yes	a, b, c	2
4	$\alpha_1\ \beta_1\ \delta_1\ \gamma_1$	$\beta_2\ \alpha_2\ \gamma_2\ \delta_2$	$\beta_3\ \alpha_3\ \gamma_3\ \delta_3$	yes	yes	no	no	a	2
5	$\beta_1\ \alpha_1\ \gamma_1\ \delta_1$	$\beta_2\ \alpha_2\ \gamma_2\ \delta_2$	$\alpha_3\ \beta_3\ \delta_3\ \gamma_3$	no	no	yes	yes	c	2
6	$\beta_1\ \alpha_1\ \gamma_1\ \delta_1$	$\alpha_2\ \beta_2\ \delta_2\ \gamma_2$	$\beta_3\ \alpha_3\ \gamma_3\ \delta_3$	no	no	no	no	b	2
7	$\beta_1\ \alpha_1\ \delta_1\ \gamma_1$	$\alpha_2\ \beta_2\ \gamma_2\ \delta_2$	$\alpha_3\ \beta_3\ \gamma_3\ \delta_3$	no	yes	yes	no	—	1
8	$\beta_1\ \alpha_1\ \delta_1\ \gamma_1$	$\beta_2\ \alpha_2\ \delta_2\ \gamma_2$	$\beta_3\ \alpha_3\ \delta_3\ \gamma_3$	no	yes	no	yes	—	1

Now, it can easily be seen that a passes OO' (Fig. 1) externally if $b+c > a+d$ or $\alpha_1 > \beta_1$, and internally if $a \sim d > b \sim c$ or $\delta_1 > \gamma_1$, and

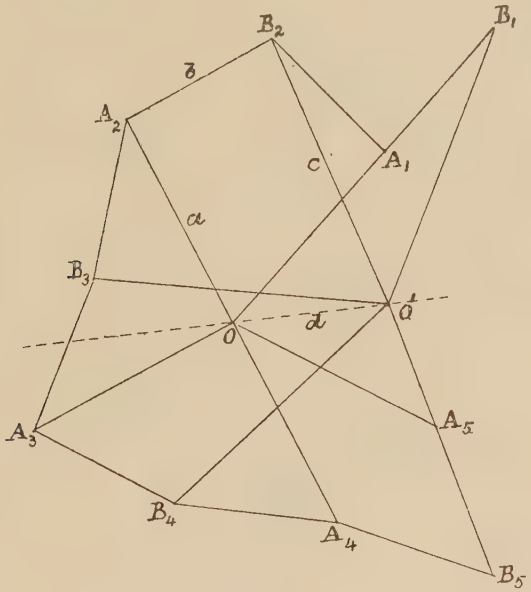

FIG. 1.

therefore revolves if $\alpha_1 > \beta_1 > \delta_1 > \gamma_1$. Also c passes OO' externally if $a+b > c+d$ or $\alpha_3 > \beta_3$, and internally if $c \sim d > a \sim b$ or $\delta_3 > \gamma_3$, and therefore revolves if $\alpha_3 > \beta_3 > \delta_3 > \gamma_3$. Similarly the revolution of b is complete if $\alpha_2 > \beta_2 > \delta_2 > \gamma_2$.

Let us consider Class 1. Denote the angles $O'OA$, $OO'B$ by ξ and η.

The angle η cannot increase beyond a certain value given by

$$\cos \eta = \frac{c^2+d^2-(a+b)^2}{2cd}.$$

After reaching this value it diminishes. Starting with the linkage in the position when η is a maximum, build upon it a series of additional linkages in the manner described. If the last c link falls in the position which is the reflection of $O'B_1$ in OO' or, as shown in Fig. 1, in the position of maximum ξ, the linkage will then form one half of a closed linkage. Now superimpose upon this compound linkage another one similar in all respects, but lacking in the first case the last c link and in the second the last a link. Pivot the two linkages together at O, O' and B_1, and either at B_5 or A_5, as the case may be. Then move the two A_1 points in opposite directions, and the whole will open out in the manner shown in Fig. 2.

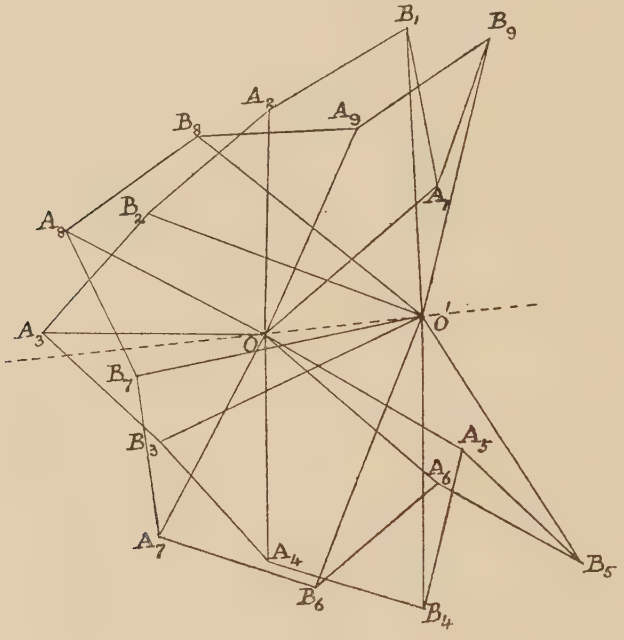

Fig. 2.

This compound linkage may then be regarded as one whole. Any motion imparted to any link will be communicated to all the others, and, when any one a or c link reaches the limit of its motion in one direction, it will turn back and become a part of the superimposed linkage, and will travel in the opposite direction to its other limiting position.

The problem of closing these linkages depends upon a principle pointed

out by Cayley * and utilized by Darboux. The theorem is *in effect* that if x_1 and x_2 denote the squares of the diagonals $O'A_1$ and $O'A_2$, x_1 and x_2 can be expressed in terms of elliptic functions of two arguments ϑ_1 and ϑ_2 whose difference is constant. This can be shown in a variety of ways. Perhaps the most natural is derived from

$$x = a^2-2ad\cos\xi+d^2 = b^2-2bc\cos\omega+c^2,$$

where ω denotes the angle ABO'. Then

$$\cos\xi = \frac{\beta_1-x-(x-\delta_1)}{\beta_1-\delta_1}, \qquad \cos\omega = \frac{\alpha_1-x-(x-\gamma_1)}{\alpha_1-\gamma_1},$$

$$\sin\xi = \frac{2\{(\beta_1-x)(x-\delta_1)\}^{\frac{1}{2}}}{\beta_1-\delta_1}, \quad \sin\omega = \frac{2\{(\alpha_1-x)(x-\gamma_1)\}^{\frac{1}{2}}}{\alpha_1-\gamma_1};$$

hence $$\frac{d\xi_1}{\sin\omega_1} = \tfrac{1}{2}(\alpha_1-\gamma_1)\frac{dx_1}{(X_1)^{\frac{1}{2}}}, \qquad \frac{d\xi_2}{\sin\omega_2} = \tfrac{1}{2}(\alpha_1-\gamma_1)\frac{dx_2}{(X_2)^{\frac{1}{2}}},$$

where $$X = (\alpha_1-x)(\beta_1-x)(x-\gamma_1)(x-\delta_1),$$

and, as the small displacements of A_1 and A_2 are proportional to $\sin\omega_1$ and $\sin\omega_2$,

$$d\xi_1/\sin\omega_1 = d\xi_2/\sin\omega_2;$$

hence $$\vartheta_2-\vartheta_1 = \text{a constant, say } \mu,$$

where $$\frac{1}{M}\vartheta = \int_\gamma \frac{dx}{X^{\frac{1}{2}}} \quad (\alpha>\beta>x>\gamma>\delta),$$

and similarly $$\vartheta_3-\vartheta_2 = \mu,$$

so that $$\vartheta_n-\vartheta_1 = (n-1)\mu = 4K-\mu,$$

since the linkage closes in $4K$.

Now μ can be determined from the condition that when

$$\cos\xi = \frac{(a+b)^2-c^2+d^2}{2d(a+b)},$$

A_n coincides with A_1, $x_n = x_1$ and $\vartheta_n = 4K-\vartheta_1$, so that $2\vartheta_1 = \mu$, and now

$$x_1 = a^2+d^2-\frac{a\{(a+b)^2-c^2+d^2\}}{a+b} = \frac{\gamma_1(\beta_1-\delta_1)-\delta_1(\beta_1-\gamma_1)\,\mathrm{sn}^2\vartheta_1}{\beta_1-\delta_1-(\beta_1-\gamma_1)\,\mathrm{sn}^2\vartheta_1},$$

the modulus being $\dfrac{(\beta_1-\gamma_1)(\alpha_1-\delta_1)}{(\alpha_1-\gamma_1)(\beta_1-\delta_1)}$ which is the same as $\dfrac{(\beta_2-\gamma_2)(\alpha_2-\delta_2)}{(\alpha_2-\gamma_2)(\beta_2-\delta_2)}$;

* *Phil. Trans. Roy. Soc.*, 1861, p. 225.

from which we obtain

$$\mathrm{sn}^2\vartheta = \frac{\beta_2-\delta_2}{\alpha_2-\delta_2}, \qquad \mathrm{cn}^2\vartheta = \frac{\alpha_2-\beta_2}{\alpha_2-\delta_2}, \qquad \mathrm{dn}^2\vartheta = \frac{\alpha_2-\beta_2}{\alpha_2-\gamma_2};$$

and hence

$$\mathrm{sn}\,\mu = \frac{2\{(\alpha_2-\gamma_2)(\beta_2-\delta_2)\}^{\frac{1}{2}}}{\alpha_2+\beta_2-\gamma_2-\delta_2}, \qquad \mathrm{cn}\,\mu = \pm\frac{\alpha_2-\beta_2-\gamma_2+\delta_2}{\alpha_2+\beta_2-\gamma_2-\delta_2},$$

$$\mathrm{dn}\,\mu = \pm\frac{\alpha_2-\beta_2+\gamma_2-\delta_2}{\alpha_2+\beta_2-\gamma_2-\delta_2},$$

and the condition that the linkage should close is that μ must be an aliquot part of $4K$. Now the ϑ that we have just evaluated is the interval between the minimum of ξ and the maximum of η. This interval is therefore $\frac{1}{2}\mu$; accordingly the interval between the B_1 and B_5 of Fig. 1 is $4\frac{1}{2}\mu$. There is, of course, the same interval from B_5 back to the starting point. Fig. 2 therefore closes in 9.

The other classes can be dealt with in a similar manner. They naturally resolve themselves into two groups: the first, which includes classes 1, 2, 7, and 8, consist of linkages which give rise to unipartite three-bar curves and which close in $4K$; the second comprises classes 3, 4, 5, and 6, and is composed of those which produce bipartite curves and which close in $2K$, the motion in the two other elliptic quadrants being represented by the other branch of the curve and being only realizable by an actual disjointing of the mechanism and by its reconstruction in another position. The value of μ can be obtained by a similar process; for, although it is not always possible to effect the coincidence of the points A_n and A_1 in the manner of the foregoing example, yet, when it is not, it is always possible to make these points occupy positions symmetrical with respect to OO' and where the value of $\cos\xi$ is known. Then $x_n = x_1$ and $\vartheta_n = 4K-\vartheta_1$ or $2K-\vartheta_1$ as the case may be, so that in every case $\mu = 2\vartheta$ and can be determined. We shall, in fact, find as follows:—

	$\mathrm{sn}^2\vartheta$	$\mathrm{cn}^2\vartheta$	$\mathrm{dn}^2\vartheta$
Classes 1 and 7 ...	$\frac{\beta_2-\delta_2}{\alpha_2-\delta_2}$	$\frac{\alpha_2-\beta_2}{\alpha_2-\delta_2}$	$\frac{\alpha_2-\beta_2}{\alpha_2-\gamma_2}$
Classes 2 and 8 ...	$\frac{\beta_2-\delta_2}{\beta_2-\gamma_2}$	$\frac{\delta_2-\gamma_2}{\beta_2-\gamma_2}$	$\frac{\delta_2-\gamma_2}{\alpha_2-\gamma_2}$
Classes 3 and 6 ...	$\frac{\beta_2-\gamma_2}{\alpha_2-\gamma_2}$	$\frac{\alpha_2-\beta_2}{\alpha_2-\gamma_2}$	$\frac{\alpha_2-\beta_2}{\alpha_2-\delta_2}$
Classes 4 and 5 ...	$\frac{\alpha_2-\delta_2}{\beta_2-\delta_2}$	$\frac{\beta_2-\alpha_2}{\beta_2-\delta_2}$	$\frac{\beta_2-\alpha_2}{\beta_2-\gamma_2}$

which give

	$\operatorname{sn}\mu$	$\operatorname{cn}\mu$	$\operatorname{dn}\mu$
Classes 1, 2, 7 and 8...	$\pm\frac{2\{(\alpha_2-\gamma_2)(\beta_2-\delta_2)\}^{\frac{1}{2}}}{\alpha_2+\beta_2-\gamma_2-\delta_2}$	$\pm\frac{\alpha_2-\beta_2-\gamma_2+\delta_2}{\alpha_2+\beta_2-\gamma_2-\delta_2}$	$\pm\frac{\alpha_2-\beta_2+\gamma_2-\delta_2}{\alpha_2+\beta_2-\gamma_2-\delta_2}$
Classes 3, 4, 5 and 6...	$\pm\frac{2\{(\alpha_2-\delta_2)(\beta_2-\gamma_2)\}^{\frac{1}{2}}}{\alpha_2+\beta_2-\gamma_2-\delta_2}$	$\pm\frac{\alpha_2-\beta_2+\gamma_2-\delta_2}{\alpha_2+\beta_2-\gamma_2-\delta_2}$	$\pm\frac{\alpha_2-\beta_2-\gamma_2+\delta_2}{\alpha_2+\beta_2-\gamma_2-\delta_2}$

Now, in Classes 1, 2, 7 and 8, in order to close in n cells, μ must be equal to $\frac{4K}{n}$ or ϑ to $\frac{2K}{n}$. If we denote $\frac{1}{\operatorname{sn}^2\frac{2K}{n}}$ by p, and $1-\operatorname{dn}^2\frac{2K}{n}$ by q, then for classes 1 and 7,

$$\frac{\alpha_2-\delta_2}{\beta_2-\delta_2}=p, \qquad \frac{\beta_2-\gamma_2}{\alpha_2-\gamma_2}=q,$$

or $$a^2-b^2+c^2-d^2+2ac+2bd=4pbd,$$

and $$-a^2+b^2-c^2+d^2+2ac+2bd=4qac;$$

therefore $$ac=\frac{p-1}{1-q}bd,$$

and $$a^2+c^2=b^2+d^2+\frac{2\{p(1-q)-q(p-1)\}}{1-q}bd;$$

$\frac{p(1-q)-q(p-1)}{1-q}$ being $\frac{k'^2\operatorname{sn}^2\frac{2K}{n}+\operatorname{cn}^2\frac{2K}{n}\operatorname{dn}^2\frac{2K}{n}}{\operatorname{sn}^2\frac{2K}{n}\operatorname{dn}^2\frac{2K}{n}}$, and always positive.

We can now assign arbitrary values to b and d, and obtain from the two foregoing equations the corresponding values of a and c, which will ensure closing in n cells. In choosing the values of b and d we must take care to make a^2+c^2 greater than $2ac$, otherwise $a-c$ will become imaginary. By giving the negative sign to $\{(a-c)^2\}^{\frac{1}{2}}$ we interchange the values of a and c, and the linkage changes to one of the associated class.

For classes 2 and 8, we put

$$\frac{\beta_2-\gamma_2}{\beta_2-\delta_2}=p, \quad \text{and} \quad \frac{\alpha_2-\delta_2}{\alpha_2-\gamma_2}=q;$$

which has the effect of simply changing the sign of

$$\frac{2\{p(1-q)-q(p-1)\}}{1-q}.$$

For classes 3 and 6,

$$\mathrm{sn}^2 \vartheta = \frac{1-\mathrm{cn}\,\frac{2K}{n}}{1+\mathrm{dn}\,\frac{2K}{n}},$$

$$\mathrm{dn}^2 \vartheta = \frac{\mathrm{cn}\,\frac{2K}{n}+\mathrm{dn}\,\frac{2K}{n}}{1+\mathrm{cn}\,\frac{2K}{n}},$$

$$\frac{1}{1-\mathrm{dn}^2 \vartheta} = \frac{1+\mathrm{cn}\,\frac{2K}{n}}{1-\mathrm{dn}\,\frac{2K}{n}},$$

so that, denoting $\mathrm{cn}\,\frac{2K}{n}$ by r, and $\mathrm{dn}\,\frac{2K}{n}$ by s, we get

$$ac = \frac{1+s}{1-s}\,bd,$$

$$a^2+c^2 = b^2+d^2+\frac{4r}{1-s}\,bd.$$

Classes 4 and 5 have the peculiarity that the alternate a links in class 4 and c links in class 5 are moving in opposite directions. If, in class 4, we consider the motion of $A_1, A_3, \ldots$ as positive, and that of $A_2, A_4, \ldots$ as negative, then $\vartheta_2, \vartheta_4, \ldots$ are negative arguments, and we get

$$\vartheta_1+\vartheta_2 = -2K+\mu,$$

$$\vartheta_2+\vartheta_3 = -\mu,$$

where μ is the positive value of ϑ_2 when $\vartheta_1 = 0$; so that

$$\vartheta_3-\vartheta_1 = 2K-2\mu,$$

$$\vartheta_5-\vartheta_3 = 2K-2\mu,$$

$$\ldots \quad \ldots \quad \ldots$$

$$\vartheta_{n+1}-\vartheta_{n-1} = 2K-2\mu,$$

whence $$\vartheta_{n+1}-\vartheta_1 = nK-n\mu\,;$$

and if ϑ_{n+1} coincides with ϑ_1, that is to say, is $2K+\vartheta_1$, then

$$\mu = \frac{n-2}{n}K = K-\frac{2K}{n}\,;$$

from which it is plain that these classes cannot close in an odd number of cells. Then denoting $\mathrm{cn}\left(K-\frac{2K}{n}\right)$ by r', and $\mathrm{dn}\left(K-\frac{2K}{n}\right)$ by s', we get

$$ac = \frac{1-s'}{1+s'}\,bd,$$

$$a^2+c^2 = b^2+d^2-\frac{4r'}{1+s'}\,bd.$$

The tables given in Sir George Greenhill's memoir on "The Third Elliptic Integral and the Ellipsotomic Problem" * afford a ready means of obtaining the values of p, q, r, and s for certain values of n in terms of a parameter which may be chosen arbitrarily and which may be considered to be a function of the modulus.

If the links a and b are equal, in the unipartite classes

$$c = \frac{p-1}{1-q}\,d,$$

and

$$a = \pm\frac{(p-q)(p+q-2)}{2(1-q)\{p(1-q)-q(p-1)\}}\,d;$$

the upper sign being taken for classes 1 and 7 and the lower for 2 and 8. Now $p-1$, $1-q$, $p-q$ and $p(1-q)-q(p-1)$ are all positive quantities. The possibility of such a linkage depends, therefore, on the sign of $p+q-2$. It can easily be verified that $p+q=2$ when $n=4$. Accordingly a linkage in which $a=b$ is only possible in classes 1 and 7 when n is greater than 4, and in classes 2 and 8 when n is less than 4. In classes 3 and 6

$$c = \frac{1+s}{1-s}\,d, \quad \text{and} \quad a = \frac{s}{r(1-s)}\,d;$$

and in classes 4 and 5,

$$c = \frac{1-s'}{1+s'}\,d, \quad \text{and} \quad a = \frac{s'}{r'(1+s')}\,d;$$

all essentially positive as μ lies in the first elliptic quadrant. There is, therefore, no impossibility in such cases.

With regard to the formation of the related poristic polygons, they can most readily be constructed by drawing tangents to the circle $A_1A_2\ldots$ through the points $A_1, A_2, \ldots$. These tangents will intersect on a conic

* *Phil. Trans. Roy. Soc.*, Series A, Vol. 203, pp. 217–304.

whose equation can be determined by eliminating z from

$$r = \frac{a}{\cos A_1 O B_1} = \frac{2a^2 z}{z^2 + a^2 - b^2}, \quad \text{and} \quad d\cos\theta = \frac{z^2 - c^2 + d^2}{2z};$$

where z represents the diagonal OB_1. This conic becomes a circle if $a = b$. It is only when $a = b$ or $b = c$ that poristic polygons with circumscribing circles can be obtained from a closed linkage. If $a = b$ the conic becomes a circle whose centre is on OO' at a distance $\frac{2a^2 d}{c^2 - d^2}$ from O, and whose radius is $\frac{2a^2 c}{c^2 - d^2}$; and this circle is the one into which the circle $B_1 B_2 \ldots$ inverts with reference to a circle concentric with, and of twice the radius of, $A_1 A_2 \ldots$, but drawn to half the scale. If $b = c$, the tangents through B_1, B_2, ..., intersect on a circle whose centre is $\frac{2c^2 d}{a^2 - d^2}$ from O', and whose radius is $\frac{2ac^2}{a^2 - d^2}$.

Fig. 3 gives an instance of a closed linkage belonging to class 5, selected because this class produces a poristic polygon inscribed in and circumscribed to circles which are completely external to one another.

It is interesting to note that the polygon $A_1 A_2 \ldots$ is always circumscribed to a fixed conic, as also is the polygon $B_1 B_2 \ldots$. For Fig. 3 clearly shows that the envelope of $A_1 A_2$ is the reciprocal polar of the locus of B_1 with respect to the circle $A_1 A_2 \ldots$. If the locus of S_1 is a circle, and s_1, s_2, ... denote the points where the lines OS_1, OS_2, ... again meet the circle, then the polygon $s_1 s_2 \ldots$ is similar to the polygon $B_1 B_2 \ldots$. For when $a = b$, $S_1 O . O s_1$ and $S_1 O . O B_1$ are both constant, therefore $O s_1 / O B_1$ is constant.

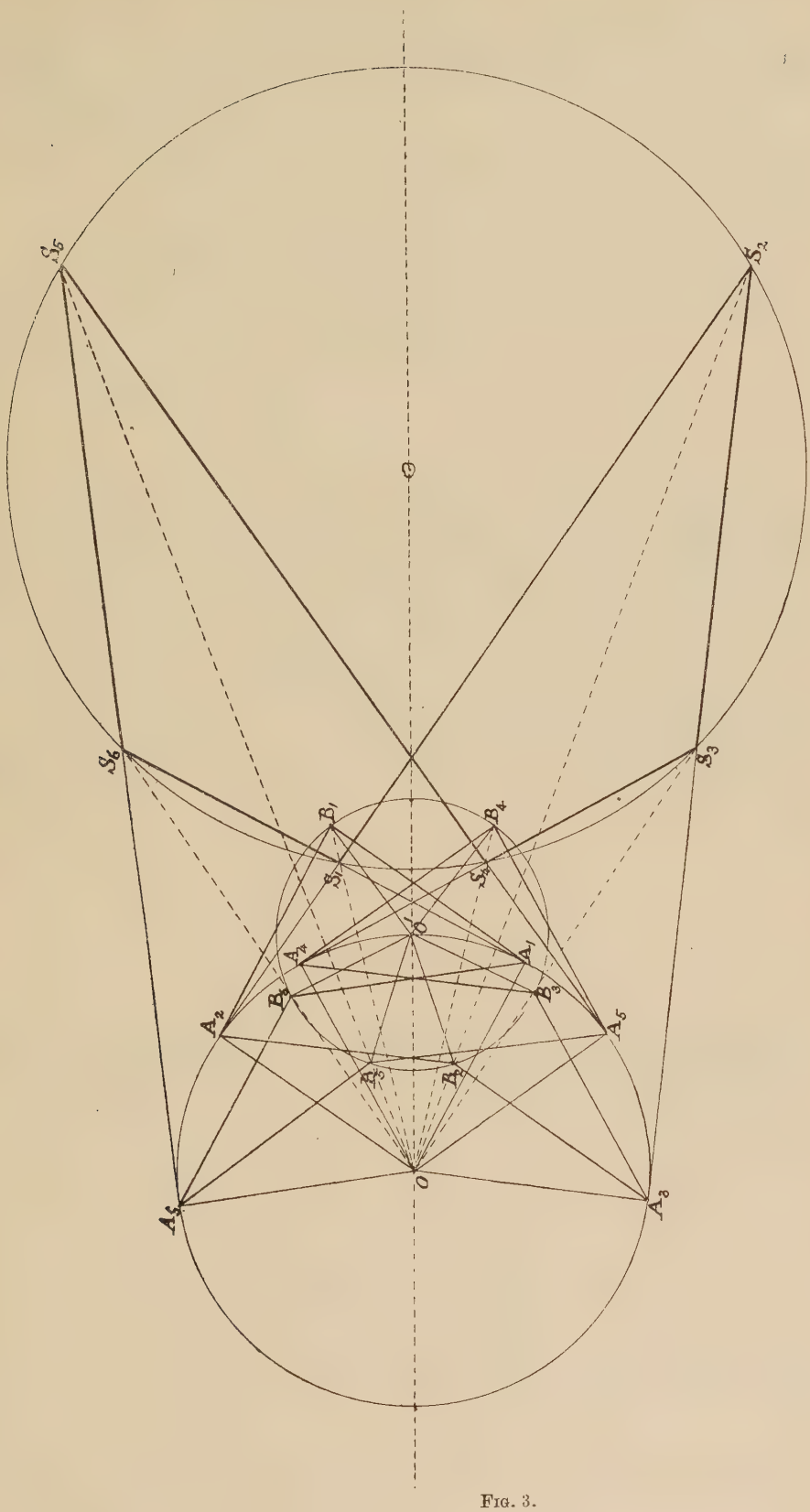

FIG. 3.

ON THE OUTER ISOMORPHISMS OF A GROUP

By W. BURNSIDE.

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WHILE preparing the second edition of my *Theory of Groups* for the press I made many ineffectual attempts to determine whether an outer isomorphism of a group necessarily permutes some of its conjugate sets, or, in the alternative, if groups exist some outer isomorphisms of which change every operation into a conjugate operation.

I have since succeeded in constructing comparatively simple examples showing that of the two suppositions the latter is the correct one. One of the simplest of these is given below.

Taking p^3 symbols $x_{\alpha,\beta,\gamma}$, where the suffixes are reduced (mod p), the four substitutions

$$A_1 \quad \text{or} \quad x'_{\alpha,\beta,\gamma} = x_{\alpha+1,\beta,\gamma},$$

$$A_2 \quad \text{or} \quad x'_{\alpha,\beta,\gamma} = x_{\alpha,\beta+1,\gamma},$$

$$A_3 \quad \text{or} \quad x'_{\alpha,\beta,\gamma} = \omega^{\alpha+\beta} x_{\alpha,\beta,\gamma-\beta},$$

$$A_4 \quad \text{or} \quad x'_{\alpha,\beta,\gamma} = \omega^{\beta} x_{\alpha,\beta,\gamma+\beta-\alpha},$$

where ω is an assigned primitive p-th root of unity, generate a group of monomial substitutions of finite order.

The substitution that corresponds to $A_3^{-1}A_1^{-1}A_3A_1$ is determined by eliminating the intermediate symbols between

$$x'_{\alpha,\beta,\gamma} = \omega^{-\alpha-\beta} x_{\alpha,\beta,\gamma+\beta},$$

$$x''_{\alpha,\beta,\gamma} = x'_{\alpha-1,\beta,\gamma},$$

$$x'''_{\alpha,\beta,\gamma} = \omega^{\alpha+\beta} x''_{\alpha,\beta,\gamma-\beta},$$

$$x''''_{\alpha,\beta,\gamma} = x'''_{\alpha+1,\beta,\gamma}.$$

Hence $A_3^{-1}A_1^{-1}A_3A_1$ is $x'_{\alpha,\beta,\gamma} = \omega x_{\alpha,\beta,\gamma}$.

In the same way it is shewn that

$$A_4^{-1}A_1^{-1}A_4A_1 \text{ is } x'_{\alpha,\beta,\gamma} = x_{\alpha,\beta,\gamma-1},$$
$$A_3^{-1}A_2^{-1}A_3A_2 \text{ is } x'_{\alpha,\beta,\gamma} = \omega x_{\alpha,\beta,\gamma-1},$$
$$A_4^{-1}A_2^{-1}A_4A_2 \text{ is } x'_{\alpha,\beta,\gamma} = \omega x_{\alpha,\beta,\gamma+1},$$

while both $A_1^{-1}A_2^{-1}A_1A_2$ and $A_3^{-1}A_4^{-1}A_3A_4$ are E, the identical substitution.

Denoting the two substitutions

$$x'_{\alpha,\beta,\gamma} = \omega x_{\alpha,\beta,\gamma},$$

and

$$x'_{\alpha,\beta,\gamma} = x_{\alpha,\beta,\gamma-1},$$

by P and Q, which are obviously independent substitutions of order p, it may be at once verified that both P and Q are permutable with A_1, A_2, A_3, A_4; while, on repeating them, it is found that A_1, A_2, A_3, A_4 are all of order p. It follows that every substitution of the group is expressible in the form

$$A_1^{x_1}A_2^{x_2}A_3^{x_3}A_4^{x_4}P^yQ^z,$$

where the indices take all values from 0 to $p-1$, and that the p^6 substitutions contained in this form are all distinct, so that the group is one of order p^6.

The existence of this group of linear substitutions proves that the relations

$$A_1^p = A_2^p = A_3^p = A_4^p = P^p = Q^p = E,$$
$$A_1^{-1}A_2^{-1}A_1A_2 = A_3^{-1}A_4^{-1}A_3A_4 = E,$$
$$A_3^{-1}A_1^{-1}A_3A_1 = P, \qquad A_4^{-1}A_1^{-1}A_4A_1 = Q,$$
$$A_3^{-1}A_2^{-1}A_3A_2 = PQ, \qquad A_4^{-1}A_2^{-1}A_4A_2 = PQ^{-1},$$

and relations expressing that both P and Q are permutable with A_1, A_2, A_3, A_4 are the complete defining relations of an abstract group G of order p^6.

Denoting $A_1^{x_1}A_2^{x_2}A_3^{x_3}A_4^{x_4}$ by R, the general inner isomorphism of G is given by

$$R^{-1}A_1R = A_1P^{-x_3}Q^{-x_4}, \qquad R^{-1}A_2R = A_2P^{-x_3-x_4}Q^{-x_3+x_4},$$
$$R^{-1}A_3R = A_3P^{x_1+x_2}Q^{x_2}, \qquad R^{-1}A_4R = A_4P^{x_2}Q^{x_1-x_2}.$$

This changes $A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}$ into $A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}P^yQ^z$, where

$$\left.\begin{aligned} y &\equiv -u_1x_3-u_2(x_3+x_4)+u_3(x_1+x_2)+u_4x_2 \\ z &\equiv -u_1x_4-u_2(x_3-x_4)+u_3x_2+u_4(x_1-x_2) \end{aligned}\right\} \pmod p.$$

Now $A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}$ is obviously not a self-conjugate operation of G. If it is one of p conjugate operations, then y/z must be independent of x_1, x_2, x_3, x_4. Now, if y/z is λ, then

$$x_4[\lambda(u_1-u_2)-u_2]+x_3(\lambda u_2-u_1-u_2)+x_2[\lambda(u_4-u_3)+u_3+u_4]$$
$$+x_1(-\lambda u_4+u_3)\equiv 0 \pmod{p}.$$

Hence, if λ is independent of x_1, x_2, x_3, x_4, then

$$\begin{vmatrix} \lambda & -\lambda-1 & . & . \\ -1 & \lambda-1 & . & . \\ . & . & -\lambda+1 & \lambda+1 \\ . & . & 1 & -\lambda \end{vmatrix} \equiv 0 \pmod{p},$$

or

$$[(\lambda-1)^2-2]^2 \equiv 0 \pmod{p}.$$

It follows that if 2 is a quadratic non-residue (mod p), then $A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}$ is one of a set of p^2 conjugate operations of G; or, in other words, that

$$A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}P^iQ^j \quad (i, j = 0, 1, \ldots, p-1)$$

constitute a conjugate set for G for all values of u_1, u_2, u_3, u_4, except

$$u_1 = u_2 = u_3 = u_4 = 0.$$

Now the defining relations of G are unaltered if $A_1P^{i_1}Q^{j_1}$, $A_2P^{i_2}Q^{j_2}$, $A_3P^{i_3}Q^{j_3}$, $A_4P^{i_4}Q^{j_4}$ are written for A_1, A_2, A_3, A_4, while P and Q are unchanged. Hence

$$\begin{pmatrix} A_1, & A_2, & A_3, & A_4 \\ A_1P^{i_1}Q^{j_1}, & A_2P^{i_2}Q^{j_2}, & A_3P^{i_3}Q^{j_3}, & A_4P^{i_4}Q^{j_4} \end{pmatrix}$$

gives an isomorphism of G, whatever the i's and j's may be. Moreover, any two of these isomorphisms carried out successively give a third isomorphism of the same form. There arises thus an Abelian group of isomorphisms of G of order p^8 and type $(1, 1, \ldots, 1)$; and every isomorphism contained in this group changes any operation of G into a conjugate operation. The order of the group of inner isomorphisms of G is p^4. Hence G admits outer isomorphisms which change every operation into a conjugate operation.

The above group of isomorphisms of order p^8 is the greatest which changes every operation into a conjugate operation, since it changes the four generating operations into any four which are conjugate with each of them respectively. Every isomorphism of G not contained in this group must permute some of the conjugate sets.

ON SUCCESSIONS OF INTEGRALS AND FOURIER SERIES

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1. In a paper* published in the *Proceedings* of the Society I have explained a variety of circumstances under which it is legitimate to replace a factor of the integrand of a definite integral by a converging or non-converging series, and then, after carrying out the multiplication, performing the integration term-by-term. The theorems there given were selected with a view to the simplicity of their demonstration or the facility of their application. I propose now to give a number of results which include those already given as particular cases. Two modes of extension are employed: in the one, inequalities take the place of equalities, or, less generally, lower or upper limits take the place of limits; in the other mode of generalisation the guiding idea is that, just as a single function, when repeated an infinite number of times, may be regarded as a sequence, or, more generally, as a succession of functions, so, conversely, successions of functions may be classified by more or less the same kind of process as individual functions. No attempt has been made to ascertain within what limits such an analogy is capable of development; but the possible existence of such an analogy itself suggests a variety of extensions of the theorems in the paper above referred to.

There is one other respect in which certain of the previous results are capable of generalisation. The most general conditions under which the product of two functions is an integral in the Lebesgue sense involve each of these functions being an integral. In practice, however, it will constantly occur that uv is an integral when one at least of the two functions u and v is not. Thus, the formula for integration by parts will often actually hold good when the general theorem fails. In this way various extensions of my previous results of considerable practical interest are possible, but I do not propose to enter into this matter here.

As regards the first mode of generalisation above explained, it indicates *inter alios* how a series may be employed when even its integrated series does

* "On the Theory of the Application of Expansions to Definite Integrals," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 463–485; cited below as the Expansions paper.

not converge, but has an upper or lower function which is an integral and takes the place of the unique limiting function in the ordinary theory. In the second of these modes of generalisation the problem consists in determining under what conditions, if f_n and g_n be representative functions of two successions whose integrals converge to the integrals of two functions f and g, we are at liberty to assert that the double limit

$$\operatorname*{Lt}_{(m\to\infty,\ n\to\infty)} \int_a^z f_n(x)\, g_m(x)\, dx$$

exists, and is equal to $\int_a^z f(x)\, g(x)\, dx$.

This is virtually, it will be seen, a discussion of the circumstances under which a certain double series converges, however it be summed, to the required integral. It is found that the process is allowable under various circumstances which are enumerated. The results arrived at take a particularly interesting form when interpreted in the language of integrals. We then obtain a number of sets of sufficient conditions that

$$\int_a^b f(x, t)\, g(y, t)\, dt$$

should be a continuous function of the ensemble (x, y). Any of the following sets of conditions are sufficient; they are only special cases of those actually obtained :—

(i) $f(x, t)$ and $g(y, t)$ are both bounded, while $f(x, t)$ and the integral of $g(y, t)$ with respect to t are continuous functions of x and of y respectively;

(ii) $g(y, t)$ is bounded and its integral with respect to t is a continuous function of y, while $f(x, t)$ is positive, is a continuous function of x, and has an integral with respect to t which is a continuous function of x;

(iii) $g(y, t)$ is bounded and is a continuous function of y, while the integral of $f(x, t)$ with respect to t is a continuous function of x, and the integral of the $(1+p)$-th power of its modulus is, for some positive value of p, a bounded function of (x, t);

(iv) $g(y, t)$ is a function of t of bounded variation, whose total variation is a bounded function of (y, t) and whose integral with respect to t is a continuous function of y, while the integral of $f(x, t)$ with respect to t is a continuous function of (x, t);

(v) $g(y, t)$ is a monotone increasing function of t and a bounded function of (y, t), and its integral with respect to t is a continuous function of y,

while $f(x, t)$ has a Harnack-Lebesgue integral with respect to t, which is a continuous function of (x, t);

(vi) $g(y, t)$ is a continuous function of (y, t), while the integral of $f(x, t)$ with respect to t is a continuous function of x, and the integral of its modulus is bounded;

(vii) $g(y, t)$ and the $(1+1/p)$-th power of its modulus have integrals with respect to t, which are continuous functions of y, while the integral of $f(x, t)$ with respect to t is a continuous function of x, and the integral of the $(1+p)$-th power of its modulus is bounded;

(viii) the integral of $g(y, t)$ with respect to t is a continuous function of y, and the integral of the $(1+1/q)$-th power of its modulus is bounded for some value of q less than p, while $f(x, t)$ is a continuous function of x, and the $(1+p)$-th power of its modulus has a bounded integral with respect to t.

If, instead of considering the double limit, we confine our attention to the repeated limits, the extension involved is of a simpler character, both as regards formulation and proof. As regards the proof, little more is required than a repeated application of theorems already enunciated for single successions. We have, however, in this case an interesting application to the mixed differential coefficients of the integral just considered. Denoting this integral by u, the theorems in question assert, for example, that

$$\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx} = \int_a^b \left(\frac{d}{dt}\frac{d}{dx}\int_a^t f(x, t)\,dt \,.\, \frac{d}{dt}\frac{d}{dy}\int_a^t g(y, t)\,dt\right) dt,$$

under a variety of sets of conditions which correspond to those (i)–(viii) just written down, being, however, somewhat reduced in number. Thus, to give one example only, the first set of conditions is replaced by the condition that the derivates of f and of g with respect to x and y respectively should be bounded functions of (x, t) and (y, t), while the integrals in the integrand should possess differential coefficients with respect to x and y respectively.

In the course of the investigation I have found it convenient to employ certain extensions of some of the results of the Semi-integrals* paper. The proofs of these are given in a preliminary section. The step taken in passing from the Semi-integrals paper to the Expansions paper, and from thence to the present paper, is marked by a decrease in the attention given

* W. H. Young, "On Semi-Integrals and Oscillating Successions of Functions," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 286–324.

to the character of the succession of integrands. This fact is well illustrated by reference to the list of conditions above enumerated for the continuity of $\int f(x, t)g(y, t)dt$. As far as I know, the first theorem of this type was given by myself in a paper on "Parametric Integration," written in the early summer of 1909 and published in the *Monatshefte für Mathematik und Physik* in January, 1910, viz., that if g is a bounded function of (t, y), whose discontinuities with respect to the ensemble (t, y) form, for each fixed value of y, a set of zero content, while f is a not necessarily bounded positive function of (t, y), whose integral with respect to t over the interval (a, b) is a continuous function of y, then

$$\int_a^b f(t, y)\, g(t, y)\, dt$$

is a continuous function of y.

A special case of this, translated into the language of successions, asserts that if a uniformly convergent sequence be multiplied by a succession of positive functions whose integrals converge, the integrals of the resulting succession converge, a theorem from which, by integration by parts, we may deduce another, in which the one succession has its integrals converging uniformly to an integral and the other is itself a succession of integrals, each of which is a monotone increasing function. If we further make one of the successions reduce to a single function, the connection of these theorems with, for example, Theorems 4 and 6 of the Expansions paper, or in their general form with the sets of conditions 4 and 5 above given and the theorems about successions from which they are deduced, is obvious.

I have already in the papers cited, and in that on "The Integration of Fourier Series," acknowledged my obligations to the work of Lebesgue. As regards the Expansions paper, my obligations are even greater than I realised at the time. In his *Intégrales Singulières*,* the suggestive character of which has now been appreciated by more than one writer, several of the theorems of that paper are, indeed, implicitly contained. This is notably the case with Theorems 5 and 6.†.

* H. Lebesgue, *Intégrales Singulières*, 1910, Fac. de Toulouse, pp. 25-128.

† The reader may compare Lebesgue's Theorem II on p. 56 of the *Intégrales Singulières*, which, suitably interpreted, gives a theorem in the integration of oscillating successions whose integrals converge to integrals. Bearing in mind that it follows from the Semi-integrals paper that, if the integrals of such a succession converge, they necessarily converge to an integral, this theorem includes Theorem 3 of the Expansions paper as a particular case. It will be seen, however, that the problem that interested me was not quite the same as that with which Lebesgue was mainly occupied, and he does not interpret his Theorem II, and the

With regard to the scope of the theorems of the paper, it is evident that they have an application in all questions involving the continuity, differentiability, and integrability of functions defined by integrals or series. But this is not all. The fundamental nature of the questions discussed is such that every new result obtained is bound to have its place in the general scheme of analysis. Want of space has prevented my dealing with more than one type of application; but this I have selected in a domain that has a special interest, namely, the Theory of Fourier Series. The fourth and last section of the paper is devoted to these applications.

I first apply the results of Section II, which deals with single successions of integrals, to obtain, in more than one way, and in each case almost intuitively, all the known results with regard to the possibility of integrating the Fourier series of a function $f(x)$ when it is multiplied term-by-term by another function $g(x)$. These results, properly interpreted, give us theorems with regard to the repeated limits of $\int f_n(x) g_m(x) dx$, where $f_n(x)$ and $g_m(x)$ denote the ordinary or the Cesàro partial summations of the Fourier series of $f(x)$ and $g(x)$.

I then apply the results of Section III, which deals with double successions of integrals, to obtain sets of conditions under which the above integral has a unique double limit whose value is $\int f(x) g(x) dx$.

These latter sets of conditions differ from the former sets in a remarkably simple manner. Each set differs from the corresponding set in that one, and only one, of the ordinary partial summations has to be replaced by a Cesàro partial summation. Thus, where in the first set of theorems $f_n(x)$ and $g_m(x)$ were both ordinary partial summations, in the latter set one is an ordinary and the other a Cesàro partial summation;

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corresponding theorems, in this way. Thus, though there is no difficulty in deducing Fatou's extension of Parseval's Theorem from Theorem II, so interpreted, the conclusion being almost immediate, Lebesgue does not adopt this course, but proves the extension by an entirely different method on pp. 105–107.

In the Expansions paper considerable stress was laid on the case when the interval of integration is infinite, and some of the results arrived at were applied in the companion paper on the "Integration of Fourier Series." In one instance the application was illegitimate, as I have already pointed out in an erratum slip. The reader may either delete the passage in question, viz., lines 16–20 of p. 459, or correct them so that they may read as follows:— "When $f(x)$ is the function of bounded variation, the result of the preceding article is no longer true, since the argument used at the end of § 3 of the companion paper for an infinite interval of integration is not applicable. Thus the theorem does not hold for an infinite interval of integration, when $f(x)$ is the function of bounded variation." In fact, $f(x)$, being periodic, cannot be expressed as the difference of two suitable monotone functions, each of which has a unique finite limit at infinity.

where in the former set one was ordinary and the other Cesàro, both have now to be Cesàro summations. The remaining conditions are the same in the two sets.

There is, however, one additional set of conditions for the convergence of the double limit to the required integral, in which $f_n(x)$ and $g_m(x)$ are both ordinary partial summations, as in the original set. The only difference is here the additional stipulation that one of the Fourier series should converge to its function, except at a set of content zero.

It should be added that other results required for the proof of the validity of these conditions, relating to the Fourier series themselves, are obtained. Several of these appear not to have been remarked hitherto and to be not without importance.

It is also well, perhaps, to point out that by putting, for example, $m = n$, the double succession gives rise to a *single* uniformly converging series for the calculation of $\int f(x)\, g(x)\, dx$ in terms of the Fourier constants of $f(x)$ and $g(x)$. This series contains in addition only integrals of products of sines and cosines of multiples of x.

I. *On Certain Modes of Oscillation of Successions of Integrals.* (§§ 2–8.)

2. One of the main objects of the Semi-integrals paper was to explain how the mode of oscillation of a succession of integrals depended on that of the succession of integrands. The conditions obtained were, however, merely sufficient conditions. We might, indeed, say, without sensible exaggeration, that there are no necessary conditions. We may illustrate this fact by remarking that whereas it is sufficient, for example, for the existence of the unique double limit

$$\operatorname*{Lt}_{(E\to 0,\ n\to\infty)} \int_E f_n(x)\, dx = 0$$

that the original succession should be bounded both above and below, or that, for some value of $p > 0$, $\int_a^b |f_n(x)|^{1+p}\, dx$ should be a bounded function of n,* the hypothecation of the existence and vanishing of the double limit in question does not allow us even to assert that the upper and lower functions of the original succession are summable, or, indeed, integrable in any sense whatever.

Consider, for, instance, the succession formed as follows :—

Suppose the segment (0, 1) divided in turn into 2, 2^2, 2^3, ..., 2^r, ... equal

* See below, § 6.

parts, and let us form our succession of functions in corresponding groups of $2, 2^2, 2^3, \ldots, 2^r, \ldots$.

Thus, f_1 and f_2 are to be each either 0 or $2^{\frac{1}{2}}$; f_3, f_4, f_5, f_6 are each to assume the values 0 and 2 only; the functions f_7 up to f_{14} are to have the values 0 and $2^{\frac{3}{2}}$ only, and so on. Moreover, these values are to be so apportioned that f_1 is zero in the first half of the interval and $2^{\frac{1}{2}}$ in the second half; f_2 is to be $2^{\frac{1}{2}}$ in the first half and 0 in the second half; f_3 is to be zero except in the last quarter, where it is to be 2; f_4 is to be zero except in the third quarter, where it is to be 2; f_5 is to be zero except in the second quarter, where it is to be 2; f_6 is to be zero except in the first quarter, where it is to be 2, and so on.

Here the upper function is everywhere $+\infty$, although

$$\int_0^1 \{f_n(x)\}^2 \, dx = 1,$$

and *a fortiori* the double limit in question is zero, as follows, for example, from a theorem given below.

It is worth remarking that when the limits of integration are not 0 and 1 the succession of integrals does not converge and that the lower and upper functions of the succession of integrals are independent of the interval of integration chosen between the extremes 0 and 1, the lower function being always zero and the upper function always unity.

3. When we come, however, to various applications of the theory of successions of functions, for example, to such as those indicated in the Expansions paper, we find that the properties of the successions of integrals are often alone involved, even when the succession employed in the calculation is the original and not the integrated succession. It becomes therefore of the first importance to classify successions of integrals from this point of view. Thus, the succession of integrals may converge boundedly, or uniformly, or, without forming a sequence, it may oscillate uniformly and homogeneously. Or, again, we may be able to assert either of these properties, not merely of the succession of integrals of the functions of the original succession, but the same may be true when we substitute for these functions their squares or, more generally, their $(1+p)$-th powers.

Among such conditions that which requires the existence and vanishing of the double limit already alluded to, viz.,

$$\operatorname*{Lt}_{(E\to 0,\, n\to\infty)} \int_E f_n(x) \, dx = 0,$$

or, more generally, that all the double limits should be $\leqslant 0$ or $\geqslant 0$, though not always easy to test in practice, occupies a very important position. It is, in fact, one of the widest of the conditions which have so far been shown to lead to positive results. A less stringent condition than the vanishing of the double limit is that which requires that all the unique limiting functions of sub-sequences of the succession of integrals should themselves be integrals. A less wide condition is one which requires that the succession of the integrals of the $(1+p)$-th powers of the integrands should be bounded for some values of p.

I propose in the present section to begin by proving certain results with regard to this double limit, or, more generally, the triple limits

$$\underset{(E\to 0,\ m\to\infty,\ n\to\infty)}{\text{Llt}} \int_E f_n(x)\, g_m(x)\, dx.$$

These results may be regarded as completing others already stated and proved in the Semi-integrals paper. In the first theorem I consider the effect of an assumption as to the character of the limits, as being either all $\leqslant 0$, or all $\geqslant 0$, or all zero. The nature of the results to be stated are such that no generality is gained by employing the triple limits instead of the double limits. In the remaining theorem I prove a result required in the sequel, namely, that the triple limit is unique and has the value zero under circumstances there stated.

Throughout the paper where inequalities occur we prove explicitly the lower inequality. The corresponding upper inequality may be at once deduced. When the limits considered are unique, the corresponding theorem may be still more simply deduced.

4. I consider first the effect of the hypothesis

$$\underset{(E\to 0,\ n\to\infty)}{\text{Llt}} \int_E f_n(x)\, dx \geqslant 0, \qquad (\alpha)$$

where the double limits here considered are obtained as the content E of the set over which we integrate approaches zero, while n approaches infinity.

I have already shewn in the Semi-integrals paper that the supposition that the lower function $l(x)$ of the original succession $f_n(x)$ is summable involves the absolute semi-integrability below of the original succession; in other words, it implies as a consequence that if S denote any set of points over which we integrate

$$\int_S l(x)\, dx \leqslant \underset{n\to\infty}{\text{Llt}} \int_S f_n(x)\, dx. \qquad (\beta)$$

The lower and upper functions of the succession of integrals are then upper semi-integrals.* Whether we assume the summability of the lower function or not, the following theorem holds good † :—

THEOREM.—*If* $\int_E f_n(x)\,dx$ *has no negative double limit, as* $E \to 0$ *and* $n \to \infty$, *then the succession of integrals* $\int_a^x f_n(x)\,dx$ *is bounded below, and oscillates uniformly and homogeneously below. Moreover, if for any subset of values of the integer* n *the integrals converge to an unique limiting function, that function is everywhere continuous on the left, being an upper semi-integral and a lower semi-continuous function.*

For
$$\int_a^{x+h} f_n(x)\,dx = \int_a^x f_n(x)\,dx + \int_x^{x+h} f_n(x)\,dx.$$

By hypothesis the second integral on the right has no negative double limit when n increases indefinitely and $h \to 0$. But the lower double limit of the integral on the left is the chasm function $\chi(x)$ of the succession of integrals, and the lower limit of the first integral on the right is the lower function $L(x)$ of the succession of integrals. Thus,

$$\chi(x) \geqslant L(x).$$

But the chasm function is never greater than the lower function; therefore

$$\chi(x) = L(x);$$

that is to say, the succession of integrals oscillates uniformly below. Since the hypothesis applies equally to the whole succession or any of its sub-successions, this proves the second part of the first statement in the enunciation.

To prove that the lower bound of $\int_a^x f_n(x)\,dx$ regarded as a function of (n, x) is not $-\infty$, let us determine E_0 and n_0, so that, for all values of $E < E_0$, and $n > n_0$,

$$\int_E f_n(x)\,dx \geqslant -e, \qquad (\gamma)$$

this is possible by (α). Next let us determine an integer k, so that

$$kE_0 > (b-a).$$

Then if S denote the content of any set in the interval (a, b), and we divide the whole interval into k equal parts, each of these parts, and

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* *Semi-Integrals*, § 16, p. 301.

† Part of this theorem is given in *Semi-Integrals*, § 25.

therefore each of the sub-sets of the set S in these parts, is less than E_0. Hence, by the above,

$$\int_S f_n(x)\,dx \geqslant -ke,$$

which proves that, from and after $n = n_0$, the lower bound of the succession of integrals is not $-\infty$. Since the lower bound of the first n_0 integrals is necessarily finite, this proves that the succession of integrals is bounded below.

To prove the final statement, let us determine E_0 and n_0 as before.

Let us take any set of non-overlapping intervals $(x, x+h)$, finite in number, and of content less than E_0, as our set E. Then, denoting $\int_a^x f_n(x)dx$ by $F_n(x)$, and its unique limiting function by $F(x)$, we get from the inequality (γ), which may in this case be written

$$\Sigma[F_n(x+h)-F_n(x)] \geqslant -e,$$

the relation

$$\Sigma[F(x+h)-F(x)] \geqslant -e.$$

Since this is true when the number of intervals is finite, it is true when it is infinite, provided their content is less than E_0. But this is the condition* that $F(x)$ should be an upper semi-integral.

That $F(x)$ is lower semi-continuous follows, of course, from the uniformity of the lower oscillation.†

COR.—*When the double limit in question has the unique value zero, viz.,*

$$\operatorname*{Lt}_{(E\to 0,\ n\to\infty)} \int_E f_n(x)\,dx = 0, \qquad (\delta)$$

then the succession of integrals $\int_a^x f_n(x)dx$ *oscillates uniformly and homogeneously both above and below, and there is in every succession of the integrals a sequence which converges uniformly to an integral.*

If further the lower and upper functions $l(x)$ *and* $u(x)$ *of the original succession are summable, all the upper and all the lower functions of the succession of integrals, and of its sub-successions, are themselves integrals.*

Moreover

$$\int_S l(x)\,dx \leqslant \operatorname*{Llt}_{n\to\infty} \int_S f_n(x)dx \leqslant \int_S u(x)dx,$$

where the integration is taken over any set of points S. If either of the

* *Semi-Integrals*, § 9, p. 297.

† See my paper "On Homogeneous Oscillation of Successions of Functions," § 4, 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 357.

functions $l(x)$ and $u(x)$ is not summable over the set S, we can still maintain the inequality, as it may be shewn that in this case

$$\int_S l(x)\,dx = -\infty, \quad \text{or} \quad \int_S u(x)\,dx = +\infty.$$

Finally when (δ) holds, it holds also when we replace $f_n(x)$ by its absolute value, so that all the above properties hold equally for $f_n(x)$ and $|f_n(x)|$.

After what has already been said, the only statements, beside the final one, that require remark are that with regard to the existence of sequences of the integrals, for which the reader may refer to my paper on "Homogeneous Oscillation," § 4, and the assertion that, if $l(x)$, or $u(x)$, is not summable, its integral may be said to be infinite with determinate sign.

To prove the final statement in the corollary, we divide the set E up into the two sets E_n and E'_n, the former consisting of all the points of E at which $f_n(x)$ is $\geqslant 0$, and the latter of the remaining points of E. Thus E_n and E'_n are both $\leqslant E$, and therefore vanish with E, however n may vary. But

$$\int_E |f_n(x)|\,dx = \int_{E_n} f_n(x)\,dx - \int_{E'_n} f_n(x)\,dx.$$

As E approaches zero, and n approaches infinity in any manner, each of the integrals on the right approaches zero, since E_n and E'_n do so. Hence $\int_E |f_n(x)|\,dx$ has, like $\int_E f_n(x)\,dx$, the unique double limit zero.

This proves the final statement in the corollary. To prove the truth of the assertion that if $l(x)$ is not summable over the set S, it may be said to have $-\infty$ for integral, we remark that, given e, we can determine E_0 and n, so that for all values of $E < E_0$ and of $n > n_0$,

$$-e \leqslant \int_E f_n(x)\,dx \leqslant e.$$

Thus, if k is such an integer that $kE_0 > b-a \geqslant S$, where S is the content of any set in the interval (a, b),

$$-ke < \int_S f_n(x)\,dx < ke$$

for all values of $n > n_0$.

But if at every point where $l(x) < -A$, we replace the value of each $f_n(x)$ by $-A$, we have a succession of functions bounded below, and whose integrals, by the last inequality, nowhere diverge properly. Hence the lower function of this succession is summable, and the succession is abso-

lutely semi-integrable below.* But the lower function of this succession is the same as before, except at the set of points at which we altered the values of the functions $f_n(x)$. Integrating therefore over the set of points G at which $l(x) \geqslant -A$,

$$\int_G l(x)\,dx \leqslant \operatorname*{Llt}_{n\to\infty} \int_G f_n(x)\,dx < ke.$$

This shews that, either, when $A \to \infty$, the integral on the left has $-\infty$ as unique limit, or else $l(x)$ is summable, and its integral is less than ke. Hence the statement made above as to $l(x)$, and similarly that with regard to $u(x)$ follows.

It may be noticed that from the preceding corollary we have at once the following result :—

If $\int_E f_n(x)\,dx$ *has the unique double limit zero, and the functions* $f_n(x)$ *have an unique limiting function,*† *then the limiting function is summable, and the sequence of functions is absolutely integrable term-by-term.*

An exceptional set of points of content zero at which the functions have a plurality of limits, does not, of course, affect this result.

It should be remarked that the condition considered in the present article corresponds so closely to the condition for a summable function $f(x)$, that $\int_E f(x)\,dx$ should have the unique limit zero, that we are not surprised to find, in the sequel, successions of the type here considered taking the place of summable functions in a variety of theorems. This suggests also that, when the interval of integration is infinite, we must include in this condition the stipulation that $\int_X^\infty |f_n(x)|\,dx$ should have the unique double limit zero, when $X \to \infty$, and $n \to \infty$, since, when $f(x)$ is summable over the infinite interval, $\int_X^\infty |f(x)|\,dx$ must have the unique limit zero, when $X \to \infty$; in other words, the condition that the double limits of $\int_E^\infty |f_n(x)|\,dx$ should be zero, must be understood to hold both when the content of E vanishes, and when the set itself recedes to an infinite distance.

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* *Semi-Integrals*, §§ 15 and 22.

† We do not here assume *a priori* that this limiting function is finite anywhere, still less that it is bounded anywhere.

The analogy here mooted between successions and functions will be found to suggest itself constantly in the present paper.

5. In the corollary of the preceding article, nothing is said as to the nature of the integral to which it is shewn that the subsequences of integrals converge. In two of the cases with which we shall be occupied later we can at once state something further.

Thus, *if* $f_n(x)$ *describes a bounded succession, the unique limiting integrals are the integrals of bounded functions*, for they lie between the integrals of the bounded limiting functions $l(x)$ and $u(x)$, whence the statement easily follows.

In the other important case in which we can deduce the uniqueness and vanishing of the double limit, namely, when

$$\int_a^x |f_n(x)|^{1+p}\,dx, \text{ for } p > 0,$$

describes a bounded succession, we have the additional fact that *the limits of sequences of the integrals* $\int_a^x f_n(x)\,dx$ *are the integrals of functions whose* $(1+p)$*-th powers are summable.* For convenience I repeat here F. Riesz's proof of this result, which is due to him.* The proof is of the same nature as that of the final statement of the theorem of § 4, and with the same notation we have, by F. Riesz's condition† that a function should be the integral of a function whose $(1+p)$-th power is summable,

$$\Sigma \frac{|F_n(x+h)-F_n(x)|^{1+p}}{h^p} \leqslant A,$$

where A is a constant. Hence proceeding to the limit

$$\Sigma \frac{|F(x+h)-F(x)|^{1+p}}{h^p} \leqslant A,$$

which shews that $F(x)$ like $F_n(x)$ is the integral of a function whose $(1+p)$-th power is summable.

6. We now proceed to the consideration of certain circumstances under which the double limit, or, more generally, the triple limit in question is unique and has the value zero.

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* F. Riesz, "Untersuchungen über Systeme integrierbarer Funktionen," 1910, § 7, *Math. Ann.*, LXIX, pp. 449–496.

† *Ibid.*, § 5.

THEOREM.—*The triple limit*

$$\operatorname*{Lt}_{(E\to 0,\ m\to\infty,\ n\to\infty)} \left\{ \int_E f_m(x)\, g_n(x)\, dx \right\} = 0$$

in each of the two following cases :—

(i) *The succession of functions* $g_n(x)$ *is any bounded succession, and the succession of functions* $f_m(x)$ *is such that each function is summable and the double limit*

$$\operatorname*{Lt}_{(E\to 0,\ m\to\infty)} \int_E f_m(x)\, dx = 0.$$

(ii) *Each function* $f_m(x)$ *is such that its* $(1+p)$*-th power is summable, and each function* $g_n(x)$ *is such that its* $(1+1/p)$*-th power is summable, and, while the double limit*

$$\operatorname*{Lt}_{(E\to 0,\ m\to\infty)} \int_E |f_m(x)|^{1+p}\, dx = 0,$$

we have

$$\int_a^b |g_n(x)|^{1+1/p}\, dx < B,$$

where B *is a finite constant independent of* n.

Indeed in case (i), the quantity whose triple limit is considered is numerically

$$\leqslant G \int_E |f_m(x)|\, dx,$$

where G is the upper bound of $|g_m(x)|$ for all values of m and x. This has therefore the unique triple limit zero, since by hypothesis $\int_E f_m(x)\, dx$ has the unique double limit zero.

In case (ii) we use the Riesz-Schwarz inequality*

$$\left| \int_E f_m(x)\, g_n(x)\, dx \right| \leqslant \left\{ \int_E |f_m(x)|^{1+p}\, dx \right\}^{1/(1+p)} \left\{ \int_E |g_n(x)|^{1+1/p}\, dx \right\}^{1/(1+1/p)}.$$

The first factor, as before, vanishes, and the second remains finite, so that the left-hand side has the unique triple limit zero.

This proves the theorem.

COR.—*The double limit*

$$\operatorname*{Lt}_{(E\to 0,\ n\to\infty)} \int_E f_n(x)\, g(x)\, dx = 0, \qquad (\epsilon)$$

in the following cases :—

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* F. Riesz, *loc. cit.*, § 2, p. 456.

(i) *$g(x)$ is any bounded function, the functions $f_n(x)$ are any summable functions whatever, such that the double limit*

$$\operatorname*{Lt}_{(E\to 0,\ n\to\infty)} \int_E f_n(x)\,dx = 0. \qquad (\delta)$$

(ii) *$g(x)$ is any summable function, while the succession of functions $f_n(x)$ is a bounded one.*

(iii) *$\int_a^b |f_n(x)|^{1+p}\,dx$ exists and is a bounded function of n, while $g(x)$ has its $(1+1/p)$-th power summable.*

The following special cases of the theorem are of particular simplicity :—

(i*a*) *$f_n(x)$ and $g_m(x)$ both describe bounded successions.*

(i*b*) *$g_m(x)$ describes a bounded succession, the functions $f_n(x)$ are positive,* and converge to $f(x)$, while $\int_a^b f_n(x)\,dx$ converges to $\int_a^b f(x)\,dx$.*

(ii*a*) *$\int_a^b |f_n(x)|^{1+q}\,dx$ and $\int_a^b |g_m(x)|^{1+1/p}\,dx$ are bounded, where*

$$0 < p < q.$$

In case (ii*a*), in fact, by the Riesz-Schwarz inequality,

$$\int_E |f_n(x)|^{1+p}\,dx \leqslant E^{(q-p)/(1+q)} \left\{ \int_a^b |f_n(x)|^{1+q}\,dx \right\}^{(1+p)/(1+q)}.$$

7. In extending the results of the Expansions paper to the case when the succession of integrals does not converge, we are met by the difficulty that the difference of the lower functions of the successions

$$\int_a^{x+h} f_n(x)\,dx \quad \text{and} \quad \int_a^x f_n(x)\,dx$$

is not in general the lower function of the succession

$$\int_x^{x+h} f_n(x)\,dx.$$

That this may, however, occur, without the integrals describing a sequence, is evident. Indeed, if $l(x)$ is the lower function of the

* See below, § 9, Cor.

succession of integrands, the equations

$$\int_x^{x+h} l(x)\,dx = \text{lower } \lim_{n\to\infty} \int_x^{x+h} f_n(x)\,dx,$$

$$\int_x^{x+h} u(x)\,dx = \text{upper } \lim_{n\to\infty} \int_x^{x+h} f_n(x)\,dx,$$

may hold in the case of a non-converging succession of integrals, for every value of x and h, as we see from the following example. It may be noted that the succession of integrals, though non-convergent, is bounded.

8. *Ex.*—Let $f_1(x) = 1$ from $\frac{1}{2}$ to 1, and zero elsewhere,

$f_2(x) = 1$ from 0 to $\frac{1}{2}$, ,,

$f_3(x) = 1$ from $\frac{3}{4}$ to 1, ,,

$f_5(x) = 1$ from $\frac{1}{2}$ to $\frac{3}{4}$, ,,

$f_7(x) = 1$ from $\frac{1}{4}$ to $\frac{1}{2}$, ,,

$f_9(x) = 1$ from 0 to $\frac{1}{4}$, ,,

and so on for all the odd indices, the division of the segment (0, 1) into 2^n equal parts being effected stage by stage, and each function being unity in one part only and zero elsewhere.

Next let each even function be so chosen that it is zero where the preceding function is not zero, and is unity where the preceding function is zero. The lower function $l(x)$ is evidently 0, and the upper function 1 everywhere.

That the lower function of the succession of integrals $\int_x^{x+h} f_n(x)\,dx$ is zero, and the upper function is h may be seen as follows. Let 2^{-k} be the first negative integral power of 2 which is less than the greater of the intervals $(0, x)$ and $(x+h, 1)$. Then at the k-th and all subsequent stages the first or the last of the divisions into which the whole segment (0, 1) is divided will lie outside the interval $(x, x+h)$. Therefore one of the subsequent functions will be zero, and one will be unity, throughout the interval $(x, x+h)$, while the other functions will lie between these values. Thus the lower limit of the integral $\int_x^{x+h} f_n(x)\,dx$ is zero, and the upper limit is h, whence the required equalities follow.

9. This example is a special case of the following theorem :—

THEOREM.—*If* $\int_E f_n(x)\,dx$ *has no negative double limit, as* $E \to 0$ *and*

$n \to \infty$, and the lower function $l(x)$ of the f_n-succession is summable, and if

$$\int_a^b l(x)\,dx = \text{lower } \lim_{n\to\infty} \int_a^b f_n(x)\,dx,$$

then, for every set S in (a, b), and in particular for every interval $(x, x+h)$,

$$\int_S l(x)\,dx = \text{lower } \lim_{n\to\infty} \int_S f_n(x)\,dx.$$

For, by the conditions of the problem, the f_n-succession is absolutely semi-integrable below, therefore, if the required equation did not hold, we should have

$$\int_S l(x)\,dx < \text{lower } \lim_{n\to\infty} \int_S f_n(x)\,dx,$$

but, if T denote the complementary set of S, we have, for the same reason,

$$\int_T l(x)\,dx \leqslant \text{lower } \lim_{n\to\infty} \int_T f_n(x)\,dx.$$

Hence, adding,

$$\begin{aligned}\int_a^b l(x)\,dx &< \text{lower } \lim_{n\to\infty} \int_S f_n(x)\,dx + \text{lower limit} \int_T f_n(x)\,dx \\ &< \text{lower } \lim_{n\to\infty} \left(\int_S f_n(x)\,dx + \int_T f_n(x)\,dx\right) \\ &< \text{lower } \lim_{n\to\infty} \int_a^b f_n(x)\,dx,\end{aligned}$$

since this is not true, the required equation holds.

COR.—*If $\int_E f_n(x)\,dx$ has no negative (positive) double limit, and, as $n \to \infty$, $f_n(x)$ has an unique limit except at a set of content zero, and if the succession of integrals $\int_a^x f_n(x)\,dx$ is such that, for $x = b$, it converges to the integral of the limiting function, then the succession is absolutely integrable term-by-term, that is, the limit of the integrals is the integral of the limit, integration being over any interval or set of points in (a, b), and therefore $\int_E f_n(x)\,dx$ has the unique double limit zero.*

For, by the above,

$$\text{lower limit}_{n\to\infty} \int_S f_n(x)\,dx + \text{lower limit}_{n\to\infty} \int_T f_n(x)\,dx$$

$$= \operatorname{Lt}_{n\to\infty} \left\{ \int_a^b f_n(x)\,dx \right\} \geqslant \text{upper limit}_{n\to\infty} \int_S f_n(x)\,dx + \text{lower limit}_{n\to\infty} \int_T f_n(x)\,dx,$$

whence $\int_S f_n(x)\,dx$ must have an unique limit when $n \to \infty$.

The most important case of this corollary is the theorem given by Vitali, that if $f_n(x)$ is a positive (negative) function [or the succession $f_n(x)$ is bounded below (above)] and converges to $f(x)$, while its integral from a to b converges to the integral of $f(x)$, the sequence is absolutely integrable.*

It may be noticed that the argument of the preceding proof serves also to establish the following theorem :—

THEOREM.—*If it is known that for all points x and $x+h$ in the interval (a, b),*

$$\int_x^{x+h} q(x)\,dx \leqslant \text{lower limit} \int_x^{x+h} f_n(x)\,dx,$$

and that the sign of equality holds when the limits of integration are a and b, then the sign of equality holds always.

Moreover, if, in addition, $\int_E f_n(x)\,dx$ has the unique double limit zero, when $E \to 0$, and $n \to \infty$, then we have for all intervals or sets of points S in (a, b),

$$\int_S q(x)\,dx = \text{lower limit} \int_S f_n(x)\,dx.$$

The concluding statement is proved below (§ 10), in the form of an inequality ; the equality then follows by the above reasoning.

10. To see the nature of the difficulty that occurs in the general case, let us suppose that all the upper functions and all the lower functions of the succession of integrals are integrals. Consider, in particular, the upper and lower functions of the succession $\int_a^x f_n(x)\,dx$, and let us denote them by $L(x)$ and $U(x)$. Then we have

$$L(x+h) = \text{lower function of} \int_a^{x+h} f_n(x)\,dx,$$

* G. Vitali, "Sull' Integrazione per Serie," § 10, 1907, *Rend. di Palermo*, Vol. XXIII, p. 149.

and we have for all values of x,

$$\int_a^x L'(x)\,dx = L(x) = \text{lower } \lim_{n\to\infty} \int_a^x f_n(x)\,dx.$$

But from this equation all that we can prove is the inequality

$$\int_x^{x+h} L'(x)\,dx \geqslant \text{lower } \lim_{n\to\infty} \int_x^{x+h} f_n(x)\,dx.$$

More generally, if $q(x)$ is such a function that

$$\int_a^x q(x)\,dx \leqslant \text{lower } \lim_{n\to\infty} \int_a^x f_n(x)\,dx,$$

we can in the general case assert nothing about the relation when the limits of integration are x and $x+h$.

On reference, however, to the method adopted in proving the theorem of § 11, it will be seen that all turns on the legitimacy of splitting up the interval (a, b) into any number of intervals we please, in each of which the properties hypothecated hold. It is with the object of avoiding this difficulty that we make the assumption contained in equation (1) of that enunciation, an assumption which, as we have seen, may, for instance, be fulfilled when the inequality becomes an equality and $q(x)$ is $l(x)$, and which is, of course, fulfilled when the integrals form a sequence, and $q(x)$ is the integrand of the integral to which they then necessarily converge.

The following theorem states properties of such a function $q(x)$:—

THEOREM.—*If $q(x)$ is such that, for all points x and $x+h$ in the interval (a, b)*

$$\int_x^{x+h} q(x)\,dx \leqslant \text{lower. } \lim_{n\to\infty} \int_x^{x+h} f_n(x)\,dx, \qquad (1)$$

where $\int_E f_n(x)\,dx$ has the unique double limit zero, then, if S is any set of points in (a, b),

$$\int_S q(x)\,dx \leqslant \text{lower } \lim_{n\to\infty} \int_S f_n(x)\,dx, \qquad (2)$$

the sign of equality holding in (2) if it holds in (1).

Also, if the succession $f_n(x)$ is bounded above, $q(x)$ may be taken to be bounded above, and, if the sign of equality holds, and the succession $f_n(x)$ is a bounded one, $q(x)$ may be taken to be a bounded function.

Again, if $f_n(x)$ has, for all values of n, its $(1+p)$-th power summable, where $p > 0$, the function $q(x)$ is such that, over any set of points in

which $q(x)$ is bounded below, or is greater than a function whose $(1+p)$-th power is summable, its $(1+p)$-th power is summable.

For, if D denote any finite set of non-overlapping intervals, we have, by (1), since the sum of two lower limits is not greater than the lower limit of the corresponding sum,

$$\int_D q(x)dx \leqslant \text{lower } \lim_{n\to\infty} \int_D f_n(x)dx, \tag{3}$$

the sign of equality holding in (3) if it holds in (1).

Now let S be any closed set in the intervals D, and E the set of the remaining points of D; we know that the content E may be made as small as we please, by suitably choosing the intervals D relative to the set S. Our last inequality may now be written

$$\int_S q(x)dx+\int_E q(x)dx \leqslant \text{lower } \lim_{n\to\infty} \left(\int_S f_n(x)dx+\int_E f_n(x)dx\right).$$

Again, if S is any measurable set, we can find a closed set inside it, such that the complementary set E_1 has content as small as we please. We have therefore

$$\int_S q(x)dx+\int_E q(x)dx-\int_{E_1} q(x)dx$$
$$\leqslant \text{lower } \lim_{n\to\infty} \left(\int_S f_n(x)dx+\int_E f_n(x)dx-\int_{E_1} f_n(x)dx\right).$$

Since $q(x)$ is summable, the second and third integrals on the left vanish, and by hypothesis the corresponding integrals on the right vanish, when E and E_1 vanish, whence (2) follows.

Now, if B denote the upper bound of the f_n-succession, we have, from (2),

$$\int_S q(x)dx \leqslant SB.$$

Thus, if S be the content of the set of points at which $q(x)$ is greater than $B+e$, where e is as small as we please, the content S must be zero. In other words, $q(x)$ may be taken to lie below the upper bound B of the f_n-succession.

Again, if the sign of equality holds in (1), and therefore in (2), and $f_n(x)$ lies for all values of n between A and B, we have

$$\int_S q(x)dx = \text{lower } \lim_{n\to\infty} \int_S f_n(x)dx \geqslant AS,$$

whence the set of points at which $q(x) \leqslant A-e$, where e is as small as we please, must have zero content.

This proves the statements when $f_n(x)$ is a bounded function of (n, x). Next let $f_n(x)$ have its $(1+p)$-th power summable, and the required double limit be zero, and let us integrate only over such sets of points that $q(x)$ is positive. It is the same thing to put all the functions zero except at those points where $q(x) \geqslant 0$. Let us now pick out such a set of the n's that the integrals converge, and therefore converge to the integral $F(x)$ of a function whose $(1+p)$-th power is summable. Denoting the integral of $q(x)$ by $Q(x)$, we have

$$Q(x+h)-Q(x) \leqslant F(x+h)-F(x).$$

But since $q(x)$ is positive, $Q(x+h)-Q(x)$ is positive, therefore we have also

$$\Sigma \frac{|Q(x+h)-Q(x)|^{1+p}}{h^p} \leqslant \frac{|F(x+h)-F(x)|^{1+p}}{h^p}.$$

This, by Riesz's condition, already quoted, proves that $Q(x)$ is the integral of a function whose $(1+p)$-th power is summable.

Now, if $q(x) \geqslant k(x)$ over the set S, where the $(1+p)$-th power of $k(x)$ is summable, the subtraction of $k(x)$ from all the functions makes $q(x)-k(x) \geqslant 0$, and changes the succession into another succession, whose generic functions $f_n(x)-k(x)$ have the same property as before, namely, that their $(1+p)$-th powers are summable, since*

$$\left\{\int |f_n(x)-k(x)|^{1+p}\,dx\right\}^{1/(1+p)} \leqslant \left\{\int |f_n(x)|^{1+p}\,dx\right\}^{1/(1+p)} + \left\{\int |k(x)|^{1+p}\,dx\right\}^{1/(1+p)}.$$

Hence the above shews that $q(x)-k(x)$ has its $(1+p)$-th power summable over the set S, and therefore, by application of another inequality like that just used, $q(x)$ has its $(1+p)$-th power summable over the set S.

Thus all the statements in the theorem have now been proved.

II. *On Upper and Lower Limits of Single Successions of Integrals.* (§§ 11–17.)

11. The successions of integrals to be considered in the present section are such that the integrands consist of the product of the generic function of a certain succession by a certain fixed function. We have now

* F. Riesz, *loc. cit.*, § 2, p. 456.

to apply the remarks made in Section I to obtain inequalities corresponding to, and including, the equalities of the Expansions paper. I take occasion at the same time to remove certain limitations from the statements of that paper.

THEOREM.—*If $f_n(x)$ describes a succession of summable functions, and $q(x)$ is such a function that, for all points x and $x+h$ in the interval (a, b),*

$$\int_x^{x+h} q(x)\,dx \leqslant \text{lower limit}_{n\to\infty} \int_x^{x+h} f_n(x)\,dx, \tag{1}$$

then

$$\int_a^b g(x)\,q(x)\,dx \leqslant \text{lower limit}_{n\to\infty} \int_a^b g(x)\,f_n(x)\,dx, \tag{2}$$

where $g(x)$ is a positive function and the sign of equality is supposed to be taken in (2) when it is taken in (1), provided one of the following sets of conditions holds good :—

(i) *$g(x)$ is bounded, and the succession of functions $f_n(x)$ is such that $\int_E f_n(x)dx$ has the unique double limit zero, when $E\to 0$ and $n\to\infty$.*

(ii) *The f_n-succession is bounded, while $g(x)$ denotes any summable function.*

(iii) *$g(x)^{1+1/p}$ and $f_n(x)^{1+p}$ are summable, and the succession of integrals $\int_a^b |f_n(x)|^{1+p}\,dx$ is bounded.*

These statements are still true when b is infinite, provided in case (ii) *the condition that $\int_E f_n(x)dx$ should have the unique double limit zero be understood to include the condition that $\int_X^\infty |f_n(x)|\,dx$ should have the unique double limit zero, when $n\to\infty$ and $X\to\infty$.*

It appears from § 6 that in all three cases

$$\int_E f_n(x)\,dx \quad \text{and} \quad \int_E f_n(x)\,g(x)\,dx$$

each have the unique double limit zero. Hence, by § 10, we have, by (1),

$$\int_S q(x)\,dx \leqslant \text{lower limit}_{n\to\infty} \int_S f_n(x)\,dx,$$

where S denotes any measurable set of points in the finite interval (a, b). This shows that (2) holds when $g(x)$ is unity at each point of any measurable set of points, and zero elsewhere.

Now, since the sum of two lower limits is not greater than the lower limit of the sum, we see from the form of (2) that, if it holds when $g(x)$ is $g_1(x)$, $g_2(x)$, ..., $g_n(x)$, it holds when $g(x)$ is the sum of these, multiplied by any positive constants. Thus, (2) holds when $g(x)$ is the sum of a finite number of functions each of which has a constant positive value at the points of a corresponding closed set, and zero elsewhere, except the last, which is zero at the points of all those closed sets and has a constant value in the remaining intervals.

But, if $g(x)$ be any bounded function, we can find a function $g_r(x)$ of the type just considered, differing from $g(x)$ at every point by less than $1/r$. Hence

$$\int_a^b g(x)\,q(x)\,dx - \int_a^b \{g(x)-g_r(x)\}\,q(x)\,dx$$

$$\leqslant \text{lower limit}_{n\to\infty} \left\{\int_a^b g(x)\,f_n(x)\,dx - \int_a^b [g(x)-g_r(x)]\,f_n(x)\,dx\right\};$$

and therefore *a fortiori*,

$$\int_a^b g(x)q(x)\,dx - \frac{1}{r}\int_a^b |\,q(x)|\,dx$$

$$\leqslant \text{lower limit}_{n\to\infty} \left\{\int_a^b g(x)\,f_n(x)\,dx + \frac{1}{r}\int_a^b |f_n(x)|\,dx\right\}.$$

Now, as pointed out in § 4, $\int_a^b |f_n(x)|\,dx$ is a bounded function of n, since $\int_E |f_n(x)|\,dx$, like $\int_E f_n(x)\,dx$, has the unique double limit zero. Hence proceeding to the limit $r\to\infty$, we get (2), when $g(x)$ is any bounded function. Thus case (i) is proved.

Again, whatever positive summable function $g(x)$ may be, we have only to define a bounded function $g_r(x)$ to be equal to $g(x)$, except where $g(x) > r$, and to be zero at these points, and, since (2) holds for $g_r(x)$, we have

$$\int_a^b g(x)\,q(x)\,dx - \int_a^b [g(x)-g_r(x)]\,q(x)\,dx$$

$$\leqslant \text{lower limit}_{n\to\infty} \left\{\int_a^b g(x)\,f_n(x)\,dx - \int_a^b [g(x)-g_r(x)]\,f_n(x)\,dx\right\},$$

which may be written

$$\int_a^b g(x)q(x)\,dx - \int_E g(x)\,q(x)\,dx$$

$$\leqslant \text{lower limit}_{n\to\infty} \left\{ \int_a^b g(x)\,f_n(x)\,dx - \int_E g(x)\,f_n(x)\,dx \right\},$$

where E denotes the set of points at which $g(x)$ is greater than r, so that, since $g(x)$ is a summable function, the content of E decreases without limit, as r increases without limit. Hence, since we must, of course, suppose $g(x)q(x)$ to be summable,

$$\text{Lt}_{r\to\infty} \int_E g(x)\,q(x)\,dx = 0;$$

also, by § 6,

$$\int_E g(x)f_n(x)\,dx$$

has, as r and n each approach infinity, the unique double limit zero. Hence, proceeding to the limit $r\to\infty$ in the inequality last given, we get (2). We notice that, throughout the argument, the sign of equality holds, if it holds in (1).

This proves the inequality or equality (2) in all cases (ii) and (iii), and so completes the proof of the theorem when the interval (a, b) is finite.

It should be remarked that, *in all the cases when the sign of equality holds in* (1), *and therefore in* (2), *the oscillation is uniform below; that is,*

$$\int_c^z g(x)\,q(x)\,dx = \text{lower double limit}_{(h\to 0,\ n\to\infty)} \int_c^{z+h} g(x)\,f_n(x)\,dx,$$

since, by § 6, $\int_z^{z+h} g(x)\,f_n(x)\,dx$ has the unique double limit zero.

In particular, *when the integrals converge, we have here uniform convergence.*

12. In order to extend the result to the case of an infinite interval, it will be sufficient to show that

$$\text{Lt}_{X\to\infty} \left(\text{lower limit}_{n\to\infty} \int_a^X g(x)\,f_n(x)\,dx \right) = \text{lower limit}_{n\to\infty} \int_a^\infty g(x)\,f_n(x)\,dx. \qquad (\xi)$$

We have evidently

$$\text{lower limit}_{n\to\infty}\int_a^X g(x)f_n(x)\,dx+\text{lower limit}_{n\to\infty}\int_X^\infty g(x)f_n(x)\,dx$$

$$\leqslant \text{lower limit}_{n\to\infty}\int_a^\infty g(x)f_n(x)\,dx$$

$$\leqslant \text{lower limit}_{n\to\infty}\int_a^X g(x)\,f_n(x)\,dx+\text{upper limit}_{n\to\infty}\int_X^\infty g(x)f_n(x)\,dx. \qquad (\eta)$$

Now, in case (ii), $\left|\int_X^\infty g(x)\,f_n(x)\,dx\right| \leqslant B\int_X^\infty g(x)\,dx$;

and therefore has, as $X\to\infty$ and $n\to\infty$, the unique double limit zero. Hence, proceeding in (η) to let $X\to\infty$, the repeated limits of $\int_X^\infty g(x)\,f_n(x)\,dx$ vanish, and we are left with the desired equality (ξ).

In case (iii),

$$\left|\int_X^\infty g(x)\,f_n(x)\,dx\right| \leqslant \left(\int_X^\infty |f_n(x)|^{1+p}\,dx\right)^{1/(1+p)}\left(\int_X^\infty |g(x)|^{1+1/p}\,dx\right)^{1/(1+1/p)}$$

$$\leqslant K\left(\int_X^\infty |g(x)|^{1+1/p}\,dx\right)^{1/(1+1/p)};$$

and therefore the same result follows.

In case (i), we have, by the gloss on the condition that $\int_E f_n(x)\,dx$ should have the unique double limit zero,

$$\underset{(X\to\infty,\ n\to\infty)}{\text{Lt}}\int_X^\infty |f_n(x)|\,dx = 0.$$

Thus, since $\left|\int_X^\infty g(x)\,f_n(x)\,dx\right| \leqslant K\int_X^\infty |f_n(x)|\,dx,$

the integral on the left has the unique double limit zero when $X\to\infty$ and $n\to\infty$, whence, as in the other cases, (η) leads to the required result (ξ).

This proves the extension of the theorem of § 11 to an infinite interval.

COR.—*The theorem holds if $g(x)$ is any bounded function, and the functions $f_n(x)$ converge to $f(x)$, provided* this sequence is bounded either*

* See above, § 7, Cor.

above or below $\left(or,\ more\ generally,\ provided\ \int_E f_n(x)\,dx\ has\ not\ both\ positive\ and\ negative\ double\ limits\right)$, *and* $\int_a^b f_n(x)\,dx$ *converges to* $\int_a^b f(x)\,dx$.

13. The theorem proved in §§ 11, 12 contains in itself a generalisation of Theorems 2, 3, and 4 of the Expansions paper. This is evident in the case of Theorems 2 and 3. As regards Theorem 4, the same is equally clear, if we bear in mind that the boundedness and integrability of the sequence $|f_n(x)|$ ensures the existence and vanishing of the usual double limit. It would be easy to give other special cases of the theorem just proved. I prefer, however, to pass at once to the generalisation of the first of the theorems of the Expansions paper, which, so far as I am aware, has not been considered, either directly or indirectly, by other writers. Instead of, as in that paper, supposing the function $f_n(x)$ to trace out a sequence, we may, without impairing the truth of the theorem, suppose it to be any non-converging succession of the type contemplated, whose integrals form a sequence. We shall, however, prove a theorem which includes even this generalisation as a particular case.

THEOREM.—*If* $g(x)$ *is any function possessing a Harnack-Lebesgue integral, and* $f_n(x)$ *is a monotone descending (ascending) function of* x *which describes a bounded succession such that the integrand of the integral, which is the lower or the upper limit of the succession of the integrals of the* f_n*-succession, is independent of the particular interval over which we integrate—viz.,*

$$\int_x^{x+h} L'(x)\,dx = \text{lower limit}_{n\to\infty} \int_x^{x+h} f_n(x)\,dx,$$

or

$$\int_x^{x+h} U'(x)\,dx = \text{lower limit}_{n\to\infty} \int_x^{x+h} f_n(x)\,dx,$$

then this equation remains true when we multiply the integrands by $g(x)$. *In particular, if the succession of integrals* $\int_a^x f_n(x)\,dx$ *converges, and* $F(x)$ *is the integral to which it necessarily converges uniformly,*

$$\int_x^{x+h} g(x)\,F'(x)\,dx = \text{Lt}_{n\to\infty} \int_x^{x+h} g(x)\,f_n(x)\,dx,$$

the convergence being also uniform.

We may evidently, without loss of generality, assume the functions $f_n(x)$ to be positive and monotone decreasing.

Using the theorem * relating to the uniformity of approach of a defining sequence of integrals to a Harnack integral, we assume any positive quantity e, and find a finite number of intervals d surrounding the closed set of the Harnack points, and such that if g_d denotes the function which is equal to g except in the intervals d, where it is zero, then

$$\left| \int_x^{x+h} (g - g_d)\, dx \right| \leqslant e,$$

for all points x, $x+h$ in (a, b).

Then, since $f_n(x)$ is monotone decreasing, we have, by the Second Theorem of the Mean,

$$\left| \int_x^{x+h} (g - g_d) f_n(x)\, dx \right| \leqslant e\,|f_n(x)| \leqslant e\,|f_n(a)| \leqslant eM,$$

where M is any quantity greater than the upper bound of $f_n(a)$. Similarly, since $L'(x)$ is † a monotone decreasing function,

$$\left| \int_x^{x+h} (g - g_d)\, L'(x)\, dx \right| \leqslant eQ,$$

where Q is any quantity greater than the upper bound of $|L'(a)$.

But since g_d has no Harnack points, it is summable. Hence, by the theorem of § 11,

$$\int_x^{x+h} g_d L'(x)\, dx = \text{lower } \lim_{n \to \infty} \int_x^{x+h} g_d f_n(x)\, dx$$

$$= \text{lower double } \lim_{(n \to \infty,\ k \to 0)} \int_x^{x+h+k} g_d(x) f_n(x)\, dx,$$

since the oscillation is uniform below.

Hence, by the above,

$$\int_x^{x+h} g(x) L'(x)\, dx - eQ_1 = \text{lower } \lim_{n \to \infty} \left(\int_x^{x+h} g(x) f_n(x)\, dx + eM_1 \right)$$

$$= \text{lower double } \lim_{(n \to \infty,\ k \to 0)} \left(\int_x^{x+h+k} g(x) f_n(x)\, dx + eM_1 \right),$$

where Q_1 and M_1 are numerically less than Q and M.

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* See Hobson's *Theory of Functions of a Real Variable*, p. 383.

† Proved below, § 14.

Since e is at our disposal, this shews that

$$\int_x^{x+h} g(x)L'(x)dx = \text{lower limit}_{n\to\infty} \int_x^{x+h} g(x)f_n(x)dx,$$

and that the oscillation is still uniform below.

Similarly the other equation involving the upper limit may be proved.

An alternative, but longer, proof may be given, after the pattern of the proof of the corresponding theorem for sequences in my paper "On a Theorem in the Harnack Integration of Series," § 2.* The alterations are so trivial that it is unnecessary to give the proof at length.†

The extension to an infinite interval follows precisely as in § 12, using the Second Theorem of the Mean instead of the First.

14. It remains to prove that the function $L'(x)$ of the preceding article is a monotone decreasing (increasing) function. We shall require the following Lemma :—

LEMMA.—*If the values of a function $f(x)$ are only assigned at all the points of an everywhere dense set, and satisfy the condition of monotony with respect to that set, the values at the remaining points may be so assigned that the function remains monotone everywhere.*

We have, in fact, at each of the remaining points P to attribute to the function, for example, a value equal to the upper bound of the assigned values to the left of P, if, as we may suppose, the function is monotone increasing. This upper bound, being not less than all the upper bounds to the left, the value at P is not less than the values at the new points to the left of P. Moreover, since it is not greater than the assigned values to the right of P, and therefore not greater than the upper bounds of those values, the function is monotone increasing.

THEOREM.—*If for all values of n, $f_n(x)$ is a monotone decreasing (increasing) function, and for all points c and z in the interval (a, b), $\int_c^z f_n(x)dx$ has for lower limit $\int_c^z L'(x)dx$, the integrals being Lebesgue*

* *Messenger of Mathematics*, November, 1910, pp. 101–106.

† I take this opportunity of correcting a misprint in lines 19–23 on p. 102. The sentence "we can . . . determine p and q so that, if d_p denotes any interval less than (a, p), with a as left-hand end-point and p as right-hand end-point," *should read* "we can . . . determine P and Q so that, if d_p denotes any interval less than (a, P)," &c.

integrals, then $L'(x)$ may be taken to be a monotone decreasing (increasing) function.

For, if $f_n(x)$ is monotone decreasing

$$\int_x^{x+h} f_n(x)\,dx \geqslant \int_y^{y+h} f_n(x)\,dx \quad (x < y).$$

Therefore the lower limit of the left-hand side is not less than the lower limit of the right-hand side; that is

$$\int_x^{x+h} L'(x)\,dx \geqslant \int_y^{y+h} L'(x)\,dx.$$

Dividing both sides by h, and letting h approach zero, we have, provided x and y do not belong to the exceptional set of content zero at which $L'(x)$ is not the differential coefficient of its integral,

$$L'(x) \geqslant L'(y).$$

Thus, except at a set of content zero, $L'(x)$ obeys the condition of monotony. Therefore, by the Lemma, we may so adjust the values of $L'(x)$ at the exceptional points that it is a monotone decreasing function of x. Since this does not alter its integral, the theorem is proved.

15. The extension of Theorem 5 of the Expansions paper will now be given. The proof involves no new difficulty; but, as it is short, I give it in full.

THEOREM.—*If* $\int_a^b |f_n(x)|\,dx$ *is less than a finite constant* U, *independent of* n, *and* $g(x)$ *is a positive bounded function having only discontinuities of the first kind, then if* $q(x)$ *is such that, for all points* x *and* $x+h$ *in the closed interval* (a, b),

$$\int_x^{x+h} q(x)\,dx \leqslant \text{lower } \lim_{n\to\infty} \int_x^{x+h} f_n(x)\,dx, \tag{1}$$

we have also

$$\int_c^z g(x)\,q(x)\,dx \leqslant \text{lower } \lim_{n\to\infty} \int_c^z g(x) f_n(x)\,dx, \tag{2}$$

c *and* z *being any points in the closed interval* (a, b). *Moreover, if the sign of equality holds in* (1), *it holds also in* (2), *and the lower oscillation is bounded.*

For such a function $g(x)$ is the limit of a succession of bounded functions $g_p(x)$, each of which is constant at all the internal points of a finite

number of abutting intervals filling up (a, b), and is equal to $g(x)$ at the points of division. We have, in fact,*

$$|g(x)-g_p(x)| < 1/p;$$

and therefore, by the First Theorem of the Mean,

$$\left| \int_c^z \{g(x)-g_p(x)\} f_n(x)\,dx \right| < U/p.$$

Hence

$$\text{lower } \lim_{n\to\infty} \int_c^z g(x) f_n(x)\,dx = \operatorname*{Lt}_{p\to\infty} \left\{ \text{lower } \lim_{n\to\infty} \int_c^z g_p(x) f_n(x)\,dx \right\}$$

$$= \operatorname*{Lt}_{p\to\infty} \left\{ \text{lower } \lim_{n\to\infty} \Sigma M_i \int_{d_i} f_n(x)\,dx \right\},$$

where M_i is the value of $g_p(x)$ in the interval d_i, and the summation is over all the intervals d_i in (c, z).

Using (1), the last expression is

$$\geqslant \operatorname*{Lt}_{p\to\infty} \Sigma M_i \int_{d_i} q(x)\,dx \geqslant \operatorname*{Lt}_{p\to\infty} \int_c^z g_p(x) q(x)\,dx,$$

the sign of equality holding if it holds in (1). But the functions $g_p(x)$ form a bounded sequence, and $q(x)$ is summable; therefore $g_p(x)\,q(x)$ describes an integrable sequence, so that the limit last written down is $\int_c^z g(x)\,q(x)\,dx$. This proves (2) in the case of a finite interval (a, b).

The boundedness of the lower oscillation follows at once from the relation

$$\left| \int_z^{z+h} g(x) f_n(x)\,dx \right| \leqslant M \left| \int_z^{z+h} f_n(x)\,dx \right| \leqslant MU,$$

where M is the upper bound of $g(x)$.

The further conclusion that the theorem is true for an infinite interval follows as in § 12, b now being $+\infty$, provided we suppose $g(z)$ to have an unique limit at infinity. This supposition, which is clearly natural, then reduces to that in which we suppose this unique limit to be zero. It should be noted, as already proved in the Expansions paper, that, if the original succession of integrals converges uniformly, the new succession also converges uniformly.

* *Intégrales Singulières*, p. 59 *seq.*, Expansions paper, p. 473.

16. We now come to the extension of Theorem 6 of the Expansions paper.

We require to modify the enunciation in more than one particular.

THEOREM.—*If the succession of integrals* $\int_a^x f_n(x)dx$ *is bounded below, and* $g(x)$ *is any positive continuous monotone decreasing function of* x, *then* $q(x)$ *being such a function that*

$$\int_a^x q(x)dx \leqslant \text{lower } \lim_{n\to\infty} \int_a^x f_n(x)dx, \tag{1}$$

we have also

$$\int_a^x q(x)\,g(x)dx \leqslant \text{lower } \lim_{n\to\infty} \int_a^x f_n(x)\,g(x)dx, \tag{2}$$

the sign of equality holding in (2), *if it holds in* (1). *Moreover, in this latter case, the lower oscillation is bounded.* (2) *still follows from* (1) *for an infinite interval, provided only* $\int_a^x f_n(x)\,dx$ *is bounded in the whole infinite interval.*

For, as in the proof of the corresponding theorem (Theorem 6) in the Expansions paper, we take as new variable t the arc of the curve $y = g(x)$, and so get

$$\int_a^x g(x)\,f_n(x)dx = \int f_n[x(t)]\frac{dx}{dt}\,g[x(t)]\,dt$$

$$= g(x)F_n(x) - \int F_n[x(t)]\frac{d}{dt}\,g[x(t)]\,dt.$$

But since $F_n[x(t)]$, regarded as a function of (n, t) is bounded below, and $\left(-\frac{d}{dt}\,g[x(t)]\right)$ is positive and summable, their product is semi-integrable below,* and has for lower function $\left(-\frac{d}{dt}g[x(t)]\right)$ times the lower function of $F_n(x, t)$, that is a function which is

$$\geqslant Q(x)\left(-\frac{d}{dt}\,g[x(t)]\right),$$

where $Q(x)$ is $\int_a^x q(x)\,dx.$

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* See *Semi-Integrals*, § 27, p. 312.

Taking the lower limit of both sides of the preceding equation, and using the fact that the lower limit of a sum is not less than the sum of the lower limits, we have, therefore,

$$\underset{n\to\infty}{\text{lower limit}} \int_a^x g(x) f_n(x)\,dx \geqslant g(x)\,Q(x) + \int Q[x(t)] \left\{ -\frac{d}{dt} g[x(t)] \right\} dt$$

$$\geqslant \int_a^x g(x)\,q(x)\,dx,$$

which proves (2), the sign of equality holding when it does so in (1).

The statement about the lower oscillation is at once seen to be true, using the Second Theorem of the Mean, since

$$\left| \int_x^{x+h} g(x) f_n(x)\,dx \right| = g(x) \left| \int_x^{x+k} f_n(x)\,dx \right|;$$

and is therefore bounded.

The extension to an infinite interval follows as in § 13, using the additional condition and supposing (1) to hold for the infinite interval. We write $g(x) = A + h(x)$ if the limit of $g(x)$ is A at infinity. If A be zero we may dispense with (1), except for finite intervals. If the old integrals converge uniformly, the uniformity of the convergence of the new integrals follows of course. (See Expansions paper.)

17. It should be remarked that the function $q(x)$ of the preceding article need not be a summable function; it is sufficient that it should possess a Harnack-Lebesgue integral, since the theorem of integration by parts holds when one of the functions, here $Q[x(t)]$, is a Harnack-Lebesgue integral and the other, here $g[x(t)]$, is a Lebesgue integral.*

This remark applied equally to Theorem 6 of the Expansions paper, for the proof, in so far as it relates to a continuous function of bounded variation, is the same as the preceding, and, in so far as it relates to a discontinuous function of bounded variation, it depends on the Second Theorem of the Mean. The enunciation of the theorem with this extension is as follows:—

If the sequence of Lebesgue integrals $\int s_n(x)\,dx$ *converges boundedly to the Harnack-Lebesgue integral* $\int f(x)\,dx$, *and* $g(x)$ *is any function of*

* For a proof of this theorem see my paper "On the Conditions that a Trigonometrical Series should have the Fourier Form," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 432.

bounded variation in the interval considered, then

$$\operatorname*{Lt}_{n\to\infty} \int s_n(x)g(x)dx = \int f(x)g(x)dx.$$

Further, the theorem remains true when one or both of the limits of integration are infinite, provided only that either (i) $\operatorname*{Lt}_{n\to\infty} \int s_n(x)dx$ *is continuous at the point, or points, infinity, i.e., provided* $\int f(x)dx$ *exists, and is equal to the sum of the limits of the separate terms in the expansion of* $\int f(x)dx$; *or* (ii) $g(x)$ *converges to zero as* x *increases. Moreover the convergence of* $\int s_n(x)g(x)dx$ *is necessarily bounded, and it is uniform, if that of* $\int s_n(x)dx$ *is uniform.*

III. *On Integrals of Double Successions.* (§§ 18–24.)

18. We now pass to a different class of theorems—namely, to the consideration of successions of integrals, whose integrands involve two varying quantities m and n. These quantities may be supposed either to have a countably infinite number of values or to vary continuously. It follows, however, from the theory of limits that all cases may be reduced to that in which m and n have only a countably infinite number of values, and we may, without loss of generality, suppose these to be the integers in their natural order. It is equally open to us to consider repeated limits, or double limits, of these successions. While, however, the former step is found to involve no fresh difficulty, we need further help in the discussion of the double limits. For this purpose we have to make use of the conditions for the uniqueness and vanishing of the triple limit of § 6, or some equivalent process. It will be seen that we also make use of the theorem given by Borel and myself, and that usually the reasoning adopted requires us to suppose one of the two successions whose products form the integrands to converge, except at a set of content zero. Each of our theorems requires the use of two subsidiary theorems, in a certain sense dual to one another, and given* in Section II. It has been the necessity of having two such theorems that has led to certain extensions there given of previous enunciations for single successions of integrals. This is the case with those results involving Harnack-Lebesgue integration.

* The statement is for our present purpose, of course, unnecessarily general in Section II. We shall only require those cases in which the successions of integrals converge.

19. THEOREM.—*If $g_m(x)$ describes a bounded succession and $f_n(x)$ is summable and describes such a succession that*

$$\underset{(E\to 0,\ n\to\infty)}{\mathrm{Lt}} \int_E f_n(x)\,dx = 0, \tag{1}$$

*and if one of the successions converges * while the integrals of the other converge, then, with the usual glosses,† equally whether z be finite or infinite,*

$$\underset{(m\to\infty,\ n\to\infty)}{\mathrm{Lt}} \int_c^z f_n(x)\,g_m(x)\,dx = \int_c^z f(x)\,g(x)\,dx, \tag{2}$$

where $\int_a^x f(x)\,dx$ and $\int_a^x g(x)\,dx$ are the integrals to which the integrals of $f_n(x)$ and $g_m(x)$ necessarily converge. Moreover the convergence in (2) *is uniform.*

To prove this we use the identity

$$f_n(x)\,g_m(x) - f(x)\,g(x)$$
$$= [f_n(x)-f(x)][g_m(x)-g(x)] + f(x)[g_m(x)-g(x)] + g(x)[f_n(x)-f(x)].$$

By the theorem of § 11 the second and third terms on the right, when integrated, converge to zero. Thus we need only occupy ourselves with the first term. Now since $\int_E f_n(x)\,dx$ has the unique double limit zero, and $f(x)$ is summable, $\int_E [f_n(x)-f(x)]\,dx$ has the unique limit zero. Also $g_m(x)$ being bounded, $g(x)$ is also bounded, and therefore $[g_m(x)-g(x)]$ is bounded. Hence, by § 6, the triple limit

$$\underset{(E\to 0,\ m\to\infty,\ n\to\infty)}{\mathrm{Lt}} \int_E [f_n(x)-f(x)][g_m(x)-g(x)]\,dx = 0.$$

That one of the two given successions which converges is, moreover, such that the corresponding double limit is zero and the limiting function summable; hence it is absolutely integrable term-by-term, so that $f(x)$ or $g(x)$, as the case may be, may be taken to be the limiting function.

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* An exceptional set of content zero at which the convergence does not take place does not, of course, affect the result. This remark must be understood to apply wherever in the course of the paper the expression "convergence" in relation to the integrands occurs.

† See § 4.

Again, since $\int_E [f_n(x)-f(x)]\,dx$ and $\int_E [g_m(x)-g(x)]\,dx$ have both the unique limit zero, $\int_a^b |f_n(x)-f(x)|\,dx$ and $\int_a^b |g_m(x)-g(x)|\,dx$ are both bounded, say both less than M.

Suppose now it is the g_m-succession which converges; then the set of points E which is such that at each point of it

$$|g_m(x)-g(x)| > e,$$

cannot be such that for all values of m the content E is greater than a positive fixed quantity c. For, if this were the case, there would be * a set of points of content $\geqslant c$ at each of which $|g_m-g|$ would, for a certain succession of values of m, be always greater than e. Since this is not true, we have

$$\operatorname*{Lt}_{m\to\infty} E = 0.$$

Similarly, if it is the function $f_n(x)$ which converges, and E now denote the set of points at which

$$|f_n(x)-f(x)| > e,$$

we get

$$\operatorname*{Lt}_{n\to\infty} E = 0.$$

Thus, in either case, E has the unique double limit zero when m and n increase indefinitely in any manner. Therefore, the triple limit having become a double limit,

$$\operatorname*{Lt}_{(m\to\infty,\, n\to\infty)} \int_E [f_n(x)-f(x)][g_m(x)-g(x)]\,dx = 0.$$

Since throughout the complementary set of E one of the factors of the integrand is numerically $\leqslant e$, and the integral of the absolute value of the other factor is less than M, the integral of $[f_n(x)-f(x)][g_m(x)-g(x)]$ over the complementary set is numerically less than eM. Thus,

$$\operatorname*{Lt}_{(m\to\infty,\, n\to\infty)} \left| \int_a^b [f_n(x)-f(x)][g_m(x)-g(x)]\,dx \right| < eM.$$

Since e may be as small as we please, the double limit on the left here is zero. Integrating the identity first written down, and taking the double limit, the required result at once follows.

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* See **Young's** *Theory of Sets of Points*, **p.** 91. The enunciation was emended in Hobson's *Theory of* ***Functions*** *of a Real Variable*.

That the convergence is uniform follows from § 6, since the triple limit

$$\operatorname*{Lt}_{(h\to 0,\ m\to\infty,\ n\to\infty)} \int_z^{z+h} g_m(x) f_n(x)\, dx = 0.$$

If the interval be infinite we require to suppose that $\int_z^\infty |f_n(x)|\, dx$ exists, and has the unique double limit zero as $z \to \infty$ and $n \to \infty$. The extension then follows by the First Theorem of the Mean.

COR. 1.—*If $g_m(x)$ describes a bounded succession whose integrals converge, and $f_n(x)$ is summable and describes a sequence which is integrable term-by-term, then if the f_n-sequence is bounded either above or below, or, more generally, if $\int_E f_n(x)\, dx$ has not both positive and negative double limits, then, whether z be finite or infinite, with a suitable gloss in the latter case,**

$$\operatorname*{Lt}_{(m\to\infty,\ n\to\infty)} \int_c^z f_n(x)\, g_m(x)\, dx = \int_c^z f(x)\, g(x)\, dx,$$

$\int f(x)\, dx$ and $\int g(x)\, dx$ being the integrals to which, by the conditions of the problem, the integrals of the two successions converge. Moreover the convergence is uniform.

For in these cases, as already mentioned in § 9, $\int_E f_n(x)\, dx$ has the unique double limit zero.

COR. 2.—*In the enunciation of the theorem and of Cor.* 1 *we may omit the condition as to the convergence of one of the successions of integrands if we substitute repeated limits for the double limit.*

20. THEOREM. — *If for all values of the indices $f_n(x)$ is summable and $g_m(x)$ has bounded variation, and the successions of their integrals converge, the former by hypothesis to an integral $\int_a^x f(x)\, dx$, and the latter in virtue of the other conditions necessarily to an integral $\int_a^x g(x)\, dx$, then if the g_m succession is bounded and $g(x)$ has bounded*

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* See above. The hypothesis $f_n(x) = 1$ would not be allowable for an infinite interval of integration.

variation, and the convergence of the integrals of $f_n(x)$ is also bounded, the repeated limits

$$\operatorname*{Lt}_{m\to\infty}\ \operatorname*{Lt}_{n\to\infty}\int_c^z f_n(x)g_m(x)\,dx, \quad \operatorname*{Lt}_{n\to\infty}\ \operatorname*{Lt}_{m\to\infty}\int_c^z f_n(x)\,g_m(x)\,dx$$

are equal to one another, and to

$$\int_c^z f(x)\,g(x)\,dx.$$

Further, if the succession of the variations of the functions $g_m(x)$ is bounded and the convergence of the integrals of $f_n(x)$ is uniform, the double limit is unique, viz.,

$$\operatorname*{Lt}_{(m\to\infty,\,n\to\infty)}\int_c^z f_n(x)\,g_m(x)\,dx=\int_c^z f(x)\,g(x)\,dx.$$

Moreover, the convergence is uniform in this equality, and z may be infinite if we make the natural gloss that the uniformity of convergence of $\int f_n(x)\,dx$ is to hold for the closed infinite interval.

We now use the identity

$$f_n(x)g_m(x)-f(x)g(x)=g_m(x)[f_n(x)-f(x)]+f(x)[g_m(x)-g(x)].$$

As in the preceding proof, we are concerned only with the first term on the right; for $g_m(x)$ and $g(x)$ are bounded, so that $[g_m(x)-g(x)]$ describes a bounded succession, whence, by the theorem of § 11, the integral of the second term has the unique double limit zero.

But, if we let $m\to\infty$,

$$\int_c^z g_m(x)[f_n(x)-f(x)]\,dx \text{ approaches } \int_c^z g(x)[f_n(x)-f(x)]\,dx,$$

since $g_m(x)$ describes a bounded succession, and $[f_n(x)-f(x)]$ is a summable function; this, again, by Theorem 6 of the Expansions paper, approaches zero, when $n\to\infty$, since $g(x)$ has bounded variation, and $\int_c^z [f_n(x)-f(x)]\,dx$ converges boundedly to zero. Thus one of the repeated limits of $\int_c^z g_m(x)[f_n(x)-f(x)]\,dx$ is zero. That the other repeated limit is zero is evident, since the single limit $n\to\infty$ is, by the theorem last quoted, zero, $g_m(x)$ having, like $g(x)$, bounded variation.

Hence, by the identity first written down, the equality of the repeated limits with one another and with $\int_c^z f(x)\,g(x)\,dx$ immediately follows.

To prove the second statement of the enunciation we may write

$$g_m(x) = P_m(x) - N_m(x),$$

where P_m and N_m are monotone decreasing functions of x and bounded functions of (m, x). We have, by the Second Theorem of the Mean,

$$\int_c^z g_m(x)[f_n(x) - f(x)]\,dx$$
$$= P_m(c)\int_c^{c+k}[f_n(x) - f(x)]\,dx - N_m(c)\int_c^{c+q}[f_n(x) - f(x)]\,dx,$$

where k and q lie between 0 and $(z-c)$.

Now, as m and n increase indefinitely, k may have a plurality of limits, but if K be any one of these, K lies between 0 and $(z-c)$ both inclusive. Also since we know that the succession of integrals $\int_a^x f_n(x)\,dx$ converges uniformly to $\int_a^x f(x)\,dx$, it follows that $\int_a^{c+k} f_n(x)\,dx$ for the chosen succession of values of n converges to $\int_a^{c+K} f(x)\,dx$. The integral $\int_a^{c+k} f(x)\,dx$ evidently has the same limit, and $\int_a^c f_n(x)\,dx$ converges to $\int_a^c f(x)\,dx$. Thus the first term on the right of the preceding equation converges to zero, since P_m remains bounded. But this is true whatever succession of values of n and m be chosen so as to give a unique limit for k, and is therefore true always.

Similarly the second term has the unique double limit zero, and therefore the left-hand side has also the unique double limit zero. This, therefore, proves the theorem.

The uniformity of the convergence follows from the fact that, by the Second Theorem of the Mean, $\int_z^{z+h} f_n(x)\,g_m(x)\,dx$ has the unique triple limit zero.

The case in which the limit z of integration is infinite follows from the equation

$$\int_X^\infty P_m(x) f_n(x)\,dx = P_m(X)\int_X^{X_1} f_n(x)\,dx,$$

where P_m is measured from infinity.

21. In the theorem just proved we have not always made use of all our facts. If for convenience we replace the functions of bounded varia-

tion by monotone increasing functions, the theorem will still be true if $f(x)$, instead of being summable, possesses a Harnack-Lebesgue integral only. Instead of the theorems used we have now, in fact, only to employ the two theorems in the second section of the present paper in which Harnack-Lebesgue integrals appear, viz., in § 13 and in § 17. In the remaining part of the argument the Second Theorem of the Mean, and not the First, is the one employed, so that everything holds as before. We thus have the following theorem :—

THEOREM.—*If, for all values of the indices, $f_n(x)$ is summable and $g_m(x)$ is a monotone positive increasing function, and the successions of their integrals converge, the former to a Harnack-Lebesgue integral $\int_a^x f(x)dx$, and the latter, in consequence of the other conditions, to a Lebesgue integral $\int_a^x g(x)dx$, then, if the g_m-succession is bounded and the convergence of the integrals of $f_n(x)$ is also bounded, the repeated limits*

$$\operatorname*{Lt}_{m\to\infty}\ \operatorname*{Lt}_{n\to\infty}\int_c^z f_n(x)g_m(x)dx \quad \text{and} \quad \operatorname*{Lt}_{n\to\infty}\ \operatorname*{Lt}_{m\to\infty}\int_c^z f_n(x)\,g_m(x)\,dx$$

are equal to one another and to

$$\int_c^z f(x)\,g(x)\,dx.$$

Further, if the convergence of the integrals of $f_n(x)$ is uniform, the double limit is unique, viz.,

$$\operatorname*{Lt}_{(m\to\infty,\,n\to\infty)}\int_c^z f_n(x)g_m(x)dx = \int_c^z f(x)g(x)dx.$$

Moreover, the convergence is uniform in this equality, and the upper limit z may be infinite if the uniform convergence of the integrals of $f_n(x)$ relates to the whole closed infinite interval.

22. THEOREM.—*If the $(1+p)$-th power of $f_n(x)$, and the $(1+1/p)$-th power of $g_m(x)$, are, for all values of the index, summable, and while $\int_a^b |f_n(x)|^{1+p}dx$ is a bounded function of n, $\int_E |g_m(x)|^{1+1/p}dx$ has the unique double limit zero, when $E\to 0$ and $m\to\infty$, and if one of the two successions converges, while the integrals of the other converge, then*

*whether z is finite or infinite with, in the latter case, the usual glosses**

$$\operatorname*{Lt}_{(m\to\infty,\, n\to\infty)} \int_c^z f_n(x)\, g_m(x)\, dx = \int_c^z f(x)\, g(x)\, dx,$$

where $f(x)$ and $g(x)$ denote the integrands of the integrals to which, by the conditions of the problem, the integrals of $f_n(x)$ and $g_m(x)$ converge. Moreover, the convergence is uniform in this equality.

First we must see that the succession which is a sequence may be taken to converge to $f(x)$ or to $g(x)$, as the case may be. To be sure of this we have only to shew that it is summable. For, using the inequality of Riesz-Schwarz,

$$\int_E |f_n(x)|\, dx \leqslant E^{1/(1+1/p)} \left(\int_E |f_n(x)|^{1+p}\, dx\right)^{1/(1+p)}$$

$$\leqslant E^{1/(1+1/p)} \left(\int_a^b |f_n(x)|^{1+p}\, dx\right)^{1/(1+p)},$$

the hypotheses made ensure that $\int_E f_n(x)\, dx$, and, by similar reasoning, $\int_E g_m(x)\, dx$, have the unique double limit zero. Hence, if the unique limiting function in question is summable, the corresponding sequence is absolutely integrable term-by-term, and therefore has $f(x)$, or $g(x)$, as the case may be, for limiting function.

Now, by § 4, $\int_a^b |g_m(x)|^{1+1/p}\, dx$ is bounded, and $\int_a^b |f_n(x)|^{1+p}\, dx$ is so, by hypothesis.

Therefore the lower functions of the successions formed by these integrands are summable.† Since one of these is evidently the $(1+p)$-th or $(1+1/p)$-th power of the unique limiting function in question, this is summable, and, *a fortiori*, that function itself is summable, so that we may take it to be $f(x)$, or $g(x)$, as the case may be.

Now, by Riesz's result, already given,‡ whether $f(x)$ is or is not this unique limiting function, its $(1+p)$-th power is summable. Therefore,

* See § 4. Evidently in the enunciation b must now be infinity.

† *Semi-Integrals*, § 15.

‡ § 5 above.

using another of Riesz's inequalities,*

$$\left\{\int_a^x |f_n(x)-f(x)|^{1+p}\,dx\right\}^{1/(1+p)} \leqslant \left\{\int_a^x |f_n(x)|^{1+p}\,dx\right\}^{1/(1+p)} + \left\{\int_a^x |f(x)|^{1+p}\,dx\right\}^{1/(1+p)},$$

we see that $[f_n(x)-f(x)]$ has its $(1+p)$-th power summable, and that its integral from a to b is a bounded function of n.

Similarly $|\,g_m(x)-g(x)\,|$ has its $(1+1/p)$-th power summable, and its integral over a set E has the unique double limit zero when $E \to 0$ and $m \to \infty$.

Hence, by the theorem of § 6, the triple limit

$$\operatorname*{Lt}_{(E\to 0,\ m\to\infty,\ n\to\infty)} \int_E [f_n(x)-f(x)][g_m(x)-g(x)]\,dx = 0.$$

We now, as in the proof of the theorem of § 19, take E to be the set of points at which that one of the factors $|f_n-f|$ and $|g_m-g|$ which converges to zero, when the index increases indefinitely, is greater than e. Then we know that E has zero as unique limit when m and n approach infinity in any manner.

Hence the above triple limit becomes a double limit, and we have

$$\operatorname*{Lt}_{(m\to\infty,\ n\to\infty)} \int_E [f_n(x)-f(x)][g_m(x)-g(x)]\,dx = 0.$$

Over the complementary set one of the factors of the integrand is numerically $\leqslant e$, while the integral of the absolute value of the other factor is bounded. Therefore, as in § 19,

$$\operatorname*{Lt}_{(m\to\infty,\ n\to\infty)} \left|\int_x^{x+h} [f_n(x)-f(x)][g_m(x)-g(x)]\,dx\right| \leqslant eM.$$

Since e is as small as we like, this proves that the double limit on the left is zero.

Hence, using the same identity as in § 19, the required result follows. The uniformity of the convergence follows from § 6. The extension to an infinite interval is obvious. In fact, we have only to apply the Schwarz-Riesz inequality to $\int_z^\infty f_n(x)\,g_m(x)\,dx$ to shew that with the conditions of the problem the triple limit vanishes.

COR.—*If we are only concerned with the repeated limits, we need not*

* F. Riesz, *loc. cit.*, p. 456 (6).

assume that either succession converges, but only that the integrals of both successions converge. We have then

$$\operatorname*{Lt}_{n\to\infty,\ m\to\infty}\int_c^z f_n(x)\,g_m(x)\,dx = \operatorname*{Lt}_{m\to\infty,\ n\to\infty}\int_c^z f_n(x)\,g_m(x)\,dx = \int_c^z f(x)\,g(x).$$

The following special case of the preceding theorem is of interest. Here also we may omit the condition of convergency of the succession $g_m(x)$, if we replace the double limits by repeated limits.

THEOREM.—*If the* $(1+p)$*-th power of* $f_n(x)$ *and the* $(1+1/p)$*-th power of* $g_m(x)$ *are for all values of the indices summable, and the functions* $g_m(x)$ *converge to* $g(x)$*, and if the integrals* $\int_a^x f_n(x)\,dx$ *and* $\int_a^x g_m(x)\,dx$ *of both successions converge, and* $\int_a^b |f_n(x)|^{1+p}\,dx$ *and* $\int_a^b |g_m(x)|^{1+1/p}\,dx$ *are bounded functions of* n *and* m *respectively, then if the last mentioned integral converges to* $\int_a^b |g(x)|^{1+p}\,dx$*, and whether* z *is finite or infinite,*

$$\operatorname*{Lt}_{(m\to\infty,\ n\to\infty)}\int_c^z f_n(x)\,g_m(x)\,dx = \int_c^z f(x)\,g(x)\,dx,$$

where $f(x)$ *denotes the integrand of the integral to which, by the conditions of the problem, the integral of* $f_n(x)$ *converges. Moreover, the convergence is uniform in this equality.*

For, since the positive functions $|g_m(x)|^{1+1/p}$ converge to $|g(x)|^{1+1/p}$, we have for all points x,

$$\int_a^x |g(x)|^{1+1/p}\,dx \leqslant \operatorname*{Llt}_{m\to\infty}\int_a^x |g_m(x)|^{1+1/p}\,dx.$$

But, since this relation is an equality for $x = b$, it is by § 9 an equality always. Thus, the succession of positive functions $|g_m(x)|^{1+1/p}$ form a sequence which is integrable term-by-term, and therefore absolutely integrable * with a summable limiting function, so that $\int_E |g_m(x)|^{1+1/p}\,dx$ has the unique double limit zero. Thus these conditions lead to the conditions of the preceding theorem.

23. THEOREM.—*If* $\int_a^b |f_n(x)|\,dx$ *describes a bounded succession, and* $\int_a^x f_n(x)\,dx$ *converges to* $\int_a^x f(x)\,dx$*, and if the function* $g_m(x)$ *has only dis-*

* See above, § 9.

continuities of the first kind, and describes a bounded succession, whose integrals converge to the integral of a function $g(x)$ having only discontinuities of the first kind, then the repeated limits

$$\operatorname*{Lt}_{n\to\infty}\operatorname*{Lt}_{m\to\infty}\int_c^z f_n(x)\,g_m(x)\,dx = \operatorname*{Lt}_{m\to\infty}\operatorname*{Lt}_{n\to\infty}\int_c^z f_n(x)\,g_m(x)\,dx = \int_c^z f(x)\,g(x)\,dx.$$

Moreover, if $g_m(x)$ converges uniformly to $g(x)$, the double limit has the same value, viz., whether z is finite or infinite,

$$\operatorname*{Lt}_{(n\to\infty,\ m\to\infty)}\int_c^z f^n(x)\,g_m(x)\,dx = \int_c^z f(x)\,g(x)\,dx.$$

Finally the convergence is uniform in this equality, if that of $\int_a^x f_n(x)dx$ to $\int_a^x f(x)dx$ is so.

To prove the repeated limits equal we proceed as follows.

Since

$$f_n(x)\,g_m(x) - f(x)\,g(x) = f_n(x)\,[g_m(x)-g(x)] + g(x)\,[f_n(x)-f(x)],$$

and it has already been proved * that the integral of the last term on the right converges to zero ; we need only occupy ourselves with the first term.

Now, since $g_m(x)$ is a bounded function of (x, n), $g(x)$ is by § 5 a bounded function, and the limit $\int_x^{x+h} g(x)dx$ of the integrals lies between the integrals of $l(x)$ and $u(x)$. Hence $g_m(x)-g(x)$ is a bounded function of (m, x).

But $f_n(x)$ is a summable function ; therefore, by § 11,

$$\operatorname*{Lt}_{m\to\infty}\int_c^z f_n(x)\,[g_m(x)-g(x)]\,dx = 0.$$

Hence, letting $n\to\infty$, we see that the first of the repeated limits in question has the required value.

On the other hand, $g_m(x)$ and $g(x)$, and therefore also $[g_m(x)-g(x)]$. have only discontinuities of the first kind. Therefore, by the theorem first quoted in this proof,

$$\operatorname*{Lt}_{n\to\infty}\int_c^z f_n(x)\,[g_m(x)-g(x)]\,dx = \int_c^z f(x)\,[g_m(x)-g(x)]\,dx.$$

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* *Supra*, § 15.

Since $f(x)$ is summable, and $[g_m(x)-g(x)]$ is a bounded function of (x, n), we get, as in considering the first repeated limit, the unique limit zero when $n \to \infty$. Thus, the second repeated limit has the required value.

Next, assuming $g_m(x)$ to converge uniformly to $g(x)$, to prove that the double limit has the same value, we proceed as follows. It is clear that $g(x)$ has only discontinuities of the first kind, so that we have again only to shew that $\int_c^z f_n(x)[g_m(x)-g(x)]dx$ has the double limit zero.

But, since the g-sequence converges uniformly, we can find m_0, so that, for all values of $m > m_0$, and all values of x in (a, b),

$$|g_m(x)-g(x)| \leqslant e;$$

and therefore

$$\left| \int_c^z f_n(x)[g_m(x)-g(x)]dx \right| \leqslant e\int_c^z |f_n(x)|\,dx \leqslant e\int_a^b |f_n(x)|\,dx.$$

Since the last integral is a bounded function of n, and e may be as small as we please, this shews that the double limit of the left-hand side is zero, and so proves the theorem. It proves equally that the triple limit

$$\underset{(h\to 0,\, m\to\infty,\, n\to\infty)}{\mathrm{Lt}} \int_z^{z+h} f_n(x)[g_m(x)-g(x)]dx = 0;$$

and hence, bearing in mind that by § 15, since $\int f_n(x)dx$ converges uniformly to $\int f(x)dx$,

$$\underset{(h\to 0,\, n\to\infty)}{\mathrm{Lt}} \int_z^{z+h} g(x)[f_n(x)-f(x)]dx = 0,$$

it follows that

$$\underset{(h\to 0,\, m\to\infty,\, n\to\infty)}{\mathrm{Lt}} \int_z^{z+h} [f_n(x)\, g_m(x)-f(x)\, g(x)]dx = 0,$$

whence the uniformity given in the enunciation follows.

The extension to an infinite interval is immediate as in § 15. We have only, as before, to suppose

$$\underset{x\to\infty}{\mathrm{Lt}}\ g(x) = A,$$

and to interpret the uniformity of the convergence of $g_m(x)$ to $g(x)$ to refer to the whole closed infinite interval.

IV. *Applications to Fourier Series.* (§§ 24–30.)

24. I begin by shewing how to apply the theory of the single successions of integrals to Fourier series. There are two distinct methods

that we may employ. We may illustrate the one method by proving the extended theorem of Parseval in the case when the two functions in question have their squares summable. In this method we multiply the Fourier series of $f(x)$, or rather its partial summations, by $g(x)$, and shew that

$$\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \operatorname*{Lt}_{n\to\infty} \int_{-\pi}^{\pi} s_n(x)\, g(x)\, dx,$$

and therefore, by the very definition of Fourier constants,

$$= \pi \left\{ \tfrac{1}{2} a_0 A_0 + \sum_{i=1} (a_i A_i + b_i B_i) \right\},$$

where the small and large letters denote the Fourier constants of $f(x)$ and $g(x)$ respectively. We have merely to apply case (iii) of § 11, bearing in mind that the inequality is now an equality, that the lower limit is an unique limit, and that $q(x)$ is $f(x)$, while to p we give the value unity. In fact, $\int_{-\pi}^{\pi} [s_n(x)]^2 dx$ is bounded, as the identity

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x) - s_n(x)]^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx - \tfrac{1}{2} a_0^2 - \sum_{i=1}^{i=n} (a_i^2 + b_i^2)$$

shews.

It should be remarked that if $s_n(x)$ denote the Cesàro n-th partial summation $\int_{-\pi}^{\pi} [s_n(x)]^2 dx$ is still bounded. In fact, instead of

$$\pi \left\{ \tfrac{1}{2} a_0^2 + \ldots + a_n^2 \right\},$$

we now have $\pi \left\{ \tfrac{1}{2} a_0^2 + \ldots + \left(1 - \frac{r}{n}\right)^2 a_r^2 + \ldots + \frac{1}{n^2} a_{n-1}^2 \right\}$,

which is clearly less than before.

In both cases, of course, the succession of integrals forms a sequence converging to $\int_{-\pi}^{\pi} [f(x)]^2 dx$.

25. We may, however, employ the same method to obtain the results in Cesàro summation analogous to Parseval's theorem. In this case in the products $s_n(x)\, g(x)$ we replace the ordinary partial summations by Cesàro partial summations of the Fourier series of $f(x)$, $f_n(x)$, or $s_n(x)$ now standing, therefore, for these Cesàro partial summations. We now need, however certain properties of these Cesàro successions, which, it will be remarked, are all taken to have the index unity. We require the fact that, when summed in the Cesàro way, the Fourier series of $f(x)$

necessarily converges to $f(x)$, except at a set of content zero. This follows from the known fact that it does so wherever

$$|f(x+t)-f(x)|$$

is for $t=0$ the differential coefficient of its integral.* By means of this theorem we can shew that the succession of integrals of these Cesàro summations has the property that the usual double limit

$$\underset{(E\to 0,\, n\to\infty)}{\mathrm{Lt}} \int_E f_n(x)\,dx$$

exists and is zero. It is clearly sufficient to prove this when $f(x)$ is positive; the required result then follows by subtraction.

Now, when $f(x)$ is positive, the Cesàro partial summations are all positive, and therefore the succession is bounded below. Moreover, the integrals of these partial summations converge to the integral of $f(x)$; that is, to the common integral of their upper and lower functions. Hence, by the corollary to the theorem of § 9, the required conclusion at once follows.

Hence, applying case (i) of the theorem of § 11, we have the extension of Parseval's theorem in the case when $g(x)$ is bounded and $f(x)$ is summable. The case in which $f(x)$ possesses a Harnack-Lebesgue integral while $g(x)$ is any function of bounded variation follows from § 17, since, by a known result,† the integrated Fourier series of $f(x)$ is then the ordinary Fourier series of the continuous function $\int f(x)\,dx$, whence it follows that the integrated succession of the Cesàro partial summations of the original series converges uniformly to this Harnack-Lebesgue integral; thus, the result follows. The sole case still to be proved by this method is, then, that in which $g(x)$ has its $(1+1/p)$-th power and $f(x)$ its $(1+p)$-th power summable.

26. To obtain this remaining result we require some new theorems in the theory of the Cesàro summation of Fourier series. The following theorem appears not to have been remarked even for the case $p=1$:—

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* H. Lebesgue, *Intégrales Singulières*. See also W. H. Young, "On the Convergence of a Fourier Series and its Allied Series," 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 254–272, § 9.

† W. H. Young, "On the Conditions that a Trigonometrical Series should have the Fourier Form," 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 421–433, Supplementary Note.

THEOREM.—*The* $(1+p)$*-th power of the n-th Cesàro partial summation of the Fourier series of a positive function* $f(x)$ *whose* $(1+p)$*-th power is summable is never greater than the n-th Cesàro partial summation of the Fourier series of the* $(1+p)$*-th power of* $f(x)$ *whenever this* $(1+p)$*-th power is summable,* $(p > 0)$.

For denoting the n-th Cesàro partial summations by S_n and T_n

$$S_n = \frac{1}{n\pi}\int_0^{\frac{1}{2}\pi} (\sin nz/\sin z)^2 [f(x+2z)+f(x-2z)]\,dz$$

$$= \frac{1}{n\pi}\int_0^{\frac{1}{2}\pi} (\sin nz/\sin z)^{2p/(1+p)} (\sin nz/\sin z)^{2/(1+p)} \{[f(x+2z)]+[f(x-2z)]\}\,dz.$$

Hence, using the Schwarz-Riesz inequality, we have

$$S_n^{1+p} \leqslant \frac{1}{n^{1+p}\pi^{1+p}} \left\{\int_0^{\frac{1}{2}\pi} (\sin nz/\sin z)^2\,dz\right\}^p \times \int_0^{\frac{1}{2}\pi} (\sin nz/\sin z)^2 [f(x+2z)+f(x-2z)]^{1+p}\,dz\,;$$

and therefore, since the first integral on the right-hand side, when divided by $n\pi$, has the value $\frac{1}{2}$,

$$S_n^{1+p} \leqslant \frac{2}{n\pi}\int_0^{\frac{1}{2}\pi} (\sin nz/\sin z)^2 \left[\frac{f(x+2z)+f(x-2z)}{2}\right]^{1+p} dz.$$

But by a well-known inequality,

$$\left[\frac{f(x+2z)+f(x-2z)}{2}\right]^{1+p} \leqslant \frac{[f(x+2z)]^{1+p}+[f(x-2z)]^{1+p}}{2}.$$

Hence $$S_n^{1+p} \leqslant T_n,$$

as was to be proved.

27. We can now prove the following important theorem and corollaries. When p is unity, the first corollary is well known, or is at least an obvious consequence of the corresponding statement for ordinary partial summations. The theorem, on the other hand, seems new, even in the case $p = 1$.

THEOREM.—*If* $f_n(x)$ *denote the n-th Cesàro partial summation of the Fourier series of a function whose* $(1+p)$*-th power is summable* $(p > 0)$, *then* $\int_E |f_n(x)|^{1+p}\,dx$ *has the unique double limit zero, when* $E \to 0$, *and* $n \to \infty$.

For, using the notation of the preceding proof, and supposing first $f(x) \geqslant 0$,

$$\int_E [f_n(x)]^{1+p}\,dx = \int_E [S_n(x)]^{1+p}\,dx \leqslant \int_E T_n(x)\,dx.$$

Now, in § 25 it was proved that the latter integral had the unique double limit zero; therefore the same is true of the former integral. This proves the theorem when $f(x)$ is a positive function.

But, in the general case, $f(x)$ is the difference of two such functions, so that the theorem holds for each of the two corresponding Fourier series, and therefore, by subtraction, for the Fourier series of $f(x)$ itself.

COR. 1.—*Under the circumstances of the theorem* $\int |f_n(x)|^{1+p}\,dx$ *is a bounded function of* n.

COR. 2. — *Under the same circumstances* $\int_E |f_n(x)|^{1+q}\,dx$, *where* $0 \leqslant q \leqslant p$, *has the unique double limit zero.*

COR. 3.—*Hence,* when* $0 \leqslant q \leqslant p$,

$$\operatorname*{Lt}_{n\to\infty} \int_c^z |f_n(x)|^{1+q}\,dx = \int_c^z |f(x)|^{1+q}\,dx,$$

whatever be the limits of integration c *and* z *in the closed interval of periodicity, the convergence being, moreover, uniform.*

From Cor. 1 and the third case of the theorem of § 11, the theorem referred to at the end of § 25 follows immediately.

28. It has been already remarked that when $p = 1$ the first corollary of the preceding article holds equally whether the partial summations considered are the ordinary or the Cesàro ones. I now proceed to show that the theorem of that article is, in the particular case in which $p = 1$, true for the ordinary partial summations.

THEOREM.—*If* $f_n(x)$ *denote the* n*-th ordinary partial summation of the Fourier series of a function* $f(x)$ *whose square is summable, and which converges to* $f(x)$ *except at a set of content zero, then*

$$\int_E [f_n(x)]^2\,dx$$

has the unique double limit zero; a fortiori, the same is true of

$$\int_E |f_n(x)|^{1+q}\,dx \quad (0 \leqslant q \leqslant 1).$$

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* See the theorem at the end of § 4, p. 54.

For where $f_n(x)$ converges to $f(x)$, $[f_n(x)]^2$ converges to $[f(x)]^2$, that is, in our case, everywhere, except at a set of content zero. Moreover, by Parseval's theorem, $\int_{-\pi}^{\pi} [f_n(x)]^2\,dx$ converges to $\int_{-\pi}^{\pi} [f(x)]^2\,dx$.

Hence, by Vitali's theorem, given above in § 9, the required result follows.

This result will be required in the application of double successions to Fourier series.

COR.—*Hence, when* $0 \leqslant q \leqslant 1$,

$$\operatorname*{Lt}_{n\to\infty} \int_c^z |f_n(x)|^{1+q}\,dx = \int_c^z |f(x)|^{1+q}\,dx,$$

where $f_n(x)$ *denotes the sum of* $2n+1$ *terms of the Fourier series of* $f(x)$ *supposed to have its square summable.*

29. In the second mode of application we are again able to deduce all the known extensions of Parseval's theorem. The point of view is that explained in one of my recent papers.* By the use, however, of the theory of single successions of integrals we may so considerably shorten the work required that all these results are almost intuitive. Everything follows, in fact, at once if we first shew that, under the various circumstances required,

$$I(x) = \int_{-\pi}^{\pi} f(t)\,g(t+x)\,dt$$

is a continuous function of x. When this has been shewn the usual series involving the Fourier constants of $f(x)$ and $g(x)$ will necessarily converge, when summed in the Cesàro manner, to the value of this integral when x is zero. But the integrand has here precisely the form studied, x taking the place of the integer n and being supposed to have ascribed to it in succession any countably infinite set of values.

Now the integrated succession of $g(t+x)$ is a succession of integrals, having at the limiting or final value of x the value of that integral, and the convergence is, of course, uniform. This remains equally true if $g(t)$ is not summable, but possesses a Harnack-Lebesgue integral. Hence, by the theorem of § 17, the integral $I(x)$ is a continuous function of (x) if $f(x)$ has bounded variation and $g(x)$ possesses a Harnack-Lebesgue integral.

* "On a Class of Parametric Integrals and their Application in the Theory of Fourier Series," 1911, *Proc. Roy. Soc.*, A, Vol. 85, pp. 401–414.

If $g(x)$ is bounded and $f(x)$ is summable, the same follows directly by the first case of the theorem of § 11.

If $f(x)$ has its $(1+p)$-th power summable and $g(x)$ has its $(1+1/p)$-th power summable, the continuity follows from case (iii) of the same theorem.

Finally, we remark that, if G be the integral of g, we have in all cases

$$\frac{d}{dx}\int_{-\pi}^{\pi} f(t)\,G(t+x)\,dt = \operatorname*{Lt}_{h=0}\int_{-\pi}^{\pi} f(t)\,\frac{G(t+x+h)-G(t+x)}{h}\,dt$$

$$= \operatorname*{Lt}_{h=0}\int_{-\pi}^{\pi} f(t)\,g(t+x+\theta h)\,dt = \int_{-\pi}^{\pi} f(t)\,g(t+x)\,dt,$$

since this last integral is a continuous function of x. The required results now follow immediately.

The special cases of these results, when for the Cesàro summation we may substitute the ordinary method of summation, follow from the general ones in the usual way.

30. We have now to shew how to apply the results of the third section in the theory of Fourier series. It is clear, since the Fourier constants of a function which is a partial summation of a Fourier series are the coefficients in that partial summation, that we get no new theorem if we apply the repeated limits theorems to the interval of integration $(-\pi, \pi)$. This is equally the case if we apply the double limits theorems. The theorems of the preceding section have an application, however, whenever the interval of integration is included in, without coinciding with, the whole interval of periodicity of the functions. The repeated limits theorems give us, as was to be expected, no further information than we can get from the theorems of Section II.

Suppose $f(x)$ and $g(x)$ to be such a pair of functions as is contemplated in one of the last three articles. Then $f(x)$ and $g(x)\cos kx$ will be precisely such a pair of functions as $f(x)$ and $g(x)$ are. In all the cases there considered, the product $f(x)\,g(x)$ will have a Fourier series, and the coefficients of this Fourier series will be obtainable from the Fourier constants of $f(x)$ and $g(x)$ by an ordinary or Cesàro summation. The formulæ* are those given on p. 99 of Lebesgue's *Séries Trigonométriques*,

* Lebesgue only considers the case when these formulæ are interpreted in the ordinary sense, so that the partial summations are ordinary, and not Cesàro, ones. The truth of the formulæ in this extended sense follows from the results of the last three articles of the present paper. I have corrected some obvious misprints and interchanged the Greek letters and the capitals.

viz.,

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2}a_0A_0+\sum_{p=1}(a_pA_p+b_pB_p) \\ a_n &= \tfrac{1}{2}a_0A_n+\sum_{p=1}[a_p(A_{p+n}+A_{p-n})+b_p(B_{p+n}+B_{p-n})] \\ \beta_n &= \tfrac{1}{2}a_0B_n+\sum_{p=1}[a_p(B_{p+n}-B_{p-n})-b_p(A_{p+n}-A_{p-n})] \end{aligned}\right\} \quad (n>0).$$

There is, therefore, no difficulty in expressing the integral of the product of $f(x)$ and $g(x)$ over any interval within the interval of periodicity in terms of the Fourier constants of $f(x)$ and $g(x)$. We have only to integrate the Fourier series of the product $f(x)\,g(x)$ term-by-term. This process, however, supposes that we have already summed in the Cesàro or in the ordinary manner the formulæ for the coefficients of the Fourier series of the product; in other words, an infinite number of infinite series.

The theorems of the last section relating to the double limit enable us to express the integral in question as the sum of a double series, in which it is immaterial how m and n go off to infinity. We have merely to multiply f_n by g_m, where these denote either the ordinary or the Cesàro partial summations, according to the particular theorem used, and perform the integration of the polynomial involving sines and cosines of multiples of x in the second degree only, and then make m and n move off to infinity in any convenient manner. The process is always a convergent one, and may be stopped at any convenient stage if an approximation alone is required. It is evident from Section III that the process is allowable in the following cases :—

(i) When $f(x)$ has bounded variation, $g(x)$ has a Harnack-Lebesgue integral and $f_n(x)$ and $g_m(x)$ both stand for Cesàro partial summations of the Fourier series of $f(x)$ and $g(x)$.

Here we use the theorem of § 21, and bear in mind that the Cesàro partial summations of the Fourier series of a monotone function form a bounded sequence of monotone functions, while those of the Fourier series of a continuous function form a sequence which converges uniformly.

(ii) When $f(x)$ is bounded and $g(x)$ is summable and $f_n(x)$ and $g_m(x)$ both stand for Cesaro partial summations.

Here we use the theorem of § 19, bearing in mind that the Cesàro summations of a bounded function form a bounded succession, and, as we have seen in § 25, the Cesàro summations of any summable function have the property that the usual double limit is zero.

(iii) When $f(x)$ has bounded variation and $g(x)$ is summable, while $f_n(x)$

stand for the ordinary partial summation and $g_m(x)$ for the Cesàro partial summation of the Fourier series of $f(x)$ and $g(x)$ respectively.

Here again we use the theorem of § 19, remembering that the Fourier series of a function of bounded variation converges boundedly.*

(iv) When $f(x)$ and $g(x)$ both have their squares summable, while $f_n(x)$ stands for the ordinary partial summation and $g_m(x)$ for the Cesàro partial summation of the Fourier series of $f(x)$ and $g(x)$.

Here we use the special case of the theorem of § 22, and further take for p the value unity. We have to bear in mind first that a Fourier series when summed in the Cesàro way necessarily converges to the function except at a set of content zero, a result already cited in § 25, and, secondly that the integral of the square of the ordinary partial summation, and the Cesàro partial summation alike, traces out a bounded succession which converges to the integral of the square of the function when the function to which the Fourier series belongs has its square summable, a result remarked on in § 24.

(v) When $[f(x)]^{1+p}$ and $[g(x)]^{1+1/p}$ are summable and $f_n(x)$ and $g_n(x)$ both stand for Cesàro partial summations.

Here we use the theorem of § 22 in its full generality, making use of the theorem and corollary of § 27, and the further fact that the Fourier series necessarily converges to its function when summed in the Cesàro way, except at most at a set of content zero.

The above sets of conditions are all characterised by one distinctive feature. They differ from the corresponding conditions for the integrability of the single series developed in §§ 24–29 and previously in my paper on "The Integration of Fourier Series" and that on "A Class of Parametric Integrals and their Application in the Theory of Fourier Series," in this respect—that in each case a partial summation has been introduced of the Cesàro type. Where before the summation was of the Cesàro type, we have now two Cesàro summations; where the summation was ordinary, we have now one of the summations Cesàro.

In the remaining set, which I am about to give, I only hypothecate the additional circumstance that one of the Fourier series should converge, except at a set of content zero. Thus, the process in question is allowable in the following case.

(vi) When $f(x)$ and $g(x)$ have their squares summable, $f_n(x)$ and $g_m(x)$

* See my paper, "On the Integration of Fourier Series," § 3, 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 453.

both stand for the ordinary partial summations, and one of the Fourier series converges, except at a set of content zero.

We now use the theorem of § 22, coupled with that of § 28. Or we may use the special case of the theorem of § 22, the known facts as to the convergence and boundedness of the integrals of the squares of the partial summations.

It should be remarked, finally, that, under any one of the six sets of conditions hypothecated, the double succession of integrals converges to the required integral in a manner which is uniform for all modes of approach to the limit. This, in fact, like the convergence itself, follows from the theorems on double successions in the articles cited.

ON INVARIANTS OF A CANONICAL SUBSTITUTION

(PAPER II.)

By HAROLD HILTON.

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1. This paper is intended to complete a paper "On Invariants of a Canonical Substitution," formerly presented to the London Mathematical Society.* Suppose A is the substitution

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \quad (t = 1, 2, \ldots, m). \quad (\alpha)$$

From these equations we deduce

$$\bar{x}_t' = \bar{a}_{t1}\bar{x}_1 + \bar{a}_{t2}\bar{x}_2 + \ldots + \bar{a}_{tm}\bar{x}_m \quad (t = 1, 2, \ldots, m).\dagger \quad (\beta)$$

Suppose that the bilinear form

$$\Sigma e_{ij}\bar{x}_i x_j \quad (i, j = 1, 2, \ldots, m)$$

is an invariant of A,‡ so that

$$\Sigma e_{ij}\bar{x}_i x_j \equiv \Sigma e_{ij}\bar{x}_i' x_j'.$$

Then $\Sigma e_{ij}\bar{x}_i x_j$ is a quadratic invariant of the substitution of degree $2m$ which is obtained by taking together (α) and (β).

Applying "I.C.S.," § 3, we get the following results :—

(1) The substitution

$$\left.\begin{aligned} x_1' &= \alpha x_1 + x_2, \ \ldots, \ x_{r-1}' = \alpha x_{r-1} + x_r, \ x_r' = \alpha x_r \\ y_1' &= \beta y_1 + y_2, \ \ldots, \ y_{s-1}' = \beta y_{s-1} + y_s, \ y_s' = \beta y_s \end{aligned}\right\} \ (r \geqslant s) \qquad \text{(i)}$$

has no invariant of the form $\Sigma e_{ij}\bar{y}_i x_j$, unless $\alpha\bar{\beta} = 1$; and, if $\alpha\bar{\beta} = 1$, the most general invariant is

$$k_1 f_1(x, \bar{y}) + k_2 f_2(x, \bar{y}) + \ldots + k_s f_s(x, \bar{y}),$$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 10 (1911), p. 423. This paper will be referred to as "I.C.S."

† $\bar{x}$ is, as usual, the complex quantity conjugate to x.

‡ Usually so called; but strictly speaking a joint invariant of the substitution obtained by combining (α) and (β).

obtained by changing y into $\bar{y}$ in the invariant

$$k_1 f_1(x, y)+k_2 f_2(x, y)+\ldots+k_s f_s(x, y)$$

at the end of "I.C.S.," § 3.

(2) The substitution

$$x_1' = \alpha x_1+x_2,\ \ldots,\ x_{s-1}' = \alpha x_{s-1}+x_s,\ x_s' = \alpha x_s, \qquad \text{(ii)}$$

has no invariant of the form $\Sigma e_{ij}\bar{x}_i x_j$, unless $\alpha\bar{\alpha} = 1 \pmod{\alpha = 1}$; and if $\alpha\bar{\alpha} = 1$, the most general invariant is

$$k_1 f_1(x, \bar{x})+k_2 f_2(x, \bar{x})+\ldots+k_s f_s(x, \bar{x}),$$

obtained by changing $\bar{y}$ into $\bar{x}$ in the previous invariant.

2. When we form the determinant

$$\begin{vmatrix} e_{11} & \ldots & e_{1m} \\ \vdots & \vdots & \vdots \\ e_{m1} & \ldots & e_{mm} \end{vmatrix}$$

of the invariant $\Sigma e_{ij}\bar{x}_i x_j$ of any substitution in canonical form N,

$$x_t' = \lambda_t x_t+\beta_t x_{t+1} \quad (t = 1, 2, \ldots, m),$$

where $\beta_t = 0$ or 1, and always $= 0$, if $\lambda_t \neq \lambda_{t+1}$, we readily see that the determinant vanishes unless the invariant-factors of N can be paired, so that to each invariant-factor $(\lambda-\lambda_1)$ corresponds another invariant-factor $(\lambda-\bar{\lambda}_1^{-1})$, unless $\bmod \lambda_1 = 1$.

We shall in future suppose that this is the case. We can then factorize the determinant as in "I.C.S.," § 6.

Let the invariant-factors of N be

$$(\lambda-\lambda_1)^a,\ (\lambda-\bar{\lambda}_1^{-1})^a,\ (\lambda-\lambda_1)^b,\ (\lambda-\bar{\lambda}_1^{-1})^b,\ (\lambda-\lambda_1)^c,\ (\lambda-\bar{\lambda}_1^{-1})^c,\ \ldots,$$

where $a \geqslant b \geqslant c \geqslant d \geqslant \ldots$; and

$$(\lambda-\lambda_0)^h,\ (\lambda-\lambda_0)^k,\ (\lambda-\lambda_0)^l,\ (\lambda-\lambda_0)^m,\ \ldots,$$

where $\bmod \lambda_0 = 1$, and $h \geqslant k \geqslant l \geqslant m \geqslant \ldots$.

Then (as in "I.C.S.," § 5) the total number of independent invariants $\Sigma e_{ij}\bar{x}_i x_j$ of N is

$$\Sigma 2(a+3b+5c+7d+\ldots)+\Sigma(h+3k+5l+7m+\ldots),$$

the first summation extending over each pair of corresponding characteristic roots λ_1 and $\bar{\lambda}_1^{-1}$ of N, and the second summation extending over each characteristic root λ_0 of N with unit modulus.

3. We now consider the "Hermitian invariants of the substitution N" of § 2 (usually so-called), *i.e.*, those joint invariants $e_{ij}\bar{x}_i x_j$ of

$$x_t' = \lambda_t x_t + \beta_t x_{t+1} \quad \text{and} \quad \bar{x}_t' = \bar{\lambda}_t \bar{x}_t + \bar{\beta}_t \bar{x}_{t+1},$$

for which $$e_{ij} = \bar{e}_{ji} \quad (i, j = 1, 2, \ldots, m).$$

In § 1 the most general Hermitian invariant of (i) is

$$\{k_1 f_1(x, \bar{y}) + k_2 f_2(x, \bar{y}) + \ldots + k_s f_s(x, \bar{y})\}$$
$$+ \{\text{the conjugate complex quantity}\},$$

where the k's are real constants.

The most general Hermitian invariant of (ii) is similarly

$$k_1 f_1(x, \bar{x}) + k_2 f_2(x, \bar{x}) + \ldots + k_s f_s(x, \bar{x}),$$

k_s, k_{s-2}, k_{s-4}, ... and ik_{s-1}, ik_{s-3}, ik_{s-5}, ... being real ($i = \sqrt{-1}$).

For the substitution N of § 2 with the invariant factors there stated the number of independent Hermitian invariants is

$$\Sigma(a+3b+5c+7d+\ldots)+\Sigma(h+2k+3l+4m+\ldots).$$

4. It is known that the general Hermitian invariant of the substitution (i) or (ii), obtained in § 3, is reduced by suitable change of variables into the form

$$\pm z_1\bar{z}_1 \pm z_2\bar{z}_2 \pm z_3\bar{z}_3 + \ldots;$$

z_1, z_2, z_3, ... being homogeneous linear functions of the original variables.

We proceed to discuss the number of + and − signs in this reduced form. For this purpose we use the following result:—

"A Hermitian form $\Sigma a_{ij}\bar{x}_i x_j$ (where $a_{ij} = \bar{a}_{ji}$) in m variables $x_1, x_2, \ldots, x_m$, and their conjugate imaginaries can be put in the form

$$\lambda_1(b_{11}x_1+b_{12}x_2+\ldots+b_{1m}x_m)(\bar{b}_{11}\bar{x}_1+\bar{b}_{12}\bar{x}_2+\ldots+\bar{b}_{1m}\bar{x}_m)$$
$$+\lambda_2(b_{21}x_1+\ldots+b_{m1}x_m)(\bar{b}_{21}\bar{x}_1+\ldots+\bar{b}_{m2}\bar{x}_m)+\ldots$$
$$+\lambda_m(b_{m1}x_1+\ldots+b_{mm}x_m)(\bar{b}_{m1}\bar{x}_1+\ldots+\bar{b}_{mm}\bar{x}_m),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the characteristic roots of the Hermitian substitution A,

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \quad (t = 1, 2, \ldots, m)."$$

In fact, if A is *any* substitution, and in the form $\Sigma a_{ij}y_i x_j$, we put

$$p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m \quad \text{for} \quad x_t,$$
$$q_{t1}y_1 + q_{t2}y_2 + \ldots + q_{tm}y_m \quad \text{for} \quad y_t,$$

we get $\Sigma c_{ij} y_i x_j$, where $C = PAQ'$; C, P, Q being the substitutions

$$x_t' = c_{t1}x_1 + c_{t2}x_2 + \ldots + c_{tm}x_m,$$

$$x_t' = p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m,$$

$$x_t' = q_{t1}x_1 + q_{t2}x_2 + \ldots + q_{tm}x_m,$$

and Q' the transposed substitution to Q.

Take now $$y_t = \bar{x}_t, \quad q_{ij} = \bar{p}_{ij}.$$

Then $\Sigma a_{ij}\bar{x}_i x_j$ becomes $\Sigma c_{ij}\bar{x}_i x_j$; where $C = PA\bar{P}'$, and $\bar{P}'$ is the transposed conjugate substitution to P.

Take now A Hermitian, and for P the unitary substitution transforming the Hermitian substitution A into a multiplication.* Then $\bar{P}' = P^{-1}$, since P is unitary, and C is the multiplication

$$x_t' = \lambda_t x_t;$$

which establishes the result.

Take now the general Hermitian invariant of (i) given in § 3.

The characteristic determinant of the corresponding substitution is (taking the case $s = 3$ as an illustration)

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & 0 & * \\ 0 & -\lambda & 0 & 0 & * & * \\ 0 & 0 & -\lambda & * & * & * \\ 0 & 0 & * & -\lambda & 0 & 0 \\ 0 & * & * & 0 & -\lambda & 0 \\ * & * & * & 0 & 0 & -\lambda \end{vmatrix},$$

where the asterisks denote quantities not necessarily zero.

Expanding this by the leading diagonal, we see that it involves only even powers of λ. Hence the characteristic equation of the corresponding substitution has as many positive as negative roots, and in the reduced form of the Hermitian invariant there are s positive and s negative signs.

Take now the general Hermitian invariant of (ii) given in § 3.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 10 (1911), p. 283. This proof may with advantage replace the statements at the end of the author's *Finite Groups*, Ch. III, § 5, which only apply if $\Sigma a_{ij}\bar{x}_i x_j$ is definite.

The corresponding characteristic equation (taking as an illustration $s = 5$) is of the form

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & a\alpha^{-4} \\ 0 & -\lambda & 0 & -a\alpha^{-2} & * \\ 0 & 0 & a-\lambda & * & * \\ 0 & -a\alpha^{2} & * & *-\lambda & * \\ a\alpha^{4} & * & * & * & *-\lambda \end{vmatrix} = 0,$$

if we neglect for a moment the dotted lines.

Now when s is even, this equation has as many positive as negative roots. When s is odd,

$$(\text{number of positive roots}) - (\text{number of negative roots}) = \pm 1.$$

This is established by using as Sturmian functions the determinants of which the dotted lines are the right and bottom boundary, and proceeding as in Burnside and Panton's *Theory of Equations*, Ch. XI, Miscellaneous Examples, No. 32.

Hence in the reduced form of the Hermitian invariant the number of positive signs is at most one greater than the number of negative signs, and *vice versa*.

The results established by Loewy in *Math. Annalen*, 50 (1898), pp. 560 to 564, may be deduced from this section.

5. Consider now a "quasi-unitary substitution" U on $x_1, x_2, \ldots, x_m$ with the invariant

$$x_1\bar{x}_1 + \ldots + x_k\bar{x}_k - x_{k+1}\bar{x}_{k+1} - \ldots - x_m\bar{x}_m.$$

It will have a pole $(X_1, X_2, \ldots, X_m)$, such that

$$X_1\bar{X}_1 + \ldots + X_k\bar{X}_k - X_{k+1}\bar{X}_{k+1} - \ldots - X_m\bar{X}_m \neq 0; \qquad (\gamma)$$

if, and only if, U has a simple invariant-factor $(\lambda - \lambda_0)$, where mod $\lambda_0 = 1$; the proof of this statement being similar to that employed for orthogonal substitutions in "I.C.S.," § 7.

If we transform U by one of the quasi-unitary substitutions given on page 276 of *Proc. London Math. Soc.*, Ser. 2, Vol. 10 (1911), in which a_t is taken as $\epsilon_{tk}X_t$ (cf. bottom of p. 279), we transform U by a quasi-unitary

substitution into a quasi-unitary substitution which is the direct product of

$$x_1' = \lambda_0 x_1 \text{ and a quasi-unitary substitution on } x_2, x_3, \ldots, x_m,$$

or of $x_m' = \lambda_0 x_m$ and a quasi-unitary substitution on $x_1, x_2, \ldots, x_{m-1}$.

The process may be repeated so long as there is a pole satisfying (γ).

In particular, a unitary substitution (for which $k = m$) may be transformed by a unitary substitution into a multiplication.

Suppose now that U has no pole satisfying (γ), but that U has a pair of poles $(X_1, X_2, \ldots, X_m)$, $(\xi_1, \xi_2, \ldots, \xi_m)$ satisfying

$$X_1\xi_1 + \ldots + X_k\xi_k - X_{k+1}\xi_{k+1} - \ldots - X_m\xi_m = 2.$$

This will be the case (as in "I.C.S.," § 7) if, and only if, U has a pair of simple invariant-factors $(\lambda-\lambda_1)$, $(\lambda-\lambda_1^{-1})$. Taking

$$p_{tk} = \tfrac{1}{2}(X_t + \xi_t), \qquad p_{tk+1} = \tfrac{1}{2}(X_t - \xi_t),$$

we may form a quasi-unitary substitution P,

$$x_t' = p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m,$$

such that P^{-1} transforms U into the direct product of a quasi-unitary substitution on $x_1, \ldots, x_{k-1}, x_{k+2}, \ldots, x_m$, and of a substitution of the type

$$\left.\begin{aligned} x_k' &= e^{i\alpha}\cosh\theta\, x_k + e^{i(\delta+\epsilon)}\sinh\theta\, x_{k+1} \\ x_{k+1}' &= e^{i(\alpha-\epsilon)}\sinh\theta\, x_k + e^{i\delta}\cosh\theta\, x_{k+1} \end{aligned}\right\}.^{*}$$

Combining the results of this section we obtain a theorem similar to that stated for orthogonal substitutions at the beginning of "I.C.S.," § 7.

The proof in this present section has been given only in outline, as it is so similar to that given in the case of orthogonal substitutions.

6. In "I.C.S.," § 10, I gave a proof of the theorem that "If a substitution A is transformable into its inverse, (i) the transformation may be performed by means of a substitution of order 2 (which is identical with its inverse), and (ii) A is the product of two substitutions of order 2."

An analogous theorem is the following:—

"If a substitution A is transformable into the inverse of its conjugate, (i) the transformation may be performed by means of a substitution which is identical with the inverse of its conjugate, and (ii) A is the product of two substitutions each identical with the inverse of its conjugate."

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* If $|\lambda_1| \neq 1$, $\tanh\theta > |\sin\tfrac{1}{2}(\alpha-\delta)|$.

Theorem (ii) may be deduced from (i) as follows:—

If $P^{-1}AP = \bar{A}^{-1}$, where $P\bar{P} = \bar{P}P = 1$, then $A = P.\bar{P}A$. But

$$P\bar{P} = 1 \quad \text{and} \quad \bar{P}A.\overline{\bar{P}A} = \bar{P}A.P\bar{A} = P^{-1}AP\bar{A} = \bar{A}^{-1}\bar{A} = 1.$$

To prove (i) we notice that if (i) is true for A, it is true for any substitution $R^{-1}AR$ into which A can be transformed. In fact,

$$(R^{-1}P\bar{R})^{-1}(R^{-1}AR)(R^{-1}P\bar{R}) = \bar{R}^{-1}P^{-1}AP\bar{R} = \bar{R}^{-1}\bar{A}^{-1}\bar{R} = (\overline{R^{-1}AR})^{-1},$$

and
$$(R^{-1}P\bar{R})\overline{(R^{-1}P\bar{R})} = R^{-1}P\bar{R}.\bar{R}^{-1}\bar{P}R = R^{-1}P\bar{P}R = 1.$$

It is sufficient therefore to prove the result for a substitution in canonical form.

The proof now runs parallel to that of "I.C.S.," § 10.

Each invariant-factor $(\lambda-\alpha)^s$ of A is paired with an invariant-factor $(\lambda-\bar{\alpha}^{-1})^s$, unless mod $\alpha = 1$.

Again the substitution H,

$$\left.\begin{aligned} x_1' &= \alpha\ x_1 + x_2,\ \ldots,\ x_{s-1}' = \alpha\ x_{s-1} + x_s,\ x_s' = \alpha\ x_s \\ y_1' &= \bar{\alpha}^{-1}y_1 + y_2,\ \ldots,\ y_{s-1}' = \bar{\alpha}^{-1}y_{s-1} + y_s,\ y_s' = \bar{\alpha}^{-1}y_s \end{aligned}\right\}$$

is transformed into $\bar{H}^{-1}$ by the substitution K,

$$\left.\begin{aligned} x_1' &= -\bar{\alpha}^2y_1 + \bar{\alpha}^3y_2 - \bar{\alpha}^4y_3 + \bar{\alpha}^5y_4 - \ldots + (-1)^s\ \bar{\alpha}^{s+1}y_s \\ x_2' &= \bar{\alpha}^4y_2 - {}^2C_1\bar{\alpha}^5y_3 + {}^3C_1\bar{\alpha}^6y_4 - \ldots + (-1)^{s-1}\,{}^{s-1}C_1\bar{\alpha}^{s+2}y_s \\ x_3' &= -\bar{\alpha}^6y_3 + {}^3C_2\bar{\alpha}^7y_4 - \ldots + (-1)^{s-1}\,{}^{s-1}C_2\bar{\alpha}^{s+3}y_s \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x_s' &= (-1)^s\ \bar{\alpha}^{2s}y_s \end{aligned}\right\},$$

$$\left.\begin{aligned} y_1' &= -\alpha^{-2}x_1 + \alpha^{-3}x_2 - \alpha^{-4}x_3 + \alpha^{-5}x_4 - \ldots + (-1)^s\ \alpha^{-s-1}x_s \\ y_2' &= \alpha^{-4}x_2 - {}^2C_1\alpha^{-5}x_3 + {}^3C_1\alpha^{-6}x_4 - \ldots + (-1)^{s-1}\,{}^{s-1}C_1\alpha^{-s-2}x_s \\ y_3' &= -\alpha^{-6}x_3 + {}^3C_2\alpha^{-7}x_4 - \ldots + (-1)^{s-1}\,{}^{s-1}C_2\alpha^{-s-3}x_s \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ y_s' &= (-1)^s\ \alpha^{-2s}\ y_s \end{aligned}\right\},$$

and
$$K\bar{K} = 1.$$

Also the substitution L,

$$x_1' = \alpha x_1 + x_2, \quad \ldots, \quad x_{s-1}' = \alpha x_{s-1} + x_s, \quad x_s' = \alpha x_s,$$

where $\bmod \alpha = 1$, is transformed into $\bar{L}^{-1}$ by the substitution M,

$$\left.\begin{aligned}
x_1' &= -\alpha^{-2}x_1 + \alpha^{-3}x_2 - \alpha^{-4}x_3 + \alpha^{-5}x_4 - \ldots + (-1)^s \alpha^{-s-1}x_s \\
x_2' &= \alpha^{-4}x_2 - {}^2C_1\alpha^{-5}x_3 + {}^3C_1\alpha^{-6}x_4 - \ldots + (-1)^{s\,s-1}C_1\alpha^{-s-2}x_s \\
x_3' &= -\alpha^{-6}x_3 + {}^3C_2\alpha^{-7}x_4 - \ldots + (-1)^{s\,s-1}C_2\alpha^{-s-3}x_s \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
x_s' &= (-1)^s \alpha^{-2s}x_s
\end{aligned}\right\},$$

and

$$M\bar{M} = 1.$$

THE PRESSURE OF RADIATION ON A CYLINDRICAL OBSTACLE

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Part I.

In an earlier paper * the pressure experienced by a reflecting spherule on which plane waves impinge was examined numerically, with special reference to the maximum value of this pressure corresponding to an obstacle whose radius is of the same order of magnitude as the wave length of the incident radiation. The determination of a corresponding maximum, when it exists, for obstacles which are not spherical is a matter of importance; but the mathematical difficulties are so formidable that no solution has yet been given. As a preliminary to a more complete knowledge of the pressure which can be experienced by non-spherical bodies, the present paper takes up the only other case in which an analysis on similar lines to that used for a sphere can be employed profitably. This is the two-dimensional problem of the circular cylinder whose length is sufficiently great in comparison with a radius of the cross-section and with the wave length of light.

The general basis of the investigation is the electromagnetic theory. It appears that all possible cases can be built up from two when the incident waves are travelling perpendicularly to the axis of the cylinder. The two cases are those in which (1) the electric force and (2) the magnetic force are respectively parallel to that axis. In Part I of the paper we propose to consider the first case in detail. The second case, which is somewhat more complicated, is investigated in Part II.

Let a set of cylindrical coordinates (r, θ, z) be chosen, where z is parallel to the axis, so that r and θ are polar coordinates in the plane of a normal section of the cylinder. The problem being one of symmetry, none of the quantities concerned will be functions of z. If the wave length be $2\pi/k$, and if the incident waves are travelling along a direction x, the initial direction from which θ is measured, the magnetic

* *Monthly Notices of the R.A.S.*, May, 1910.

force in the incident waves may be taken as e^{ikx}, ignoring a time factor, acting along the direction y perpendicular to x and z. Moreover,

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The waves are incident on the cylinder on the side for which x is positive.

If the cylinder is a perfect reflector, the magnetic force at its surface must be tangential and the electric force normal. A disturbance will therefore be thrown off, in which the radial magnetic force at the surface $r = a$ (a being the radius of the cylinder) will be equal and opposite to that of the incident waves. The latter is, of course,

$$\frac{y}{r}\, e^{ikx} \quad \text{or} \quad \sin\theta . e^{ikr\cos\theta},$$

which may be expressed by a well known theorem ascribed by Nielsen* to Hansen, but probably used earlier by Jacobi, in the form

$$\sin\theta . J_0(kr) + 2\sin\theta \,\Sigma_1^\infty\, i^n J_n(kr)\cos n\theta,$$

where $J_n(kr)$ is the usual Bessel function of argument kr and order n, finite at the origin, or on the axis of the cylinder. This series may be written in the form

$$\sin\theta\,(J_0+J_2) + i\sin 2\theta(J_1+J_3) + i^2\sin 3\theta(J_2+J_4) + \ldots,$$

and, by the use of a recurrence formula for the Bessel functions, this becomes

$$a_1 = \frac{2}{kr}\,\Sigma_1^\infty\, i^{n-1}\, n\sin n\theta . J_n(kr). \tag{1}$$

This might also have been obtained by differentiating the original series with respect to θ. The electric force along the axis is

$$R = Ve^{ikx} = VJ_0(kr) + 2V\Sigma_1^\infty\, i^n J_n(kr)\cos n\theta. \tag{2}$$

Let us now consider the general equations of the external electromagnetic field. If (P, Q, R) and (a, b, c) be the components of the electric and magnetic vectors along the directions at any point in which r, θ, and z respectively increase, then, by the cylindrical form of the circuital electromagnetic relations,

$$\left.\begin{aligned} -r\dot{a} &= \frac{\partial R}{\partial\theta} - r\frac{\partial Q}{\partial z}, & \frac{r\dot{P}}{V^2} &= \frac{\partial c}{\partial\theta} - r\frac{\partial b}{\partial z} \\ -\dot{b} &= \frac{\partial P}{\partial z} - \frac{\partial R}{\partial r}, & \frac{r\dot{Q}}{V^2} &= \frac{\partial a}{\partial z} - \frac{\partial c}{\partial r} \\ -r\dot{c} &= \frac{\partial}{\partial r}(Qr) - \frac{\partial P}{\partial\theta}, & \frac{r\dot{R}}{V^2} &= \frac{\partial}{\partial r}(br) - \frac{\partial a}{\partial\theta} \end{aligned}\right\}, \tag{3}$$

* *Handbuch der Cylinderfunctionen.*

with the solenoidal conditions

$$\frac{\partial}{\partial r}(ar)+\frac{\partial b}{\partial \theta}+r\frac{\partial c}{\partial z}=\frac{\partial}{\partial r}(Pr)+\frac{\partial Q}{\partial \theta}+r\frac{\partial R}{\partial z}=0, \tag{4}$$

where V is the velocity of disturbances in the outer medium. For the present problem, $\partial/\partial z \equiv 0$, and therefore, by the solenoidal conditions, functions ϕ and ψ exist such that

$$\left.\begin{aligned} P&=-\frac{\partial \phi}{r\partial\theta}, & Q&=\frac{\partial \phi}{\partial r}\\ a&=-\frac{\partial \psi}{r\partial\theta}, & b&=\frac{\partial \psi}{\partial r}\end{aligned}\right\}. \tag{5}$$

If the disturbance has a wave-length $2\pi/k$, it will involve a time factor e^{ikVt}, and we accordingly write $\partial/\partial t = ikV$, and derive, on reduction,

$$R=ikV\psi, \qquad c=ik\phi/V, \tag{6}$$

where ϕ and ψ appear as solutions of the equation

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+k^2\psi=0. \tag{7}$$

For the most general vibration of a given period there is no essential relation between ϕ and ψ. Thus, the most general vibration of given period in or about a system of coaxial cylinders of circular section, independent of z, consists of a superposition of two others, in one of which the electric force, and in the other the magnetic force, is parallel to the axis at all points. This is a special case of a more general result true for orthogonal sets of surfaces of revolution.

The general solution of (7) finite at the origin is

$$\psi=\Sigma_0^\infty A_n J_n(kr)\cos n\theta. \tag{8}$$

Let us consider the case in which the incident waves have no magnetic component parallel to the axis. Then a complete solution may be obtained in terms of functions of type ψ only, the electric force in both incident and scattered waves having a component R only parallel to the axis, and the magnetic force having components (a, b) in the plane of the circular section at any point.

The function ψ for the incident waves is, by (2) and (6), of the form

$$\psi_1=-\frac{i}{k}J_0(kr)+\frac{2}{k}\Sigma_1^\infty i^{n-1}J_n(kr)\cos n\theta, \tag{9}$$

leading by (4) at once to the proper formula (1).

If $K_n(kr)$ be the corresponding function of distance for the scattered waves, we may write, for these waves,

$$\psi_2 = -\frac{ia_0}{k} K_0(kr) + \frac{2}{k} \Sigma_1^\infty i^{n-1} a_n K_n(kr) \cos n\theta, \tag{10}$$

and a_n may be determined from the general surface condition of the continuity of tangential components of electric and magnetic force. For a perfect reflector it is sufficient to make a or $-\partial\psi/r\partial\theta$ and R or $ikV\psi$ vanish at the surface. Both conditions are secured if ψ itself vanishes, and therefore if

$$\left.\begin{aligned} a_0 &= -J_0(ka)/K_0(ka) \\ a_n &= -J_n(ka)/K_n(ka) \end{aligned}\right\}, \tag{11}$$

a now denoting the radius, and the form of $\psi = \psi_1 + \psi_2$ becomes

$$\psi = -\frac{i}{k} \{J_0(kr) - J_0(ka) K_0(kr)/K_0(ka)\}$$
$$+ \frac{2}{k} \Sigma_1^\infty i^{n-1} \{J_n(kr) - J_n(ka) K_n(kr)/K_n(ka)\} \cos n\theta. \tag{12}$$

Let $Y_n(x)$ be the second solution of Bessel's equation of order n defined by *

$$Y_n(x) = \left\{\frac{\partial J_n}{\partial n} - (-)^n \frac{\partial J_{-n}}{\partial n}\right\}_{n=\text{integer}}. \tag{13}$$

Then
$$\frac{K_n(kr)}{K_n(ka)} = \frac{Y_n(kr) + i\pi J_n(kr)}{Y_n(ka) + i\pi J_n(ka)}. \tag{14}$$

The electric force vanishes at the surface. The magnetic component is only b or $\partial\psi/\partial r$, and becomes on the surface

$$b = -i \left\{\frac{J_0'(ka)\, Y_0(ka) - Y_0'(ka)\, J_0(ka)}{Y_0(ka) + i\pi J_0(ka)}\right\}$$
$$+ 2\Sigma_1^\infty i^{n-1} \left\{\frac{J_n'(ka)\, Y_n(ka) - Y_n'(ka)\, J_n(ka)}{Y_n(ka) + i\pi J_n(ka)}\right\} \cos n\theta. \tag{15}$$

The two kinds of Bessel function are related by the identity

$$Y_n(x)\, J_n'(x) - J_n(x)\, Y_n'(x) = -2/x, \tag{16}$$

and the surface value of b becomes, inserting the time factor,

$$b = \frac{2i}{M_0 ka} e^{ikVt - i\eta_0} - \frac{4}{ka} \Sigma_1^\infty i^{n-1} \frac{\cos n\theta}{M_n} e^{ikVt - i\eta_n}, \tag{17}$$

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* The referee points out that this definition may cause confusion. It was adopted mainly to keep the notation in line with that of earlier papers. The present function is more usually given a factor in English work, and is identical with $-2K_n$, where the latter symbol is used in the sense of Heine.

where
$$\left.\begin{aligned} M_n^2 &= Y_n^2(ka)+\pi^2 J_n^2(ka) \\ \tan\eta_n &= \pi J_n(ka)/Y_n(ka) \end{aligned}\right\}. \tag{18}$$

Determination of the Pressure.

Let E be the energy per unit volume of the incident radiation, half electric and half magnetic. Then the pressure on a unit length of the cylinder parallel to its axis has a resultant action along the direction of propagation of the incident waves, given by

$$p = E\iint b^2\cos\theta\, dS, \tag{19}$$

the surface integral being taken over the unit length. This follows at once from the Maxwellian stress system, the electric force being zero. Expressed as a line integral round the circular section, the formula becomes

$$p = 2aE\int_0^\pi b^2\cos\theta\, d\theta. \tag{20}$$

In this calculation the average value of b^2 is needed for the determination of the motive pressure; that is, the average value over a sufficiently large time, which neglects the periodic terms. It is a familiar relation that the average value of the square of a complex b is given by $\frac{1}{2}|b|^2$ or $\frac{1}{2}b\bar{b}$, where $\bar{b}$ is the conjugate complex. In the present case this average becomes B, where

$$2B = \frac{4}{k^2a^2}\,\frac{1}{M_0^2}+\frac{16}{k^2a^2}\,\Sigma_1^\infty\frac{\cos^2 n\theta}{M_n^2}+\frac{16}{k^2a^2}\,\frac{1}{M_0}\,\Sigma_1^\infty\frac{\cos n\theta}{M_n}\cos\left(\eta_0-\eta_n+\frac{n\pi}{2}\right)$$
$$+\frac{32}{k^2a^2}\,\Sigma_1^\infty\,\Sigma_1^\infty\frac{\cos r\theta\cos s\theta}{M_rM_s}\cos\left\{(\eta_r-\eta_s+(s-r)\frac{\pi}{2}\right\}, \tag{21}$$

where the double summation excludes equal values of r and s.

Now for all integral values of n, r, s,

$$\int_0^\pi\cos^2 n\theta\cos\theta\, d\theta = 0, \tag{22}$$

$$\int_0^\pi\cos r\theta\cos s\theta\cos\theta\, d\theta = \tfrac{1}{2}\int_0^\pi\{\cos(r+s)\theta+\cos(r-s)\theta\}\cos\theta\, d\theta.$$

The first integral vanishes because $r+s$ is necessarily greater than unity. The second also vanishes unless $r-s=1$, in which case

$$\int_0^\pi\cos r\theta\cos s\theta\cos\theta\, d\theta = \frac{\pi}{4}. \tag{23}$$

Moreover, when $n = 1$,

$$\int_0^\pi \cos n\theta \cos\theta \, d\theta = \frac{\pi}{2}, \tag{24}$$

being zero for other integral values of n.

Thus
$$p = 2aE \int_0^\pi B \cos\theta \, d\theta$$
$$= -\frac{8\pi aE}{k^2a^2} \Sigma_0^\infty (M_n M_{n+1})^{-1} \sin(\eta_n - \eta_{n+1}), \tag{25}$$

all the terms falling into line with one another.

But
$$-\sin(\eta_n - \eta_{n+1}) = \pi(Y_n J_{n+1} - J_n Y_{n+1})/M_n M_{n+1},$$

by (18), and by an identical relation

$$Y_n J_{n+1} - J_n Y_{n+1} = 2/ka, \tag{26}$$

we deduce
$$\frac{p}{2aE} = \frac{8\pi^2}{k^3a^3} \Sigma_0^\infty M_n^{-2} M_{n+1}^{-2};$$

or if z denotes ka, the argument of the Bessel functions,

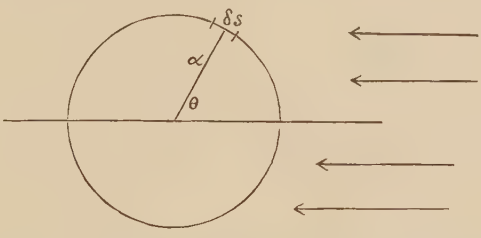
$$\frac{p}{2aE} = \frac{8\pi^2}{z^3} \Sigma_0^\infty (Y_n^2 + \pi^2 J_n^2)^{-1} (Y_{n+1}^2 + \pi^2 J_{n+1}^2)^{-1}, \tag{27}$$

gives the pressure per unit length of the cylinder. This result is of considerable elegance.

General Remarks.

In the first place, before making a detailed investigation of the formula just obtained, we may shew that it approaches the correct limiting value when the diameter of the obstacle increases indefinitely.

On the usual theory of pressure, a plate of a reflecting body, on which radiation falls at an angle θ to the normal, experiences a normal pressure $2E\cos^2\theta$ per unit area, where E is energy in a unit volume of the



incident beam. If the cylinder has a large diameter, and attention is

confined to a unit length along the axis, the pressure on the plate of unit length and breadth δs, an element of arc of the circular section, becomes $2E\delta s\cos^2\theta$. This acts towards the axis of the cylinder and has a component $2E\delta s\cos^3\theta$ parallel to the incident rays. The resultant pressure in this direction is obtained by integration over the unshadowed part, or just one half, of the surface and becomes

$$p = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2E\cos^3\theta \,.\, a\, d\theta = \tfrac{8}{3}aE \tag{28}$$

per unit length of cylinder.

Consider now the accurate formula

$$\frac{p}{2aE} = \frac{8\pi^2}{z^3}\,\Sigma_0^\infty \frac{1}{(Y_n^2+\pi^2J_n^2)(Y_{n+1}^2+\pi^2J_{n+1}^2)}\,.$$

Writing

$$\left.\begin{aligned} J_n(z) &= \left(\frac{2R_n}{\pi z}\right)^{\frac{1}{2}}\sin\phi_n \\ Y_n(z) &= -\left(\frac{2\pi R_n}{z}\right)^{\frac{1}{2}}\cos\phi_n \end{aligned}\right\}, \tag{29}$$

where the functions R_n and ϕ_n have been examined in detail in earlier papers,* then

$$\frac{p}{2aE} = \frac{2}{z}\,\Sigma_0^\infty \frac{1}{R_nR_{n+1}}\,. \tag{30}$$

Now, when z is large, and n is not so large that $z-n$ is of so low an order as $z^{\frac{1}{3}}$,

$$R_n = z/(z^2-n^2)^{\frac{1}{4}} \tag{31}$$

approximately, and the number of terms of the series for which this occurs is effectively z.

When $n-z$ or $z-n$ is of order $z^{\frac{1}{3}}$ or less, the number of terms thus concerned being also of this order, R_n and R_{n+1} are of order $z^{\frac{1}{3}}$. The sum of such terms is therefore of order $z^{-\frac{1}{3}}$ at most in the series $\Sigma(R_nR_{n+1})^{-1}$ and is negligible in comparison with the earlier terms, which have a sum of order z.

When n is greater than z, and $n-z$ is of higher order than $z^{\frac{1}{3}}$, the functions R_nR_{n+1} become large in comparison with any power of z, and the sum of the terms involving them is not significant †.

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* *Vide Phil. Mag.*, February, 1910.

† A similar problem is discussed in *Phil. Mag.*, April, 1910.

These points are all illustrated very clearly in the numerical tables which follow. Finally, then, when z is large, we may write

$$\frac{p}{2aE} = \frac{2}{z^3} \Sigma_{n=0}^{z} (z^2-n^2)^{\frac{1}{2}} \left[z^2-(n+1)^2\right]^{\frac{1}{2}}. \tag{32}$$

In determining the leading term in the sum of this series, expressed in inverse powers of z, we may replace it by an integral in the usual way, writing $n/z = x$ and $z^{-1} = dx$, so that

$$\frac{p}{2aE} = 2\int_0^1 (1-x^2)\,dx = \tfrac{4}{3},$$

or

$$p = \tfrac{8}{3}aE.$$

This is in accordance with (28), and the formula (27) therefore tends to the proper limiting value for a large obstacle.

The function R_n used above is not identical with the R_n of the earlier paper on radiation pressure. The latter, in the present notation, would be $R_{n+\frac{1}{2}}$ or R_m, m denoting $n+\frac{1}{2}$, as throughout that paper. The functions R_n and ϕ_n possess the properties

$$\left.\begin{aligned} R_{n+1} &= R_{n-1} + \frac{4n^2}{z^2} R_n - \frac{2n}{z}\sqrt{(R_n R_{n-1}-1)} \\ \sin(\phi_n-\phi_{n+1}) &= (R_n R_{n+1})^{-\frac{1}{2}} \end{aligned}\right\}, \tag{33}$$

the proofs of these relations in the earlier paper not being confined to the case in which n is half an odd integer.

The Small Obstacle.

It is also convenient, before proceeding to tabulation, to obtain a formula for the pressure on a very thin obstacle of cylindrical shape, due to its scattering action on the incident radiation. For this purpose attention may be confined to the first term of (27), and we may also write

$$\left.\begin{aligned} Y_0(z) &= 2\left\{\log\frac{z}{2}+\gamma\right\} \\ J_0(z) &= 1, \quad Y_1(z) = -2/z, \quad J_1(z) = 0 \end{aligned}\right\}, \tag{34}$$

where γ is Euler's constant. Thus

$$\frac{p}{2aE} = \frac{8\pi^2}{z^3}\,\frac{z^2}{4}\,\frac{1}{\pi^2+4\,(\gamma+\log\frac{1}{2}z)^2};$$

or, finally, since $z = 2\pi a/\lambda$,

$$p = 2\lambda\pi E/\{\pi^2+4(\gamma+\log \pi a/\lambda)^2\}, \tag{35}$$

and we note that the pressure tends to zero with the radius of the obstacle. The ratio $p/\frac{8}{3}aE$, on the contrary, tends to infinity, exhibiting a marked difference from the behaviour of the corresponding ratio in the case of the sphere.

Numerical Results.

A large amount of numerical work is necessary in the tabulation of the formula (27), and a fairly complete account of it is given, as many of the intermediate figures are useful for other purposes.

In the first two tables the values of the Bessel functions are given. The functions of the logarithmic type $Y_n(z)$ have only been tabulated hitherto for the cases $n = 0$ and $n = 1$. The values for these cases have been taken from B. A. Smith's tables,* and those for higher values of n have been deduced by the recurrence formula

$$Y_{n+1}(z) = \frac{2n}{z} Y_n(z) - Y_{n-1}(z). \tag{36}$$

The notation is different from that of Smith's tables. His function $(\log 2-\gamma)J_n(z)-Y_n(z)$ is identical with the function $-\frac{1}{2}Y_n(z)$ of this paper. As Smith's tables are restricted to a few significant figures, the last figure in the present table for $Y_n(z)$ will sometimes be incorrect to the extent of a unit.

The functions of type $J_n(z)$ involved less calculation. $J_0(z)$ and $J_1(z)$ have been taken from Meissel's tables.† For integer values of z the functions of higher orders are also taken from these tables. But, for non-integral values such as $z = 1{\cdot}5$, it was necessary to calculate them from J_0 and J_1 by the recurrence formula

$$J_{n+1}(z) = \frac{2n}{z} J_n(z) - J_{n-1}(z). \tag{37}$$

* *Phil. Mag.*, 1898, Vol. 1, p. 122. They are reprinted by Jahnke and Emde, *Functionentafeln*, Leipzig (B. G. Teubner).

† Gray and Mathews's *Treatise on Bessel Functions.*

TABLE 1.

z	·1	·4	·5	·6	·8	·9	1·0
J_0	·9975	·9604	·9385	·9120	·8463	·8075	·7652
J_1	·0499	·1960	·2423	·2867	·3688	·4059	·4401
J_2	·0012	·0197	·0306	·0437	·0758	·0946	·1149
J_3	$<10^{-3}$	·0013	·0026	·0044	·0102	·0144	·0196
$-Y_0$	4·820	1·904	1·397	·9692	·2726	−·0176	−·2772
$-Y_1$	20·29	5·595	4·623	3·960	3·073	2·743	2·454
$-Y_2$	401·0	26·07	17·095	12·23	7·410	6·113	5·186
$-Y_3$	$>10^3$	255·1	132·1	77·57	33·98	24·43	18·29
$-Y_4$	$>10^4$	$>10^3$	$>10^3$	763·5	247·4	156·7	104·5

In this and the following table, the values of the Y_n functions have been checked as far as three figures in the last function of any series by means of the relation

$$Y_n J_{n+1} - J_n Y_{n+1} = 2/z.$$

TABLE 2.

z	1·1	1·2	1·5	2	2·5	3	4	5	6
J_0	·7196	·6711	·5118	·2239	−·0484	−·26005	−·3971	−·1776	·1506
J_1	·4709	·4983	·5579	·5767	·4971	·3391	−·0660	−·3276	−·2767
J_2	·1366	·15935	·2321	·3528	·4461	·4861	·3641	·0466	−·2429
J_3	·0257	·0329	·06095	·1289	·2166	·3091	·4302	·3648	·1148
J_4	·0036	·005	·0117	·0340	·0738	·1320	·2811	·3912	·3576
J_5	$<10^{-3}$	·0006	·0016	·0070	·0195	·0430	·1321	·2611	·3621
J_6	$<10^{-4}$	$<10^{-4}$	·0004	·0012	·0042	·0114	·0491	·13105	·2458
J_7	$<10^{-5}$	$<10^{-5}$	$<10^{-3}$	$<10^{-3}$	10^{-3}	·0025	·0152	·0534	·1296
J_8	$<10^{-6}$	$<10^{-6}$	$<10^{-4}$	$<10^{-4}$	$<10^{-3}$	$<10^{-3}$	·0040	·0184	·0565
$-Y_0$	−·5096	−·7166	−1·2016	−1·6034	−1·5648	−1·1840	+·0532	+·9692	+·9054
$-Y_1$	2·193	1·951	1·295	·3364	−·4586	−1·020	−1·2502	−·4646	·5498
$-Y_2$	4·497	3·969	2·929	1·940	1·198	·5040	−·6783	−1·155	−·7221
$-Y_3$	14·16	11·28	6·515	3·544	2·375	1·692	·572	−·460	−1·031
$-Y_4$	72·74	52·42	23·13	8·69	4·50	2·88	1·536	·604	−·3091
$-Y_5$	515	338	116·8	31·22	12·03	5·99	2·50	1·426	·619
$-Y_6$	$>10^3$	$>10^3$	755	147·4	43·62	17·08	4·715	2·247	1·341

The following additional values have also been employed :—

$$Y_7(3) = -62{\cdot}3, \qquad Y_7(4) = -11{\cdot}645, \qquad Y_7(5) = -3{\cdot}968,$$

$$Y_7(6) = -2{\cdot}063,$$

$$Y_8(4) = -36{\cdot}04, \qquad Y_8(5) = -8{\cdot}863, \qquad Y_8(6) = -3{\cdot}473,$$

$$Y_9(5) = -24{\cdot}395, \qquad Y_9(6) = -7{\cdot}198,$$

$$Y_{10}(6) = -18{\cdot}12.$$

Let $$\rho_n = Y_n^2(z) + \pi^2 J_n^2(z),$$

so that $$\rho_n = \frac{2\pi}{z} R_n. \tag{38}$$

Then the values of ρ_n are exhibited in the next two tables :—

TABLE 3.

z	·1	·4	·5	·6	·8	·9	1·0
ρ_0	33·050	12·728	10·643	9·148	7·143	6·436	5·856
ρ_1	411·75	31·679	21·950	16·490	10·786	9·151	7·933
ρ_2	$16{\cdot}10^4$	679·6	292·2	149·6	54·96	37·46	28·20
ρ_3	$>10^6$	$65{\cdot}10^3$	$17{\cdot}10^3$	6017	1154	596·6	334·5
ρ_4	$>10^7$	$>10^6$	$>10^6$	$>10^5$	$>5{\cdot}10^4$	$25{\cdot}10^3$	$11{\cdot}10^3$

TABLE 4.

z	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
1·1	5·3707	6·999	20·409	200·5	5291	$26{\cdot}10^4$	$>10^7$
1·2	4·9591	6·257	16·003	127·25	2746	$13{\cdot}10^4$	$>10^6$
1·5	4·0304	4·750	9·108	42·50	535·0	$14{\cdot}10^3$	$>10^5$
2	3·0660	3·396	4·991	12·72	75·52	974·4	$16{\cdot}10^3$
2·5	2·4715	2·649	3·399	6·103	20·32	144·8	1903
3	2·0693	2·175	2·570	3·806	8·466	35·87	291·7
4	1·5590	1·606	1·7685	2·154	3·140	6·425	22·255
5	1·2505	1·275	1·355	1·525	1·875	2·706	5·220
6	1·0436	1·058	1·104	1·1935	1·358	1·677	2·394

The following additional values were used:—

$$\rho_7(4) = 135{\cdot}6, \qquad \rho_7(5) = 15{\cdot}77, \qquad \rho_7(6) = 4{\cdot}421,$$
$$\rho_8(5) = 78{\cdot}56, \qquad \rho_8(6) = 12{\cdot}09,$$
$$\rho_9(6) = 51{\cdot}82, \qquad \rho_{10}(6) = 328.$$

The fifth and sixth tables exhibit the corresponding values of $1/\rho_n\rho_{n+1}$ for $n = 0, 1, 2, 3, \ldots$.

TABLE 5.

z	$(\rho_0\rho_1)^{-1}$	$(\rho_1\rho_2)^{-1}$	$(\rho_2\rho_3)^{-1}$	z	$(\rho_0\rho_1)^{-1}$	$(\rho_1\rho_2)^{-1}$	$(\rho_2\rho_3)^{-1}$
·1	736.10^{-7}	2.10^{-8}	$<10^{-10}$	·9	·01698	·00292	·00005
·4	·00248	·00005	$<10^{-7}$	1·0	·02153	·00447	·00011
·5	·004281	·000156	2.10^{-7}	1·1	·02667	·00700	·00024
·6	·006629	·000405	10^{-6}	1·2	·03223	·00999	·00051
·8	·01298	·00169	·00002	1·5	·05223	·02311	·00258

The value of $(\rho_3\rho_4)^{-1}$ for $z = 1{\cdot}5$ is only 5.10^{-5}, and it is therefore unnecessary in the above table.

TABLE 6.

z	2	2·5	3	4	5	6
$(\rho_0\rho_1)^{-1}$	·09605	·15274	·22219	·39938	·6272	·9058
$(\rho_1\rho_2)^{-1}$	·05900	·11107	·17890	·35207	·5787	·8564
$(\rho_2\rho_3)^{-1}$	·01575	·04821	·10225	·26255	·4839	·7591
$(\rho_3\rho_4)^{-1}$	·00104	·00806	·03104	·1479	·3498	·6172
$(\rho_4\rho_5)^{-1}$	·00001	·0003	·00329	·0496	·1971	·4391
$(\rho_5\rho_6)^{-1}$	$<10^{-7}$	$<10^{-5}$	$<10^{-3}$	·0070	·0708	·2490
$(\rho_6\rho_7)^{-1}$	$<10^{-9}$	$<10^{-7}$	$<10^{-4}$	·00033	·0122	·0945

The extra values used are

(1) for $z = 5$, $\qquad (\rho_7\rho_8)^{-1} = {\cdot}0008$,

(2) for $z = 6$, $\quad (\rho_7\rho_8)^{-1} = {\cdot}0187$, $\quad (\rho_8\rho_9)^{-1} = {\cdot}0016$, $\quad (\rho_9\rho_{10})^{-1} = {\cdot}00006$.

If S denotes the series

$$S = \sum_{n=0}^{n=\infty} (\rho_n\rho_{n+1})^{-1}, \tag{39}$$

the values of S and of the pressure ratio $p/\frac{8}{3}aE$ are given in the final table, with the corresponding ratios of diameter to wave length. The pressure ratio is $6\pi^2 S/z^3$.

TABLE 7.

z	S	Diameter wave length	$p/\frac{8}{3}aE$	z	S	Diameter wave length	$p/\frac{8}{3}aE$
·1	736.10^{-7}	·032	4·358	1·2	·04273	·382	1·464
·4	253.10^{-6}	·127	2·342	1·5	·07797	·477	1·368
·5	444.10^{-5}	·159	2·102	2	·1718	·637	1·272
·6	$703{\cdot}5.10^{-5}$	·190	1·929	2·5	·3204	·796	1·214
·8	·01469	·255	1·699	3	·5377	·955	1·179
·9	·01995	·286	1·621	4	1·219	1·27	1·128
1·0	·02611	·318	1·546	5	2·3205	1·59	1·099
1·1	·03392	·350	1·509	6	3·941	1·91	1·081

The character of this table is somewhat unexpected. The ratio of the pressure to its limiting value for a large obstacle approaches the value unity in a regular manner as the size of the obstacle increases. This is especially evident if the values are plotted on a curve. The difference in behaviour of the cylinder and sphere is remarkable; for, in the present case, the pressure may be made greater than a bodily action, such as gravitation, in any arbitrarily assigned ratio by taking the radius of the cylinder small enough. This is, of course, only true in practice if a cylinder composed of so small a number of molecules could continue to approximate to a perfect reflector; but it is evident that the ratio can be made very great. The importance of this result to astrophysics is contained in the fact that the cylinder is, perhaps, a better type of obstacle than the sphere for the representation of the state of things which may obtain, for example, in the tail of a comet or in the solar corona.

In order to illustrate further the manner in which the pressure tends towards its limiting value, the case $z = 8$ has been worked out in detail. The calculation by the previous method being very laborious in this case, another method was adopted. The functions R_0, R_1, and R_2 were calculated from the asymptotic series*

$$R_n(z) = 1 + \frac{n^2 - \frac{1}{4}}{z^2}\frac{1}{2} + \frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{z^4}\frac{1.3}{2.4} + \ldots, \qquad (40)$$

* *Phil. Mag.*, February, 1910.

and the corresponding values of R_3, R_4, ..., by the recurrence formula (33). The pressure is then given by

$$p = \frac{4aE}{z} \Sigma_1^\infty (R_n R_{n+1})^{-1}. \tag{41}$$

The results of the calculation are as follows:—

$$\begin{aligned}
R_0 &= \cdot 998062, & R_1 &= 1\cdot 005831, & R_2 &= 1\cdot 021631, \\
R_3 &= 1\cdot 07016, & R_4 &= 1\cdot 12982, & R_5 &= 1\cdot 28546, \\
R_6 &= 1\cdot 4568, & R_7 &= 1\cdot 7607, & R_8 &= 2\cdot 4709, \\
R_9 &= 4\cdot 3223, & R_{10} &= 17\cdot 3521, & (R_0 R_1)^{-1} &= \cdot 99613, \\
(R_1 R_2)^{-1} &= \cdot 97315, & (R_2 R_3)^{-1} &= \cdot 91466, & (R_3 R_4)^{-1} &= \cdot 82708, \\
(R_4 R_5)^{-1} &= \cdot 68854, & (R_5 R_6)^{-1} &= \cdot 53398, & (R_6 R_7)^{-1} &= \cdot 38987, \\
(R_7 R_8)^{-1} &= \cdot 22985, & (R_8 R_9)^{-1} &= \cdot 09363, & (R_9 R_{10})^{-1} &= \cdot 01333,
\end{aligned}$$

and the next function is of order 10^{-3}.

The resulting value of the pressure is given by

$$p/\tfrac{8}{3}aE = 1\cdot 061, \tag{42}$$

and corresponds to an obstacle whose diameter is 2·54 wave lengths. The approach towards the limiting value is still regular.

Part II.

When the incident *magnetic* force is parallel to the axis of the cylinder the analysis proceeds on very similar lines, although the final tabulation is somewhat more troublesome. In the present section of the paper this case is worked out in detail. It is then possible to construct the general case of incidence, of waves travelling perpendicular to the axis, without difficulty and to submit it to examination.

The analysis is given briefly, being analogous throughout to that of the first section. It is now the function ϕ which exists instead of ψ, and the radial, transversal, and axial components of the vectors are

$$\left.\begin{aligned} a = 0, \quad b = 0, \quad c = -ik\phi/V \\ P = -\partial\phi/r\partial\theta, \quad Q = \partial\phi/\partial r, \quad R = 0 \end{aligned}\right\}. \tag{43}$$

The magnetic force in the incident waves is again e^{ikx}, but in a new direction. The time factor is e^{ikVt}, and the corresponding electric force is $-Ve^{ikx}$ perpendicular to the axis and to the incident ray. Its radial

component is $-V\sin\theta e^{ikr\cos\theta}$, which, as in (1), may be expressed in the form

$$P_1 = -\frac{2V}{kr}\Sigma_1^\infty i^{n-1} n \sin n\theta\, J_n(kr), \tag{44}$$

with the magnetic force

$$c = J_0(kr)+2\Sigma_1^\infty i^n J_n(kr)\cos n\theta, \tag{45}$$

so that the function ϕ_1 for the incident waves is given by

$$\phi_1 = \frac{iV}{k}J_0(kr)-\frac{2V}{k}\Sigma_1^\infty i^{n-1}\cos n\theta J_n(kr). \tag{46}$$

With this we associate a formula involving the function $K_n(kr)$ instead of $J_n(kr)$, so that the functions together satisfy the surface condition. The magnetic force is necessarily tangential by the vanishing of ψ. The electric force will be normal if $\partial\phi/\partial r = 0$. This is sufficient to secure the satisfaction of the surface conditions, and, determining the coefficients of the portion of ϕ representing the effect of the obstacle, the final form of ϕ for the external space is

$$\begin{aligned}\phi = {} & \frac{iV}{k}\{J_0(kr)-J_0'(ka)\,K_0(kr)/K_0'(ka)\} \\ & -\frac{2V}{k}\Sigma_1^\infty i^{n-1}\cos n\theta\{J_n(kr)-J_n'(ka)\,K_n(kr)/K_n'(kr)\},\end{aligned} \tag{47}$$

or, in the notation of Part I,

$$\begin{aligned}\phi = {} & \frac{iV}{k}\{J_0(kr)\,Y_0'(ka)-Y_0(kr)\,J_0'(ka)\}/\{Y_0(ka)+i\pi J_0(ka)\} \\ & -\frac{2V}{k}\Sigma_1^\infty i^{n-1}\cos n\theta\{J_n(kr)\,Y_n'(ka)-Y_n(kr)\,J_n'(ka)\} \\ & \qquad /\{Y_n(ka)+i\pi J_n(ka)\}.\end{aligned} \tag{48}$$

If
$$Y_n'(ka)+i\pi J_n'(ka) = M_n e^{i\eta_n}, \tag{49}$$

the radial electric force at the surface is

$$P = -(\partial\phi/r\partial\theta)_{r=a}$$

or
$$\rho = -\frac{4V}{k^2a^2}\Sigma_1^\infty \frac{ni^{n-1}\sin\theta}{M_n}e^{ikVt-i\eta_n}, \tag{50}$$

and the magnetic force is $c = -ik\phi/V$, or

$$c = \frac{2}{kaM_0}e^{ikVt-i\eta_0}+\frac{4}{ka}\Sigma_1^\infty \frac{i^n\cos n\theta}{M_n}e^{ikVt-i\eta_n}, \tag{51}$$

where the usual identical relation between the two types of Bessel function has been employed and the time factor has been inserted. Now the pressure p acting on a unit length of the cylinder, derived as before

from the resultant of the Maxwellian tractions on its outer surface, is given by

$$p = E \iint (c^2 - P^2/V^2) \cos\theta \, dS, \tag{52}$$

the integral being taken over the curved surface, E being the density of incident energy, and average values of the squares being taken. This is equivalent to a line integral round the circular section, and we obtain

$$\frac{p}{2aE} = \int_0^\pi (c^2 - P^2/V^2) \cos\theta \, d\theta. \tag{53}$$

The average values of the squares are

$$\overline{c^2} = \frac{2}{k^2a^2M_0^2} + \frac{8}{k^2a^2} \Sigma_1^\infty \frac{\cos^2 n\theta}{M_n^2} + \frac{8}{k^2a^2M_0} \Sigma_1^\infty \frac{\cos n\theta}{M_n} \cos\left(\eta_0 - \eta_n + n\frac{\pi}{2}\right)$$

$$+ \frac{16}{k^2a^2} \Sigma_1^\infty \Sigma_1^\infty \frac{\cos r\theta \cos s\theta}{M_r M_s} \cos\left\{\eta_r - \eta_s + (s-r)\frac{\pi}{2}\right\}, \tag{54}$$

$$\overline{P^2/V^2} = \frac{8}{k^4a^4} \Sigma_1^\infty \frac{n^2 \sin^2 n\theta}{M_n^2}$$

$$+ \frac{16}{k^4a^4} \Sigma_1^\infty \Sigma_1^\infty \frac{r.s \sin r\theta \sin s\theta}{M_r M_s} \cos\left\{\eta_r - \eta_s + (s-r)\frac{\pi}{2}\right\}, \tag{55}$$

with equal values of r and s excluded from the double summations. With the relations

$$\int_0^\pi (\cos^2 n\theta, \ \sin^2 n\theta) \cos\theta \, d\theta = 0 \quad (n = 0, 1, 2, \ldots),$$

$$\int_0^\pi \cos n\theta \cos\theta \, d\theta = 0 \quad (n \neq 1)$$

$$= \frac{\pi}{2} \quad (n = 1),$$

$$\int_0^\pi \sin r\theta \sin s\theta \cos\theta \, d\theta = 0 \quad (s \sim r \neq 1)$$

$$= \frac{\pi}{4} \quad (s \sim r = 1), \tag{56}$$

we find

$$\int_0^\pi (\overline{P^2/V^2}) \cos\theta \, d\theta = \frac{4\pi}{k^4a^4} \Sigma_1^\infty \frac{n(n+1)}{M_n M_{n+1}} \sin(\eta_{n+1} - \eta_n), \tag{57}$$

$$\int_0^\pi \overline{c^2} \cos\theta \, d\theta = \frac{4\pi}{k^2a^2} \Sigma_0^\infty \frac{1}{M_n M_{n+1}} \sin(\eta_{n+1} - \eta_n), \tag{58}$$

the summations starting at different points; but the first can also be

regarded as starting at $n = 0$. Thus, by (53), with $ka = z$ as usual,

$$\frac{p}{2aE} = \frac{4\pi}{z^2} \Sigma_0^\infty \left(1 - \frac{n.n+1}{z^2}\right) \frac{1}{M_n M_{n+1}} \sin(\eta_{n+1} - \eta_n). \tag{59}$$

But, by (49), $\sin(\eta_{n+1} - \eta_n) = \pi(Y_n J'_{n+1} - J'_n Y_{n+1})/M_n M_{n+1}$,

and it is not difficult to establish the identity

$$Y'_n(z)\, J'_{n+1}(z) - J'_n(z)\, Y_{n+1}(z) = \frac{2}{z}\left(1 - \frac{n.n+1}{z^2}\right), \tag{60}$$

although it has apparently not been explicitly noticed. Accordingly,

$$\sin(\eta_{n+1} - \eta_n) = \frac{2\pi}{z M_n M_{n+1}}\left(1 - \frac{n.n+1}{z^2}\right),$$

and the pressure is finally given by

$$\frac{p}{2aE} = \frac{8\pi^2}{z^3} \Sigma_0^\infty \left(1 - \frac{n.n+1}{z^2}\right)^2 \Big/ \{Y'^2_n + \pi^2 J'^2_n\}\{Y'^2_{n+1} + \pi^2 J'^2_{n+1}\}, \tag{61}$$

and this is the formula which must be tabulated.

Case of a Large Obstacle.

We must first show that this formula approximates to the correct result for a large obstacle when z becomes large. This result is, of course, again $p = \frac{8}{3}aE$. The method employed in Part I is again of service. We introduce the functions R_n and ϕ_n of (29) and of the earlier paper. Then

$$M_n^2 = Y'^2_n + \pi^2 J'^2_n = \tfrac{1}{4}\{(Y_{n-1} - Y_{n+1})^2 + \pi^2 (J_{n-1} - J_{n+1})^2\}$$

$$= \tfrac{1}{4}\left\{\frac{2\pi}{z} R_{n-1} + \frac{2\pi}{z} R_{n+1} - \frac{4\pi}{z}(R_{n-1} R_{n+1})^{\frac{1}{2}} \cos(\phi_{n-1} - \phi_{n+1})\right\}. \tag{62}$$

But for a large value of z successive functions of type R will differ from one another by practically infinitesimal amounts, and thus, when they occur in the *same term* of an infinite series like (61), they may be equated. Accordingly, it is lawful to write

$$M_n^2 = \frac{\pi}{z} R_n [1 - \cos(\phi_{n-1} - \phi_{n+1})].$$

But, under the same conditions,

$$\phi_{n-1}-\phi_{n+1} = (\phi_{n-1}-\phi_n)+(\phi_n-\phi_{n+1}) = 2(\phi_{n-1}-\phi_n),$$

so that, by the recurrence formula (33),

$$1-\cos(\phi_{n-1}-\phi_{n+1}) = 2\sin^2(\phi_{n-1}-\phi_n) = 2/R_nR_{n-1} = 2/R_n^2.$$

Finally, then, we may write in *any single term* of the series (61),

$$M_n^2 M_{n+1}^2 = 4\pi^2/z^2R_n^2; \tag{63}$$

and therefore

$$\frac{p}{2aE} = \frac{2}{z}\Sigma_0^\infty R_n^2\left(1-\frac{n.n+1}{z^2}\right)^2 \tag{64}$$

for a large obstacle. This process is, of course, only valid in the region of variation of n such that R_n does not become large. But there is no necessity to repeat the argument which shewed, in Part I, that the important part of the sum of series such as (61), when z is large, is derived from the terms in which n ranges from unity to a value differing from z by a quantity of lower order. For this range R_n is not large. We may therefore write, in order to obtain the first term of the pressure expressed as a series in descending order of z,

$$\frac{p}{2aE} = \frac{2}{z}\Sigma_0^z R_n^2\left(1-\frac{n.n+1}{z^2}\right)^2, \tag{65}$$

and under the same conditions, by (31),

$$R_n^2 = 1\Big/\left(1-\frac{n^2}{z^2}\right).$$

Replacing this summation by the asymptotically equivalent integral with $n/z = x$ as variable in the ordinary way,

$$\frac{p}{2aE} = 2\int_0^1 (1-x^2)\,dx = \tfrac{4}{3}, \tag{66}$$

or $p = \frac{8}{3}aE$. The formula is therefore verified for a large obstacle.

The Small Obstacle.

As in Part I, we proceed now to a determination of the pressure on a cylinder of small section due to its scattering action on the waves. Attention may now be confined to the first term of (61), in which we may write,

retaining only the main terms,

$$\left.\begin{aligned} Y_1(z) &= -\frac{2}{z}, & Y_0'(z) &= -Y_1 = \frac{2}{z} \\ Y_1'(z) &= \frac{2}{z^2}, & J_0'(z) &= J_0'(z) = 0 \end{aligned}\right\}; \tag{67}$$

and therefore $$\frac{p}{2aE} = \frac{8\pi^2}{z^3} \Big/ \frac{4}{z^2}\,\frac{4}{z^4} = \pi^2 z^3/2,$$

or if λ be the wave length $$p = 8\pi^5 a^4 E/\lambda^3. \tag{68}$$

In this case both the pressure itself and the ratio $p/\frac{8}{3}aE$ tend to zero with the radius of the obstacle in contradistinction to the result in Part I.

Numerical Tables.

We proceed now to a tabulation of the formula (61), the numerical work being somewhat heavier than before, but much facilitated by the intermediate tables of Part I.

Tables 8 and 9 give the values of the derivates of the Bessel functions which are necessary later. These have been calculated from the tables of the functions themselves (1 and 2) by means of the relations, for any argument z,

$$\left.\begin{aligned} 2J_n' &= J_{n-1} - J_{n+1} \\ 2Y_n' &= Y_{n-1} - Y_{n+1} \end{aligned}\right\}. \tag{69}$$

Much labour is saved later by retaining the factor 2 which appears in these relations. The last figure is approximate. J_0' and Y_0' are respectively $-J_1$, $-Y_1$. The tables have been checked by the identity (60).

TABLE 8.

z	$2J_1'$	$2J_2'$	$2J_3'$	$2Y_1'$	$2Y_2'$	$2Y_3'$
·5	·908	·240	·030	15·70	127·5	$>10^3$
·6	·868	·282	·043	11·26	73·61	751·3
·8	·7705	·359	·075	7·137	30·91	240·0
·9	·713	·3915	·093	6·131	21·69	150·6
1·0	·650	·4205	·110	5·463	15·84	99·3

TABLE 9.

z	1·2	1·5	2	2·5	3	4	5
$2J'_1$	·5118	·2797	−·1289	−·4945	−·7461	−·7612	−·2242
$2J'_2$	·4654	·4970	·4478	·2805	·0300	−·4962	−·6924
$2J'_3$	·1543	·2204	·3188	·3723	·3541	·0830	−·3446
$2J'_4$	·0323	·0593	·1219	·1971	·2661	·2981	·1037
$2J'_5$	·005	·012	·0328	·0696	·1206	·2320	·2602
$2J'_6$	$<10^{-3}$	$<10^{-2}$	$<10^{-2}$	10^{-2}	·0405	·1169	·2077
$2Y'_1$	4·686	4·131	3·543	2·763	1·688	−·7315	−2·124
$2Y'_2$	9·330	5·220	3·208	2·834	2·712	1·822	·005
$2Y'_3$	48·45	20·20	6·75	3·302	2·376	2·214	1·759
$2Y'_4$	327	110·3	27·68	9·66	4·30	1·928	1·886
$2Y'_5$	$>10^3$	732	138·7	39·12	14·20	3·179	1·643
$2Y'_6$	$>10^4$	$>10^3$	10^3	2.10^2	56·3	9·145	2·542

The following additional values were employed :—

(1) $z = 4$, $2J'_7 = ·015$, $2Y'_7 = 31·33$, $2Y'_8 = 120·9$;

(2) $z = 5$, $2J'_7 = ·113$, $2J'_8 = ·048$, $2J'_9 = ·017$,

$2Y'_7 = 6·616$, $2Y'_8 = 20·43$, $2Y'_9 = 78·96$, $2Y'_{10} = 302·5$.

Let
$$\sigma_n = 4\{(Y'_n)^2 + \pi^2(J'_n)^2\}. \qquad (70)$$

Then the values of σ_n are exhibited in the next two tables :—

TABLE 10.

z	·5	·6	·8	·9	1·0
σ_0	87·80	65·96	43·144	36·604	31·732
σ_1	254·6	134·2	56·80	42·61	34·014
σ_2	16256	5419	956·7	472·0	252·6
σ_3	$>10^6$	564.10^3	57600	22680	9861

TABLE 11.

z	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
1·1	28·00	28·425	145·2	4657	25.10^4	$>10^8$	$>10^{12}$
1·2	25·03	24·54	89·19	2348	10^5	$>10^7$	$>10^{10}$
1·5	19·00	17·84	29·69	408·5	12170	$>10^5$	$>10^8$
2	13·58	12·72	12·27	46·57	766·3	19240	$>10^6$
2·5	10·596	10·048	8·808	12·27	93·70	1530	$>4.10^4$
3	8·700	8·343	7·364	6·883	19·19	201·8	3170
4	6·424	6·254	5·750	4·970	4·594	10·637	83·77
5	5·100	5·007	4·732	4·266	3·663	3·368	6·888

The following are additional values which were required:—

(1) $z = 4$, $\sigma_7 = 981{\cdot}6$, $\sigma_8 = 15.10^3$

(2) $z = 5$, $\sigma_7 = 43{\cdot}90$, $\sigma_8 = 417{\cdot}3$,

$\sigma_9 = 6234$, $\sigma_{10} = 91500$.

With these values of the σ's the pressure is given from (61) by

$$\frac{p}{2aE} = \frac{128\pi^2}{z^3} \Sigma_0^\infty \left(1 - \frac{n.n+1}{z^2}\right)^2 \Big/ \sigma_n \sigma_{n+1}. \qquad (71)$$

We may denote $(\sigma_n \sigma_{n+1})^{-1}$ by τ_n. Then the requisite values of τ_n are given in the next tables.

TABLE 12.

z	·5	·6	·8	·9	1·0
τ_0	$4{\cdot}474.10^{-5}$	$1{\cdot}130.10^{-4}$	$4{\cdot}081.10^{-4}$	$6{\cdot}480.10^{-4}$	$9{\cdot}263.10^{-4}$
τ_1	$2{\cdot}416.10^{-7}$	$1{\cdot}375.10^{-6}$	$1{\cdot}840.10^{-5}$	$4{\cdot}972.10^{-5}$	$1{\cdot}164.10^{-4}$
τ_3	$<10^{-11}$	$3{\cdot}269.10^{-10}$	$1{\cdot}815.10^{-8}$	$9{\cdot}342.10^{-8}$	$4{\cdot}016.10^{-7}$

TABLE 13.

z	1·1	1·2	1·5	2	2·5	3	4	5
τ_0	$1{\cdot}257.10^{-3}$	$1{\cdot}628.10^{-3}$	$2{\cdot}950.10^{-3}$	$5{\cdot}787.10^{-3}$	$9{\cdot}396.10^{-3}$	·01378	·02489	·03916
τ_1	$2{\cdot}422.10^{-4}$	$4{\cdot}569.10^{-4}$	$1{\cdot}888.10^{-3}$	$6{\cdot}407.10^{-3}$	$1{\cdot}130.10^{-2}$	$1{\cdot}628.10^{-2}$	·02781	·04220
τ_2	$1{\cdot}478.10^{-6}$	$4{\cdot}775.10^{-6}$	$8{\cdot}246.10^{-5}$	$1{\cdot}750.10^{-3}$	$9{\cdot}253.10^{-3}$	$1{\cdot}973.10^{-2}$	·03499	·04954
τ_3	$8{\cdot}6.10^{-10}$	4.10^{-9}	$2{\cdot}012.10^{-7}$	$2{\cdot}802.10^{-5}$	$8{\cdot}698.10^{-4}$	$7{\cdot}571.10^{-3}$	·04380	·06399
τ_4	$<10^{-13}$	$<10^{-12}$	$<10^{-10}$	7.10^{-8}	$6{\cdot}975.10^{-6}$	$2{\cdot}583.10^{-4}$	·02046	·08106
τ_5	$<10^{-19}$	$<10^{-18}$	$<10^{-15}$	$<10^{-10}$	$<10^{-9}$	$1{\cdot}563.10^{-6}$	$1{\cdot}122.10^{-3}$	·04311

Additional values subsequently used are :—

(1) $z = 4$, $\tau_6 = 1{\cdot}216.10^{-5}$, $\tau_7 = 6{\cdot}97.10^{-8}$;

(2) $z = 5$, $\tau_6 = 3{\cdot}31.10^{-3}$, $\tau_7 = 5{\cdot}46.10^{-5}$, $\tau_8 = 3{\cdot}84.10^{-7}$, $\tau_9 = 1{\cdot}75.10^{-9}$.

The final values of $p/\frac{8}{3}aE$, calculated from these data, by the formula

$$p/\tfrac{8}{3}aE = \frac{96\pi^2}{z^3}\,\Sigma_1^\infty \left(1 - \frac{n.n+1}{z^2}\right)^2 \tau_n, \qquad (72)$$

are shewn in the last table.

TABLE 14.

z	·5	·6	·8	·9	1·0	1·1	1·2	1·5	2·0	2·5	3·0	4·0	5·0
Diameter wave length	·159	·190	·255	·286	·318	·350	·382	·477	·637	·796	·955	1·27	1·59
$p/\frac{8}{3}aE$	·429	·621	·898	·987	·998	·985	·957	·900	·940	·934	·950	·946	·965

The table shews that the pressure ratio exhibits a series of maxima and minima while tending towards the value unity on the whole. The principal maximum occurs near $z = 1$, where the ratio is almost exactly unity. This series of maxima and minima is analogous to that which occurs in the case of the sphere which is not perfectly reflecting. It could have been predicted from the type of the formula (61), in which the variation of $z^3 R_n R_{n+1}$ with z is, for any value of n, of the nature of a continuous increase with z. The values become smaller rapidly as n increases when z is small, but if z be larger they tend to the same order of magnitude.

Meanwhile the factor $(1-n.n+1/z^2)^2$ vanishes whenever

$$z^2 = n.n+1,$$

and obliterates the corresponding term, and a definite minimum of the pressure ratio follows. As z increases, the obliterated term is otherwise a smaller fraction of the total, and the minima become less pronounced. The curve of the pressure ratio resembles a wave form of decaying amplitude tending asymptotically to a straight line. The table indicates this character clearly.

A combination of the two solutions yields the solution for the general two-dimensional problem. For, if the incident magnetic force H makes any angle β with the axis, it has components (1) $H\cos\beta$ along the axis, and (2) $H\sin\beta$ perpendicular to the axis and to the direction of propagation. These may be considered separately as in Parts I and II. If for a given wave length and diameter the pressure ratio $p/\frac{8}{3}aE$ becomes A in the case of Part I and B in that of Part II, the pressure in the general case must be

$$p = (A\sin^2\beta+B\cos^2\beta)\tfrac{8}{3}aE.$$

This appears at once from the nature of the analysis. Maxima and minima of this quantity, as a function of β, can only occur when β is zero or 90°; that is to say, they occur in the two cases investigated in detail.

It follows that the cylinder can give extended support to the astrophysical theory based on radiation pressure, but only by virtue of the case discussed in Part I, and cases not too far removed from it. Here it gives valuable support in accordance with the brief discussion given in Part I.

A METHOD OF PROVING CERTAIN THEOREMS RELATING TO ADJOINTNESS

By J. C. FIELDS.

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IN a paper presented to Section A of the British Association for the Advancement of Science, at the Portsmouth meeting, the writer gave simple proofs of a number of theorems in the theory of the algebraic functions. In the paper in question the method of the *deformation of a product* played an important *rôle*. In the present paper this method does not make its appearance, but is replaced by a very simple device which, in the case of certain theorems, enables us to arrive at our results with, perhaps, even greater facility.

Let

$$f(z, u) = u^n + f_{n-1}u^{n-1} + \ldots + f_0 = 0 \tag{1}$$

be an algebraic equation of degree N in (z, u), We assume that $f(z, u)$ presents no repeated factor. In the neighbourhood of a value $z = a$ we have identically

$$f(z, u) = (u - P_1)\ldots(u - P_n) \tag{2}$$

where $P_1, \ldots, P_n$ are power-series in $z - a$. The orders of coincidence of the function $f'_u(z, u)$ with the n branches $u - P_1 = 0, \ldots, u - P_n = 0$ we shall indicate by the symbols $\bar{\mu}_1, \ldots, \bar{\mu}_n$ respectively. The order of coincidence of $f'_u(z, u)$ with the branch $u - P_t = 0$ is evidently also the order of coincidence of the product

$$(u - P_1)\ldots(u - P_{t-1})(u - P_{t+1})\ldots(u - P_n) \tag{3}$$

with this branch.

Suppose the n branches to group themselves into r cycles of orders $\nu_1, \ldots, \nu_r$ respectively. The orders of coincidence of $f'_u(z, u)$ with the branches of these cycles we shall indicate by the symbols $\mu_1, \ldots, \mu_r$ respectively. A function of (z, u) is said to be adjoint for the value $z = a$ if its orders of coincidence with the branches of the several cycles do not fall short of the numbers $\mu_1 - 1 + \frac{1}{\nu_1}$, $\ldots$, $\mu_r - 1 + \frac{1}{\nu_r}$ respectively. To

say of a rational function of (z, u) that it is adjoint for the value $z = a$ is then equivalent to saying that its orders of coincidence with the n branches $u-P_1 = 0, \ldots, u-P_n = 0$ are greater than the numbers $\bar{\mu}_1-1, \ldots, \bar{\mu}_n-1$ respectively.

Consider a rational function of (z, u) in its reduced form

$$H(z, u) = h_{n-1}u^{n-1}+\ldots+h_0. \tag{4}$$

Supposing the function to be adjoint for the value $z = a$, we shall prove that the coefficient of u^{n-1} must be integral with regard to the element $z-a$. Since $H(z, u)$ is adjoint for the value $z = a$, the orders of coincidence of the product $(z-a)^i H(z, u)$ with the n branches $u-P_1 = 0, \ldots, u-P_n = 0$ are greater than the numbers $i+\bar{\mu}_1-1, \ldots, i+\bar{\mu}_n-1$ respectively. We shall assume i to be an integer. By taking i large enough we can make the orders of coincidence of the product with all n branches as large as we like.

We have an identity of the form

$$f(z, u)+(z-a)^i H(z, u) = (u-Q_1)\ldots(u-Q_n), \tag{5}$$

where the Q's are power-series in $z-a$. With indefinite increase of i the orders of coincidence of the product on the right-hand side, with all n branches, increase indefinitely. It follows that, with indefinite increase of i, each one of the n branches must possess an indefinitely large order of coincidence with some one of the factors $u-Q$, and with only one, since two of the branches cannot simultaneously have indefinitely large orders of coincidence with the same factor.

The identity (5) then takes the form

$$f(z, u)+(z-a)^i H(z, u) = \{u-P_1-(z-a)^{\alpha_1}S_1\}\ldots\{u-P_n-(z-a)^{\alpha_n}S_n\}, \tag{6}$$

where each of the series* $S_1, \ldots, S_n$ begins with a constant term which is different from 0 and where with indefinite increase of i the exponents $\alpha_1, \ldots, \alpha_n$ all become indefinitely great. Now when the exponents α_t are all sufficiently large it is evident that the order of coincidence of the product on the right-hand side with the branch $u-P_t = 0$ is given by the lowest exponent in the product

$$(P_t-P_1)\ldots(P_t-P_{t-1})(P_t-P_{t+1})\ldots(P_t-P_n)(z-a)^{\alpha_t}S_t. \tag{7}$$

This order of coincidence is then $\bar{\mu}_t+\alpha_t$. But, from the left-hand side of

* Where $(z-a)^{\alpha_t}S_t = 0$ identically, $H(z, u)$ is divisible by $u-P_t$, and therefore has a factor in common with $f(z, u)$. Here we can write $\alpha_t = \infty$, and the reader will have no trouble in seeing how this case also may be included under the argument.

the identity, we know that this order of coincidence is $> i+\bar{\mu}_t-1$, since by hypothesis $H(z, u)$ is adjoint for the value of the variable z in question. We therefore have $a_t > i-1$. This is true of each of the n exponents a_t.

On subtracting the product $(u-P_1) \ldots (u-P_n)$ from the right-hand side of the identity (6) and $f(z, u)$ from the left-hand side, we now see that the coefficient of u^{n-1} in the reduced form remaining on the right-hand side is represented by a power-series in $z-a$, in which the exponent of the lowest term is $> i-1$. This is then true of the reduced form $(z-a)^i H(z, u)$. In this rational reduced form, therefore, the exponent of the lowest term in the coefficient of u^{n-1} must be $\geqslant i$, whence follows that the coefficient of u^{n-1} in the reduced form $H(z, u)$ must be integral with regard to the element $z-a$. The coefficient of u^{n-1} in the reduced form of a rational function of (z, u) must then be integral with regard to the element $z-a$ if the function is adjoint for the value $z = a$.

If the equation (1) is integral relatively to the element $z-a$, not only will the coefficient of u^{n-1} be integral with regard to this element, in the reduced form of a rational function of (z, u) which is adjoint for the value $z = a$, but the reduced form $H(z, u)$ itself will be integral with regard to the element. In this case, namely, the series $P_1, \ldots, P_n$ corresponding to the value $z = a$ are all integral relatively to the element $z-a$. Also the exponents a_t in the identity (6) are each $> i-1$ if we only take i sufficiently large. On subtracting the product $(u-P_1) \ldots (u-P_n)$ from the right-hand side of the identity, it is then evident that every term in the reduced form which remains is divisible by a power of $z-a$ higher than $(z-a)^{i-1}$. This is then true of the reduced form $(z-a)^i H(z, u)$ on the left-hand side, which must therefore be divisible by $(z-a)^i$, since the reduced form is rational. It follows that the reduced form $H(z, u)$ which is adjoint for the value $z = a$ must be integral with regard to the element $z-a$.

The form of representation employed in (5) can be conveniently utilized to prove the following theorem :—

A rational function of (z, u), which is adjoint relatively to all but one of the n branches corresponding to the value $z = a$, must also be adjoint relatively to the remaining branch if in its reduced form the coefficient of u^{n-1} is integral with regard to the element $z-a$. If the remaining branch in question belongs to a cycle of order > 1, its order of coincidence with a rational function of (z, u) will be the same as the orders of coincidence of the other branches of the cycle with this function, and will therefore, as a matter of course, be adjoint if the orders of coincidence of the other

branches of the cycle are adjoint. Usually, however, a branch constitutes by itself a cycle of order 1.

Suppose $H(z, u)$ to be the reduced form of a rational function which is adjoint relatively to the $n-1$ branches $u-P_2=0, \ldots, u-P_n=0$ corresponding to the value $z=a$. Suppose, furthermore, that the coefficient of u^{n-1} in the reduced form is integral with regard to the element $z-a$, and consider the product $(z-a)^i H(z, u)$, where i is a positive integer which we may choose as large as we will. By choosing i sufficiently large we can make the orders of coincidence of the product with all n branches as large as we like. The orders of coincidence of the product with the $n-1$ branches $u-P_2=0, \ldots, u-P_n=0$ will be greater than the numbers $i+\bar{\mu}_2-1, \ldots, i+\bar{\mu}_n-1$ respectively, since the orders of coincidence of $H(z, u)$ with these branches are greater than $\bar{\mu}_2-1, \ldots, \bar{\mu}_n-1$ respectively. We shall then have an identity of the form (6), in which we can make the exponents a_t as large as we please by choosing i sufficiently large. For i sufficiently large, then, the order of coincidence of the product on the right-hand side of this identity, with a branch $u-P_t=0$, will evidently be $\bar{\mu}_t+a_t$, the lowest exponent in the product (7) represented as a power-series in $z-a$. For $t=2, \ldots, n$ we then have $\bar{\mu}_t+a_t > i+\bar{\mu}_t-1$, and therefore $a_t > i-1$.

On subtracting the product $(u-P_1) \ldots (u-P_n)$ from the right-hand side of the identity (6) and considering the aggregate coefficient of u^{n-1} in the reduced form which remains, we see that the lowest power of $z-a$ which actually appears in this coefficient is $(z-a)^{a_1}$ if $a_1 \leqslant i-1$, since, in this case, the exponents $a_2, \ldots, a_n$ would all have values which are $> a_1$. From the left-hand side, however, the coefficient of u^{n-1} in the reduced form here in question contains no power of $z-a$ lower than $(z-a)^i$. It follows that we must have $a_1 > i-1$. As a consequence the order of coincidence of the product on the right-hand side of (6), with the branch $u-P_1=0$, must be $> \bar{\mu}_1+i-1$. This is then true of the order of coincidence of the function $(z-a)^i H(z, u)$ with this branch. The order of coincidence of the function $H(z, u)$ with the branch $u-P_1=0$ must therefore be $> \bar{\mu}_1-1$. The reduced form $H(z, u)$, which is adjoint relatively to the $n-1$ branches $u-P_2=0, \ldots, u-P_n=0$, must then also be adjoint relatively to the branch $u-P_1=0$ if the coefficient of u^{n-1} is integral with regard to the element $z-a$ in question.

The preceding results, obtained for any finite value $z=a$, evidently hold equally well for the value $z=\infty$ when, throughout the argument, the element $z-a$ is replaced by the element $1/z$. Furthermore, we can prove that a reduced form, which is adjoint for the value $z=\infty$, must be of

degree $\leqslant N-1$. Assuming the identity (2) to have reference to the value $z=\infty$, we see that the degree of the product of less than n of the factors $u-P_t$ must be $\leqslant N-1$, since N is the degree of the product of all n factors. Suppose $H(z, u)$ to be the reduced form of a rational function of (z, u) which is adjoint for the value $z=\infty$. The orders of coincidence of the product $\left(\frac{1}{z}\right)^i H(z, u)$ with the n branches will then be greater than the numbers $i+\bar{\mu}_1-1, \ldots, i+\bar{\mu}_n-1$ respectively. We assume i to be an integer. In analogy with (6) we have an identity of the form

$$(8)\quad f(z, u)+\left(\frac{1}{z}\right)^i H(z, u) = \left\{u-P_1-\left(\frac{1}{z}\right)^{\alpha_1} S_1\right\} \ldots \left\{u-P_n-\left(\frac{1}{z}\right)^{\alpha_n} S_n\right\},$$

where $S_1, \ldots, S_n$ are power-series in $1/z$, integral with regard to this element. In these series we may assume the constant term to be different from 0—at least so long as we do not have

$$\left(\frac{1}{z}\right)^{\alpha_t} S = 0$$

identically, in which case it suffices for our purpose to note that we can write $\alpha_t=\infty$. By choosing i sufficiently large we can make the exponents α_t all as large as we please. On choosing i sufficiently large we see that the order of coincidence of the branch $u-P_t=0$, with the product on the right-hand side of (8), will be given by the lowest exponent in the expansion of the product

$$(P_t-P_1)\ldots(P_t-P_{t-1})(P_t-P_{t+1})\ldots(P_t-P_n)\left(\frac{1}{z}\right)^{\alpha_t} S_t,$$

in powers of $1/z$. This lowest exponent is, however, evidently $\bar{\mu}_t+\alpha_t$, and this must therefore be the order of coincidence of the branch $u-P_t=0$ with the function $\left(\frac{1}{z}\right)^i H(z, u)$. The order of coincidence of the branch $u-P_t=0$ with the function $\left(\frac{1}{z}\right)^i H(z, u)$ is, however, $> i+\bar{\mu}_t-1$, whence it follows that we have $\alpha_t > i-1$. Each of the exponents α_t on the right-hand side of the identity (8) is then $> i-1$.

On subtracting the product $(u-P_1)\ldots(u-P_n)$ from the right-hand side of the identity (8), the aggregate of the terms which remain can be written as the sum of a number of elements, each of which consists of the product of less than n of the factors $u-P_1, \ldots, u-P_n$, multiplied by a power-series in $1/z$ whose first term has an exponent which is $> i-1$.

The aggregate in question must then be of degree

$$< (N-1)-(i-1) = N-i.$$

This aggregate is, however, identical with the reduced form $\left(\frac{1}{z}\right)^i H(z, u)$. The degree of this reduced form is therefore $< N-i$. It follows that the degree of the function $H(z, u)$ is $< N$, and therefore $\leqslant N-1$, since the function is rational. The degree of a rational reduced form then must be $\leqslant N-1$ if the function is adjoint for the value $z = \infty$.

ON MULTIPLE FOURIER SERIES

By W. H. YOUNG, Sc.D., F.R.S.

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and the corresponding relations

$$(s_{C-O})_{m,n} = (S_{C-O})_{m,n} - \frac{n}{m+1} \sum_{r=1}^{m} A_{r,n}, \qquad \text{(IV)}$$

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If $f(x, y)$ is, in a crucial neighbourhood of the point of continuity considered, bounded with respect to (x, y), continuous with respect to x, and has bounded variation with respect to y, the double Fourier series converges in the Cesàro-ordinary way to $f(x, y)$ 180

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is summable, the double Fourier series of $f(x, y)$ converges doubly in the ordinary way to the value $g(0, 0)$, where $g(u, v)$ is the function of § 13 182

31. Concluding remarks 184

1. In a paper published in the *Proceedings* of the Society I have considered, in some detail, the question of the convergence—in the ordinary and in the Cesàro sense—of the Fourier series of a function of a single variable and of its allied series. The principle of the method adopted is that of replacing the function to be considered by one which, except at a single point, has a very simple character, namely, that of being an absolutely convergent indefinite integral. By its means it was possible to obtain in a few pages a large number of new results, as well as fresh proofs of old ones. The full advantage of the method is not, however, completely revealed when we are dealing with functions of a single variable, owing to the fact that the more classical results had already been obtained without it with comparative ease.

When we come to functions of two variables, we are approaching what is much less trodden ground. Moreover, we are dealing with a subject which, as is well known, presents peculiar difficulties. Here, then, our method may be expected to have an opportunity of shewing itself in its real power. Further, the very nature of the considerations employed suggests that, when the step from one variable to two variables has been taken, the complete generalisation for any number of variables will follow. This is, in fact, the case. The difficulties, whether they be difficulties of

conception, or of technique, remain from henceforth the same in character. As regards the former type of difficulty, the enunciations of theorems for two variables hold *mutatis mutandis* for any number of variables. Moreover, the additional properties of the coefficients of a series of Fourier required, when we pass from one dimension to two dimensions, have also a form which is immediately generalisable. As regards difficulties of technique, as soon as we have learnt to deal with functions of bounded variation in two variables, no new difficulty is introduced, except in the matter of complication.

Corresponding to the well known theorem that the coefficients of a Fourier series converge to zero, we have now to introduce the new property that, in the case of an odd-odd function, the sum of the first m coefficients of the n-th row, each divided by the suffix indicating its place in the row, and the sum of the first n coefficients of the m-th column, each divided by the suffix indicating its place in the column, approach zero, besides the original property that the m-th coefficient of the n-th row itself also has the unique double limit zero, as m and n increase without limit. Moreover, we have to generalise these properties themselves by referring them, not to a Fourier series, but to the series got by deriving twice with respect to x and y the double Fourier series of a function of bounded variation which is a double integral only in rectangles not having points common with the axes.

It is evident that these statements have a form that admits of immediate generalisation. On the other hand, a theorem of a somewhat different and more special type, which can at once be generalised for any number of variables, is required, namely, that, if a bounded function of (x, y) be continuous with respect to x, the Cesàro sum of the first m coefficients of the n-th row of its Fourier series has the unique double limit zero.

Turning to the theorems themselves, which form the main object of the present communication, we may at once state them in a form suitable for any number of variables. The two concepts employed are bounded variation with respect to an ensemble and continuity with respect to an ensemble. Let us denote the first ensemble, for brevity, by x and the second by y. Thus, either x or y may be absent or both may be present. Similarly, let us denote by the integration sign with respect to x and y the repeated integration with respect to all the ensemble x and all the ensemble y respectively. There is no distinction in our theory between repeated integrals and double integrals, as all the functions with which we are concerned are summable in the whole domain considered. Then our general theorem asserts that if

(i). $f(x, y)$ is a bounded function, and continuous with respect to y,

(ii) the multiple integral $g(u, v)$, defined below, is at the point $u = 0$, $v = 0$ a continuous function of (u, v) and has with respect to u a total variation which is bounded,

then at such a point (x, y) the multiple Fourier series of f converges to $g(0, 0)$ multiply, when the summation is performed in the ordinary way as regards the x's and the Cesàro way as regards the y's. Here, apart from a numerical factor,

$$g(u, v) = \frac{1}{\sin v} \int_{-v}^{v} [f(x+2u, y+2v)+f(x+2u, y-2v)]\, dv,$$

where the notation is symbolic in the sense explained above and in a further sense easily understood, *e.g.* in connection with the symbolism for the integrand.

I content myself, for the reasons just explained, with giving the proof of this theorem in the case of two variables, which itself includes those in which there are two x's, one x and one y, and two y's respectively.

Besides the Fourier series themselves, there are other series closely allied with them. In the case of a single variable the Fourier series has only one allied series, properly so called. In the case of two variables there are three such allied series obtained by differentiating the given Fourier series once with respect to x, once with respect to y, and successively with respect to x and y, and in each case omitting the numerical coefficients thus introduced and retaining the new signs. The attentive reader of my earlier paper above cited will have no difficulty in stating and proving, on the lines of the present paper, the corresponding theorems for these allied series. Considerations of space have led me to omit them here, but they must be regarded as forming an essential part of the subject-matter. It was, in fact, one of the notable advantages of the mode of discussion carried out in the earlier paper, and here continued, that it dealt almost simultaneously with the Fourier series and its allied series and enabled us, with almost equal ease, to predict and obtain demonstrations of the truth of properties of the allied series analogous to known ones of the Fourier series, besides obtaining simultaneously new properties of both series.

For the same reason I have neglected to deduce the very large number of consequences which follow from the equations here obtained. Moreover, I have thought it unnecessary to do more than indicate some of the various special tests which follow from the above general test as simple consequences.

After what has been done in the paper and in the earlier one, this work of detail presents no difficulty.

Similar remarks are true, but to a much less extent, of the generalisation of the other class of test—for Cesàro convergence—of what may be called the Lebesgue type, given in the earlier paper. A discussion of these must be reserved for a subsequent communication.*

In conclusion, I should like to point out that the properties proved and required of the coefficients of a multiple Fourier series are such as to suggest of themselves various extensions to multiple series of known results for single series. Many interesting properties of single series of constants depend on the convergence or boundedness of the succession formed by multiplying the general term of the series by the index denoting its place, and it is precisely this condition which it will be seen by reference to the paper is generalised for double and multiple series. In fact, the conditions which take the place of this single condition will be actually found to suffice in a number of interesting generalisations of known theorems for single series.

2. I begin by giving the statement and proof of the generalisation for two dimensions of the theorem which I have referred to in my earlier paper as the theorem of Riemann-Lebesgue.†

THEOREM.—*If $f(x, y)$ is a summable function of (x, y) and S is any measurable plane set of points of finite plane content, the four integrals,*

$$\iint_S f(x, y) \begin{matrix} \sin mx \sin ny \\ \cos mx \cos ny \end{matrix} dx\,dy,$$

have the unique double limit zero as m and n increase without limit. In particular the coefficients $a_{m,n}$, $b_{m,n}$, $c_{m,n}$, $d_{m,n}$ of the double Fourier series of $f(x, y)$ converge to zero as m and n increase indefinitely.‡

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* The subject of double and multiple Fourier series has, as a referee kindly pointed out to me, been attacked by several earlier writers, cf. Krause, *Leipz. Ber.*, 1903, pp. 164, 239; G. H. Hardy, *Quar. Jour.*, Vol. XXXVII: A. Vergerio, *Giornale di Matem.*, 1911, p. 181. Their work follows the lines of Jordan's treatment and results in conditions included in those here given as particular cases.

† Cp. for instance Hobson's *Functions of a Real Variable*, § 454, p. 674.

‡ The theorem is still true if only one of the variables becomes infinite, *e.g.*, if $m \to \infty$, using the inequality

$$\left| \int_a^b dx \int_c^d \sin(mx+ny)\,dy \right| = \left| \frac{1}{m} \int_{ma}^{mb} dx \int_c^d \sin(x+ny)\,dy \right| \leqslant \frac{2}{m}(d-c).$$

CASE 1.—Let $f(x, y) = 1.$

Then $$\int_a^b \int_c^d \sin(x+y)\,dx\,dy = \int_a^b \{\cos(c+x) - \cos(d+x)\}\,dx$$
$$= \sin(b+c) - \sin(a+c) - \sin(b+d) + \sin(a+d).$$

Hence the integral on the left is numerically less than 5. Therefore, changing the variables,

$$\left| \int_a^b \int_c^d \sin(mx+ny)\,dx\,dy \right| = \left| \int_{ma}^{mb} \int_{nc}^{nd} \sin(x+y)\,dx\,dy/mn \right| < 5/mn.$$

The same is, of course, true if we change y into $-y$; hence, by addition,

$$\left| \int_a^b \int_c^d \sin mx \cos ny\,dx\,dy \right| < 10/mn.$$

Again, since the integrand is numerically $\leqslant 1$, we have evidently,

$$\left| \iint_S \sin mx \cos ny\,dx\,dy \right| \leqslant S.$$

Now, given e_1 and e_2, independent of each other and as small as we please, and supposing S to be a given measurable plane set, we can first find a closed plane set inside S, say S_1, so that the content of $S - S_1 \leqslant e_1$. We can then find an integer k dependent on e_2 and e_1, so that we can enclose the closed set S_1 in k rectangles $(a_i, c_i;\ b_i, d_i)$ such that the content of the points of these rectangles not belonging to $S_1 \leqslant e_2$. Hence, by our two inequalities,

$$\left| \iint_S \sin mx \cos ny\,dx\,dy \right| \leqslant e_1 + e_2 + 5k/mn.$$

Keeping e_1 and e_2 fixed, let mn increase without limit, then we have

$$\operatorname*{Llt}_{mn \to \infty} \left| \iint_S \sin mx \cos ny\,dx\,dy \right| \leqslant e_1 + e_2.$$

Since e_1 and e_2 are as small as we please, this shews that

$$\operatorname*{Lt}_{mn \to \infty} \iint_S \sin mx \cos ny = 0.$$

Thus, if m and n increase indefinitely in any manner, we get for our integral the unique limit zero.

The above shews that the theorem is true if $f(x, y)$ is constant over the set S.

Hence, if we divide the set S into any finite number k of measurable

sets and define the function $f(x, y)$ to be constant in each of these sets, or, which is the same thing, to be the sum of the k functions, each of which has a constant value at the points of one of these sets and is zero elsewhere, the theorem holds for such a function $f(x)$.

But, if $f(x, y)$ is any bounded function, we can find such a function, say $g(x, y)$, which differs from $f(x, y)$ everywhere by less than a pre-assigned small positive quantity e. Hence

$$\left| \iint_S f(x, y) \sin mx \cos ny \, dx \, dy - \iint_S g(x, y) \sin mx \cos ny \, dx \, dy \right|$$

$$< \left| \iint e \sin mx \cos ny \, dx \, dy \right| < eS.$$

Proceeding to the double limit $m \to \infty$, $n \to \infty$, we have, therefore, for the first integral on the left double limits which are all numerically less than eS. Since e is as small as we please, therefore, the integral in question has the unique double limit zero. This proves the theorem for any bounded function $f(x, y)$.

Finally, if $f(x, y)$ is unbounded, but summable, and we define $g(x, y)$ to be equal to $f(x, y)$, except where the latter is greater than M, and at such points to be zero, and the theorem holds for $g(x, y)$. Also, since $f(x, y)$ is summable, the content of the set of these latter points, say E, is as small as we please, if M is chosen sufficiently large. Hence

$$\left| \iint_S f(x, y) \sin mx \cos ny \, dx \, dy - \iint_S g(x, y) \sin mx \cos ny \, dx \, dy \right|$$

$$= \left| \iint_E f(x, y) \sin mx \cos ny \, dx \, dy \right| \leqslant \iint_E |f(x, y)| \, dx \, dy.$$

But the latter integral vanishes when E approaches zero, and therefore when M increases indefinitely. Hence, taking first the double limit $m \to \infty$, $n \to \infty$, and then letting $M \to \infty$, the first integral on the left has the unique limit zero, which proves the theorem.

COR. 1.—*The above theorem holds when the integration is over an infinite region or a set of points of infinite content.*

For, if $|f(x, y)|$ has an integral over the infinite interval, the double integral of $|f(x, y)|$ over the part of the infinite region outside a certain finite region S inside it, can be made as small as we please by suitably choosing S. Thus, denoting this part by E, and supposing the integral

over E less than e,

$$\left| \iint f(x, y) \sin mx \cos ny \, dx \, dy - \iint_S f(x, y) \sin mx \cos ny \, dx \, dy \right|$$

$$\leqslant \iint_E |f(x, y)| \, dx \, dy < e.$$

Taking the double limit $m \to \infty$, $n \to \infty$, and then letting $e \to 0$, we see that the first integral on the left, viz., that over the infinite region, has zero as unique double limit. This proves the corollary.

COR. 2.—*The convergence of the double integral to its limit zero is uniform.*

For, denoting the integrand by $Q(x, y)$, so that

$$|Q(x, y)| \leqslant |f(x, y)|,$$

$$\int_a^{b+h} \int_c^{d+k} Q(x, y) \, dx \, dy$$

$$= \int_a^b \int_c^d Q(x, y) \, dx \, dy + \int_b^{b+h} \int_c^d Q(x, y) \, dx \, dy + \int_a^b \int_d^{d+k} Q(x, y) \, dx \, dy$$

$$+ \int_b^{b+h} \int_d^{d+k} Q(x, y) \, dx \, dy,$$

replacing $Q(x, y)$ by $|f(x, y)|$ in each of the last three integrals, we do not decrease their numerical values, and we see that they all vanish with h and k, since $f(x, y)$ has a double integral. Hence, by the theorem, the integral on the left has the unique quadruple limit zero when $n \to \infty$, $m \to \infty$, $h \to 0$, $k \to 0$, which proves the corollary.

3. The increment, or double increment, of a function $f(x, y)$ of two variables over the rectangle $(x, y; x+h, y+k)$, whose sides are parallel to the axes of coordinates—to distinguish which it will, in the present paper, sometimes be called a *fundamental* rectangle—and whose left-hand bottom corner, if h and k are positive, is the point (x, y), is the expression

$$F(x+h, y+k) - F(x, y+k) - F(x+h, y) + F(x, y),$$

and may be a positive ($\geqslant 0$) increment, or a negative ($\leqslant 0$) increment, or, when considered only in absolute value, a total increment. The quantities h and k are here not zero. If one of them, say h, is zero, the expression reduces to

$$F(x, y+k) - F(x, y),$$

which, in the language of the theory of functions of a single variable, is

the increment of the function $F(x, y)$, where x is a parameter, or, in the language of the theory of functions of two variables, is the increment with respect to y of the function $F(x, y)$.

It is not proposed here to go into the theory of the double increment and the double variation, which is closely analogous to that of the corresponding quantities in one dimension, and is more or less known to those who have had occasion to use it. It is only intended here to state explicitly properties which are used in the present paper, in which I shall only use the total increment and the total variation, so that the word *total* will sometimes be omitted.

The total increment of $F(x, y)$ over any set of non-overlapping rectangles is the sum of the separate total increments, and the total variation of $F(x, y)$ over a fundamental rectangle is the upper bound of the total increments over all possible such sets of rectangles lying in that fundamental rectangle.

If (a, b) is a point at which $F(x, y)$ is continuous—that is, continuous with respect to the ensemble (x, y)—the variation of $F(x, y)$ over a rectangle $(a, b\,;\ a+h, b+k)$ will have zero as unique double limit as h and k approach zero in any manner whatever. The same is true at such a point of the variations of $F(x, y)$ with respect to x and to y separately, since, if at (a, b) the function is continuous with respect to (x, y), it is continuous with respect to x and with respect to y.*

A function of bounded variation with respect to (x, y) will also be understood to be so defined that it is of bounded variation with respect to x and with respect to y; that is, not only the upper bound of the total variation in the rectangle considered is finite, but also the upper bounds of the variation with respect to x and to y are finite, which are obviously hypotheses independent of the former. The following lemma shows that, as here fully expressed, however, these two latter hypotheses are redundant; it suffices to hypothecate that the variations on one particular parallel to each of the axes are finite, and it will follow from the boundedness of the two-dimensional variation that the one-dimensional variations are bounded throughout the rectangle considered :—

LEMMA.—*If $F(x, y)$ is in a certain square $(0, 0, e, e)$ with the origin as corner a function of bounded variation in (u, v) and has also a bounded variation with respect to y along the edge $x = 0$ of the square, then it has bounded variation with respect to y along every parallel to the axis of y in the square, and this latter variation converges uniformly to zero with e, provided $F(x, y)$ is continuous at the origin.*

* See below, § 4, first footnote.

For

$$|F(x, y+k)-F(x, y)|$$
$$\leqslant |F(x, y+k)-F(x, y)-F(0, y+k)+F(0, y)|+|F(0, y+k)-F(0, y)|.$$

Hence, denoting by V'_x the sum of the total increments on the left of this inequality, taken over any set of non-overlapping intervals on the line in question, and inside the rectangle $(0, 0; e, e)$ and by V' the two-dimensional total increment over the corresponding rectangles, namely, the sum of the two-dimensional increments which form the first of the two terms on the right, we have

$$V'_x \leqslant V'+V'_0.$$

Denoting by V_x, V, and V_0 the total variations of $F(x, y)$ along the parallel to the axis of y in the whole square and along the axis of y, the upper bound of V'_x is V_x, and of V'_0 is V_0, while the upper bound of V' is not greater than its upper bound when x varies in the square, that is, not greater than V. Thus, since the upper bound of a sum is not greater than the sum of the upper bounds,

$$V_x \leqslant V+V_0.$$

Since the right-hand side is independent of x, is bounded, and has zero as unique limit when $e \to 0$, this proves the lemma.

4. Using the words *increment and variation*, then, to mean *total increment and variation*, it is easily proved * that *the variation of $f(x, y)$ over any rectangle is the sum of the variations over any finite number of rectangles into which we may divide it.*

The increment of $f(x, y)$ over any rectangle is, of course, less than or equal to the sum of the increments over such subdivisions.

Hence we get the following lemma, the proof of which is appended, since, though closely analogous to the corresponding proof in one dimension, there are a number of points to be considered which only occur in space of dimensions higher than one.

LEMMA.—*If $f(x, y)$ is a continuous function of (x, y), with bounded variation, then the variation over a fundamental rectangle of area e depends on e in such a way that it has the unique limit zero when e approaches zero in any manner.*

* The argument in one dimension is available with the necessary alterations for space of higher dimensions. I may refer to my paper on "Functions of Bounded Variation," 1910, *Quar. Jour. of Pure and Applied Math.*, Vol. XLII, pp. 54–85.

Let us assume that, if possible, the theorem is not true, and let a countably infinite set of rectangles be given whose areas decrease without limit and over which the variations do not decrease without limit. We may then choose out a subset of the rectangles so that the limit of the variations is unique and is a positive quantity k.

Since the areas approach zero, either the widths, or the heights, of a countably infinite set of the rectangles approach zero. We have, therefore, two typical cases to discuss:

(i) the widths approach zero;

(ii) the widths and the heights both approach zero.

In the former case the vertical sides of the rectangles have one or more limiting lines, and we may choose out such a subset that there is only one such. In the latter case the rectangles have one or more limiting points, and we may choose out such a subset that there is only one such.

We shall take this latter case first. The limiting point may be taken to be internal to all the rectangles since, by enlarging the rectangles, we may ensure that this is the case, without decreasing the variations, or altering the property (ii).

Take two of the rectangles, one lying inside the other, with P internal to both, and such that the variations over both lie between k and $k+e$, all which is possible by the conditions of the case considered. Then, since the variation over the larger rectangle is the sum of that over the smaller rectangle and that over the remaining part, the latter variation must be less than e. Thus, if we divide the outer rectangle up into a finite number of rectangles, say m, so that the sum of the increments over these lies between $k-e$ and $k+e$—which is possible since the variation is the upper bound of such sums—and subdivide these rectangles by means of the bounding lines of the smaller rectangle, the sum of the (total) increments being increased, but not greater than the variation over the larger rectangle, still lies between $k-e$ and $k+e$, and the sum of those outside the smaller rectangle is less than e.

Now, since $f(x, y)$ is continuous at P, we can draw a small square, with P as centre and sides parallel to the axes, so that the increment of $f(x, y)$ over any fundamental rectangle inside this small square is less than e/m. Since the given rectangles shrink up to the point P, we can then determine the second of the small rectangles just discussed so as to lie inside this small square; it will then follow that the sum of the increments

in the sum above considered over the rectangles lying in the smaller of the two given rectangles considered will be less than e, since their number is at most m. Hence the sum of all the increments is less than $2e$, which is not in accordance with the supposition that it lies between $k-e$ and $k+e$, e being chosen beforehand sufficiently small. Thus this case cannot occur.*

We have therefore only to discuss the case when the rectangles shrink up to a stretch PQ on a certain ordinate. By weeding out the rectangles we can ensure that one of three cases occurs—first, PQ is internal to all the rectangles; second, it is external to all the rectangles. But we shall see that both these cases are reducible to the third case when the ordinate on which PQ lies is the left-hand bounding ordinate of all the rectangles. In fact, if the ordinate of PQ divides each rectangle into two rectangles, in one of these the variation is not less than the half of the variation over the whole rectangle. Hence, dividing the rectangles and choosing them suitably, we can ensure that they all have the ordinate of PQ for bounding ordinate and all lie on the same side of it, which we may for definiteness choose to be the right. The limit of the variation can, by further weeding, be taken to be unique, and is a positive quantity, say k.

If, however, PQ is external to the rectangles, we first weed them out so as to lie all on one side of PQ, and then continue them up to the ordinate PQ. This does not reduce the variations. Thus, taking a set of these enlarged rectangles so that the limit of the variations is unique, we have such a set as is sought.

Thus, the case we have to consider is that in which all the rectangles lie on the right of PQ, with the ordinate of PQ for their left-hand boundary. We can then, by weeding out the rectangles, either ensure that the sides of the rectangles which lie on that ordinate all lie in PQ, or not. We shall take the former case first, and it is then evident that we need only consider the case when PQ is itself the side of all the rectangles, since this only means enlarging the rectangles till they are bounded above and below by the parallels to the x-axis through P and Q.

As before, we now take two of the rectangles such that the corresponding variations lie between k and $k+e$, and it follows that the variation over the rectangle which is part of the larger, but not of the smaller, is less than e. We then determine m, such that we can divide up the larger rectangle into m parts, so that the sum of the corresponding increments lies between $k-e$ and $k+e$, and we subdivide these rectangles by

* This proves the statement used above, that if $f(x, y)$ is continuous at P, the variation over any rectangle in a sufficiently small square neighbourhood of P is as small as we please.

the right-hand ordinate of the smaller rectangle. Thus the sum of the increments over the rectangles to the right of this bounding ordinate is less than e.

Since $f(x, y)$ is continuous with respect to (x, y) all along PQ, we can now determine h so that the increment of $f(x, y)$ over any rectangle of width h, or less, is less than e/m. The sum of the increments over those of the m rectangles, or parts of rectangles, lying in the strip so determined is then less than e. Also we can choose the smaller of the two given rectangles, above used, to lie inside this strip. Thus the sum of all the increments over the m rectangles is less than $2e$. Thus, again, this case cannot occur.

There remains, therefore, only the case when the ordinate of PQ bounds all the rectangles on the left, but none of the sides of the rectangles on this ordinate lie inside PQ. We may evidently take it that each such side contains the whole of PQ, since this again only involves enlarging the rectangles suitably. The two rectangles, then, which complete any one of the given rectangles after removing the part cut out by the parallels to the x-axis through P and Q, form two sets of rectangles with P and Q respectively as unique limiting points. Hence the variations over these two parts have zero as limit, and therefore the variation over the part having PQ as left-hand bounding stretch must have k as unique limit. But this has already been proved impossible. Thus, this last case also cannot occur, and by a *reductio ad absurdum* the lemma is proved.*

Hence, using the preliminary inequality employed in the proof of the lemma of § 3, we have the following additional statement:—

If $f(x, y)$ is a continuous function of (x, y), with bounded variation, in a certain fundamental rectangle, then the variation with respect to y along the intercept on any ordinate between two parallels to the axis of x is a continuous function of x.

5. Denoting by $P(x, y)$ and $-N(x, y)$ the positive and negative variations of $f(x, y)$ over the rectangle $(0, 0\,;\ x, y)$, so that $P(x, y)+N(x, y)$ is the total variation of $f(x, y)$ with respect to (x, y) over this rectangle, we easily prove that we have

$$f(x, y)-f(0, y)-f(x, 0)+f(0, 0) = P(x, y)-N(x, y).$$

* This proves that if $f(x, y)$ is continuous with respect to (x, y) along any stretch, the variation of $f(x, y)$ over any rectangle lying in a sufficiently narrow rectangular neighbourhood of that stretch is as small as we please.

Hence, denoting the positive and negative variations of $f(x, y)$ with respect to x and y respectively along the axes by $P_0(x)$, $-N_0(x)$, $P_0(y)$, $-N_0(x)$, we have

$$f(x, y)-f(0, 0) = [P(x, y)+P_0(x)+P_0(y)]-[N(x, y)+N_0(x)+N_0(y)].$$

The two functions of (x, y) in brackets on the right of this equation are, by hypothesis, bounded, since $f(x, y)$ has bounded variation, and they are monotone increasing functions of each variable separately, and of the ensemble (x, y). To bring the terminology used in the sequel into accordance with that of Lebesgue in his most recent paper,* we shall denote by $V(x, y)$ the sum of these two functions, and call it for brevity *the variation*. The functions $f(x, y)$ used by Lebesgue are "functions of an interval" which accounts for the slight difference of usage here pointed out. With this terminology we can at once use Lebesgue's form of the Second Theorem of the Mean,† the enunciation of which is as follows:—

"If E denotes the rectangle of integration, supposed to contain the origin, and M denotes the upper bound of the absolute value of the double integral $\iint h(x, y)\,dx\,dy$ over any and every simple domain containing the origin and contained in E, and V is the variation, as above defined, of $f(x, y)$ in E, then

$$\left| \iint_E f(x, y)\, h(x, y)\, dx\, dy - f(0, 0) \iint_E h(x, y)\, dx\, dy \right| \leqslant MV."$$

A "simple domain containing the origin" means such a region that, if P is a point of it, the fundamental rectangle with P and the origin as opposite vertices lies entirely in the region.

Since any point may be taken as the origin, this formula allows us to approximate to the integral over any rectangle. In applying this theorem in the present paper the point chosen to take the place of the origin is always the left-hand bottom corner (ξ, η) of the rectangle, and the value of the function f there is either zero or depends on a certain quantity e in such a way that $f(\xi, \eta)$ vanishes with e. The formula used is thus

$$\left| \iint_E f(x, y)\, h(x, y)\, dx\, dy \right| \leqslant M[V + |f(\xi, \eta)|],$$

where V is the variation over E in the sense explained in the present article.

* H. Lebesgue, "Sur l'intégration des fonctions discontinues," 1910, *Ann. de l'école normale supérieure*, Série 3, Tome 27, §§ 35 *seq.*, pp. 408 *seq.*

† *Ibid.*, § 55, p. 445.

We shall also have occasion to use the theorems proved by Lebesgue, in the paper just quoted, that the variation of an integral is the integral over the same region or set of points of the absolute value of the integrand,* and that an integral has its integrand for mixed differential coefficient except at a set of points of content zero.†

6. Before proceeding to obtain the further main results of the paper, it will be necessary to enunciate certain preliminary lemmas to be constantly employed.

LEMMA.—*If* $F(x, y)$, $H(x)$, $K(y)$, *be integrals when regarded as functions of* (x, y), *of* x *and of* y *respectively, then*

$$I = \int_a^b \int_c^e F(x, y) \frac{d}{dx} H(x) \frac{d}{dy} K(y)\, dx\, dy$$

$$= F(b, e) H(b) K(e) - F(b, c) H(b) K(c)$$

$$- F(a, e) H(a) K(e) + F(a, c) H(a) K(c) + \int_a^b \int_c^e \frac{d^2}{dx\, dy} F(x, y) H(x) K(y)\, dx\, dy$$

$$- \int_a^b \left(\frac{d}{dx} F(x, e) K(e) - \frac{d}{dx} F(x, c) K(c) \right) H(x)\, dx$$

$$- \int_c^e \left(\frac{d}{dy} F(b, y) H(b) - \frac{d}{dy} F(a, y) H(a) \right) K(y)\, dy.$$

COR. 1.—*If* $H(b) = K(e) = 0$,

then $$I = F(a, c) H(a) K(c) + \int_a^b \int_c^e F_{xy} H(x) K(y)\, dx\, dy$$

$$+ \int_a^b F_x(x, c) K(c) H(x)\, dx + \int_c^e F_y(a, y) H(a) K(y)\, dy.$$

COR. 2.—*If* $K(e) = 0$,

then $$I = - F(b, c) H(b) K(c) + F(a, c) H(a) K(c) + \int_a^b \int_c^e F_{xy} H(x) K(y)\, dx\, dy$$

$$+ \int_a^b F_x(x, c) K(c) H(x) + \int_c^e F_y(a, y) H(a) K(y)\, dy.$$

7. LEMMA 1.—*If the region of integration be a simple one*‡ *and such*

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* Lebesgue, *loc. cit.*, § 19, p. 383.

† *Ibid.*, § 32, p. 405.

‡ *Supra*, § 5.

that it lies entirely between the ordinate $u = \pi$ and the axis of v, then numerically

$$I = \iint \frac{\sin u \sin v}{uv}\, du\, dv$$

is not greater than π^2.

In fact we may write

$$I = \int_{e_1}^{e_2} du \int_{f(u)}^{g(u)} \frac{\sin u \sin v}{uv}\, dv,$$

where $0 \leqslant e_1 \leqslant e_2 \leqslant \pi$, and $g(u), f(u)$ may be infinite. Hence

$$|I| \leqslant \int_{e_1}^{e_2} \frac{\sin u}{u} \left| \int_{f(u)}^{g(u)} \frac{\sin v}{v}\, dv \right| du \leqslant \pi \int_{e_1}^{e_2} \frac{\sin u}{u}\, du \leqslant \pi^2.$$

LEMMA 2.—*For a similar region we have*

$$\left| \iint \begin{matrix} \cos u \to \cos u \\ \searrow \quad \swarrow \\ \sin u \to \sin v \end{matrix} du\, dv \right| \leqslant 8.$$

LEMMA 3.— $$\left| \iint \begin{matrix} \cos u \\ \searrow \\ \sin u \end{matrix} \frac{\sin mv}{v}\, du\, dv \right| \leqslant 4\pi.$$

In fact, in each of the two last lemmas we shall, if a cosine occurs instead of a sine, have to split the integral up into two parts, by dividing the corresponding region of integration where the cosine possibly changes sign.

LEMMA 4.—*If the region of integration lie in the (+, +)-quadrant,*

$$\left| \iint \frac{1}{mu^2 v} \sin^2 mu \sin nv\, du\, dv \right| \leqslant \tfrac{1}{2}\pi^2.$$

In fact, the inner integral with respect to v is numerically $\leqslant \pi$, and the remaining factors of the integrand are positive. Therefore our integral is numerically

$$\leqslant \pi \int \frac{1}{mu^2} \sin^2 mu\, du \leqslant \pi \int \frac{1}{u^2} \sin^2 u\, du \leqslant \tfrac{1}{2}\pi^2.$$

LEMMA 5.—*If the region of integration lie in the (+, +)-quadrant,*

$$\iint \frac{1}{mnu^2 v^2} \sin^2 mu \sin^2 nv\, du\, dv = \tfrac{1}{4}\pi^2.$$

8. I now give the formulæ for the partial summation of the double Fourier series in the various possible modes, ordinary-ordinary, Cesàro-ordinary, and Cesàro-Cesàro, according as the summation with respect to one or both indices is ordinary or Cesàro (index unity). The summation may in each case be effected in a quite straightforward manner, as in the case of one variable.*

Let

$$f(x, y) \sim \tfrac{1}{4}a_{0,0}+\tfrac{1}{2}\sum_{m=1}(a_{m,0}\cos mx+b_{m,0}\sin mx)$$
$$+\tfrac{1}{2}\sum_{n=1}(a_{0,n}\cos ny+b_{0,n}\sin ny)$$
$$+\sum_{m=1}\sum_{n=1}(a_{m,n}\cos mx\cos ny+b_{m,n}\sin mx\sin ny$$
$$+c_{m,n}\cos mx\sin ny+d_{m,n}\sin mx\cos ny)$$
$$\sim \tfrac{1}{4}a_{0,0}+\tfrac{1}{2}\sum_{m=1}(a,\ b,\ x)_{m,0}+\tfrac{1}{2}\sum_{n=1}(a,b,y)_{0,n}+\sum_{m=1}\sum_{n=1}(a,b,c,d,x,y)_{m,n},$$

which we will, for shortness in the present article, write

$$f(x, y) \sim \tfrac{1}{4}a_{0,0}+\tfrac{1}{2}\sum_{m=1}A_{m,0}+\tfrac{1}{2}\sum_{n=1}A_{0,n}+\sum_{m=1}\sum_{n=1}A_{m,n}.$$

Then, writing

$$g(u, v) = f(x+2u, y+2v)+f(x+2u, y-2v)+f(x-2u, y+2v)$$
$$+f(x-2u,\ y-2v),$$

we have

$$g(u, v) \sim 4\{\tfrac{1}{4}a_{0,0}+\tfrac{1}{2}\sum_{m=1}(A_{m,0}\cos 2mu)+\tfrac{1}{2}\sum_{n=1}(A_{0,n}\cos 2nv)$$
$$+\sum_{m=1}\sum_{n=1}A_{m,n}\cos 2mu\cos 2nv\}.$$

Hence $$A_{m,n} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(u, v)\cos 2mu\cos 2nv\,du\,dv,$$

for all values of m and n including zero. We have, therefore, denoting by $s_{m,n}$ the sum of the terms of the Fourier series of $f(x, y)$ such that the indices of the coefficients do not exceed m and n,

$$s_{m,n} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(u, v)\frac{\sin(2m+1)u}{2\sin u}\frac{\sin(2n+1)v}{2\sin v}\,du\,dv$$
$$= \frac{1}{\pi^2}\int_{0}^{\frac{1}{2}\pi}\int_{0}^{\frac{1}{2}\pi} g(u, v)\frac{\sin(2m+1)u}{\sin u}\frac{\sin(2n+1)v}{\sin v}\,du\,dv.$$

* Cp., for example, Hobson's *Functions of a Real Variable*, § 440, p. 646, and § 469, p. 708.

Thus the sum of the double Fourier series, if it converges in the ordinary-ordinary manner, is the unique double limit

$$s_{0-0} = \operatorname*{Lt}_{(m\to\infty,\, n\to\infty)} \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} g(u,v) \frac{\sin mu}{\sin u} \frac{\sin nv}{\sin v}\, du\, dv, \qquad (1)$$

where m and n are odd integers.

Again, since the ordinary sum of the first n quantities, (for $t=1$ to n), $\sum_{r=1}^{m} A_{r,t}$ is $s_{m,n}$, the Cesàro partial summation of the same is

$$\frac{\frac{1}{2}s_{m,0}+s_{m,1}+\ldots+s_{m,n}}{n+1},$$

that is,

$$\frac{1}{\pi^2(n+1)} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} g(u,v) \frac{\sin(2m+1)u}{\sin u \sin v} \left(\sum_{t=1}^{n} \sin(2t+1)v\right) du\, dv$$

$$= \frac{1}{\pi^2(n+1)} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} g(u,v) \frac{\sin(2m+1)u}{\sin u} \left(\frac{\sin(n+1)v}{\sin v}\right)^2 du\, dv.$$

Thus, if this expression has a unique double limit, so that the series converges in the ordinary-Cesàro manner, the ordinary-Cesàro sum of the series is the unique double limit

$$s_{0-C} = \operatorname*{Lt}_{(m\to\infty,\, n\to\infty)} \frac{1}{n\pi^2} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} g(u,v) \sin mu \operatorname{cosec} u \sin^2 nv \operatorname{cosec}^2 v\, du\, dv, \qquad (2)$$

where m is odd, and n assumes all positive integral values.

The formula when the summation with respect to n is ordinary, and that with respect to m is Cesàro, is got from the above by the simple interchange of m and n, n being in this case odd.

Finally, summing with respect to each index in the Cesàro manner, since the sum of the first m Cesàro partial summations of the columns of our array is the ordinary-Cesàro partial summation, we get, taking the mean of these, the Cesàro-Cesàro partial summation

$$\frac{1}{(m+1)(n+1)} \sum_{r=0}^{m} \sum_{t=0}^{n} s_{r,t},$$

whose expression as a double integral is at once got by summing the double integrals expressing the ordinary-Cesàro partial summations and dividing by $(m+1)$. Thus, making the same change as before of $(m+1)$ into m, we see that, if the series converges in the double Cesàro manner,

the sum is the unique double limit

$$s_{C-C} = \underset{(m\to\infty,\, n\to\infty)}{\mathrm{Lt}} \frac{1}{mn\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} g(u, v)\sin^2 mu \operatorname{cosec} u \sin^2 nv \operatorname{cosec} v\, du\, dv, \quad (3)$$

where m and n assume all positive integral values.

9. The following is the first theorem on the convergence of a double Fourier series in the ordinary-ordinary way.

THEOREM.—*The double Fourier series of a function of bounded variation which is a double integral in every fundamental rectangle whose sides do not contain any point of either axis, converges doubly at the origin to the value of the function there, provided the origin is a point of continuity of the function.*

If we refer to the expression for the partial summation of the double Fourier series, we notice that, if for the function we put a constant value, the double integral has that constant for value. Hence, or otherwise, it is evident that we may, without loss of generality, suppose the function which we are considering to have the value zero at the origin. Further, since $uv \operatorname{cosec} u \operatorname{cosec} v$ has the properties hypothecated for the function, and the product of two such functions is a function of the same kind, it is clear that our theorem will have been proved to be correct if we shew that the integral J, where

$$J = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(u, v)\frac{\sin mu}{u}\,\frac{\sin nv}{v}\, du\, dv,$$

has the unique double limit zero, when F is any function of the type considered.

We first remark that, if the zero limits of integration be each replaced by a small positive quantity e, the convergence of the double integral to zero follows by the extended theorem of Riemann-Lebesgue, given in § 2.

We have therefore merely to consider the sum of the three integrals, denoted for brevity by (I), (II) and (III), whose rectangles of integration are

$$(0, 0;\ e, e), \quad (e, 0;\ \tfrac{1}{2}\pi, e), \quad (0, e;\ e, \tfrac{1}{2}\pi).$$

First consider (I), and split up its rectangle of integration into three parts,

$$\left(0, 0;\ \frac{\pi}{m}, e\right), \quad \left(\frac{\pi}{m}, 0;\ e, \frac{\pi}{n}\right), \quad \left(\frac{\pi}{m}, \frac{\pi}{n};\ e, e\right).$$

Denote by $H(u)$ the integral $\int_0^u \frac{\sin t}{t}\, dt$, so that $0 \leqslant |H(u)| \leqslant \pi$.

Then applying the formula of double integration by parts, given in the lemma of § 6, for integration over the third of these rectangles, which we may do, since $F(x, y)$ is a double integral throughout it, we get a linear expression involving terms of the type

$$F(e_1, e_2)\, H(me_3)\, H(ne_4), \qquad \int_{\pi/m}^{e}\int_{\pi/n}^{e} \frac{d^2F}{du\,dv}\, H(mu)\, H(nv)\, du\, dv,$$

$$\int_{\pi/n}^{e} F_v(e, v)\, H(me)\, H(nv)\, dv, \qquad \int_{\pi/m}^{e} F_u(u, e)\, H(mu)\, H(ne)\, du.$$

Now, every such term has a value which may be made as small as we please by suitably choosing e. In the case of the terms of the type of the first, this follows from the continuity of $F(u, v)$ at the origin, and, in the case of the others, it follows from the continuity of the total variation of F, whether regarded as a function of one or of two variables. Thus the double integral represented by the second term is numerically less than

$$\pi^2 \int_{\pi/m}^{e}\int_{\pi/n}^{e} \frac{d^2F}{du\,dv}\, du\, dv,$$

that is, π^2 times the total variation * of $F(u, v)$ in the rectangle of integration; *a fortiori*, therefore, it is less than π^2 times the total variation of $F(u, v)$ in the rectangle $(0, 0; e, e)$. The single integrals actually given are numerically less than a multiple of the total variations of the functions $F(u, e)$ and $F(e, v)$ with respect to u and v respectively in the interval $(0, e)$. But these variations, by our hypothesis with respect to F, and by one of the lemmas of § 4, converge to zero, as e approaches zero. There will be other similar terms, in which π/m and π/n take the place of e; but, as these quantities are less than e, our statement with regard to the magnitude of the terms of the linear expression considered holds good.

Thus, we have proved that the portion of (I) considered is as small as we please, the smallness depending on e.

We next consider the portion of (I) obtained from integration over one of the three remaining parts into which its rectangle has been divided. It will be seen from the nature of the argument that it will be sufficient if we confine our attention to, say, the first, viz.,

$$\int_0^{\pi/m} du \int_0^{e} F(u, v)\, \frac{\sin mu}{u}\, \frac{\sin nv}{v}\, dv.$$

* *Supra*, § 5.

Since $F(u, v)$ is of bounded variation and continuous at the origin, its total variation in the region of integration of this integral is a quantity as small as we please. But, by one of the lemmas of § 6, the upper bound of the double integral of $(\sin mu \sin nv)/uv$ over any simple domain in the region of integration has an upper bound which is certainly not greater than π^2. Hence, by the extended form of the Second Theorem of the Mean,* the integral considered may be made as small as we please by sufficiently diminishing e. Summing up our results so far, we have found that the integral (I) may be made as small as we please by sufficiently diminishing e.

We have now to consider the integrals (II) and (III), and it will evidently be sufficient to consider one of them, say (II).

In its region of integration u does not vanish. Therefore, as $1/u$ has the same properties in this case as F, we may suppose the factor $1/u$ absorbed into F. In other words, it will be sufficient to show that

$$\int_e^{\frac{1}{2}\pi} du \int_0^e F(u, v) \sin mu \frac{\sin nv}{v} dv$$

has the unique double limit zero.

We break up the interval of integration with respect to v into two parts, $(0, \pi/n)$ and $(\pi/n, e)$, and so obtain two double integrals, whose sum is the integral considered. Over the region of integration of one of them F is a double integral, and therefore we may apply the theorem of double integration by parts. In this case, however, as distinguishing it from the previous application, the values of F and of its variation considered are not all of them as small as we please. On the other hand $\sin mu$ has taken the place of $(\sin mu)/u$, and accordingly $(1-\cos mu)/m$ plays now the part of $H(mu)$. As, in consequence, every term of our linear expression contains $1/m$ as a factor and is otherwise bounded, the portion of the integral considered has the unique double limit zero, when m and n increase without limit.

It remains therefore to consider

$$\int_e^{\frac{1}{2}\pi} du \int_0^{\pi/n} F(u, v) \sin mu \frac{\sin nv}{v} dv.$$

Applying to it the extended form of the Second Theorem of the Mean, and

* *Supra*, § 5, $F(\xi, \eta) = F(0, 0)$ is here zero. For the other two integrals treated similarly, $F(\xi, \eta) = F(\pi/m, 0)$ and $F(\xi, \eta) = F(\pi/m, \pi/n)$ respectively, and hence vanish with $1/m$ and $1/n$.

remembering that $F(u, v)$ has bounded variation in the region of integration, while the integral of $(\sin mu \sin nv)/v$ over every simple domain containing the point $(e, 0)$ in the region of integration has an upper bound which is certainly less than $2\pi/m$, we see that the integral has the unique double limit zero, as m and n increase without limit.

Thus, as m and n increase without limit, J has, as we see by summing all the results obtained so far, limits which are all numerically less than a quantity which may be made as small as we please by sufficiently diminishing e; that is, J has the unique double limit zero.

10. From the preceding theorem it at once follows, precisely as for ordinary Fourier series * that *the necessary and sufficient condition that a double trigonometrical series*

$$\sum_{r=1}\sum_{s=1}(a, b, c, d, x, y)_{r,s} = \sum_{r=1}\sum_{s=1}(a_{r,s}\cos rx\cos sy + b_{r,s}\sin rx\sin sy + c_{r,s}\cos rx\sin sy + d_{r,s}\sin rx\cos sy) \quad (1)$$

should have the Fourier form, is that the integrated series, that is the series

$$\sum_{r=1}\sum_{s=1}\frac{1}{rs}(b, a, -d, -c, x, y)_{r,s} = \sum_{r=1}\sum_{s=1}(b_{r,s}\cos rx\cos sy + a_{r,s}\sin rx\sin sy - d_{r,s}\cos rx\sin sy - c_{r,s}\sin rx\cos sy), \quad (2)$$

should converge throughout the whole closed rectangle $(-\pi, -\pi; \pi, \pi)$ *to a function which is an absolutely convergent integral.*

In this case the function which appears in the Fourier form of the coefficients $a_{r,s}$, $b_{r,s}$, $c_{r,s}$, *and* $d_{r,s}$ *may be taken to be any one of the mixed derivatives of the second order of this integral; in other words, the integral in question is the double integral of the function which appears in the Fourier form of the coefficients.*

First, to shew that the condition is necessary, let (1) be a double Fourier series, to prove that (2) converges to the integral of the function $f(x, y)$ corresponding to (1), let us denote by $F(x, y)$ the indefinite integral

$$F(x, y) = \int_0^x dx \int_0^y f(x, y)\, dy. \quad (3)$$

* W. H. Young, "On the Conditions that a Trigonometrical Series should have the Fourier Form," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 421–433.

Then, expressing the Fourier constants in their Fourier form, we have

$$a_{r,s} = \pi^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)\cos rx\cos ry, \tag{4}$$

and similar identities for $b_{r,s}$, $c_{r,s}$, and $d_{r,s}$.

Also, since the terms with index zero are all absent in (1),

$$\begin{aligned} 0 &= \pi^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)\,dx\,dy \\ &= F(\pi, \pi)-F(\pi, -\pi)-F(-\pi, \pi)+F(-\pi, -\pi), \end{aligned} \tag{5}$$

and the right-hand side of (4), and the similar identities, have the value zero when either r or s is zero.

Integrating (4) by parts after multiplying by π^2/rs, we get, using (5),

$$a_{r,s}/rs = \pi^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x, y)\sin rx\sin sy\,dx\,dy = B_{r,s}, \tag{6}$$

using capital letters for the double Fourier constants of $F(x, y)$.

Similarly,
$$b_{r,s}/rs = A_{r,s}, \tag{7}$$
$$-c_{r,s}/rs = D_{r,s}, \tag{8}$$
$$-d_{r,s}/rs = C_{r,s}, \tag{9}$$

which proves that the coefficients in (2) are the double Fourier constants of $F(x, y)$, and therefore that (2) is the double Fourier series of $F(x, y)$, since, as follows from the equation (5) and the remaining identities mentioned immediately after it, the Fourier constants of $F(x, y)$ which have one or both indices zero are themselves zero.

But $F(x, y)$ is a double integral, and therefore, by § 9, equal to its double Fourier series. This proves the necessity of the condition.

Next to prove its sufficiency—that is, assuming (2) to converge to $F(x, y)$, which is the double integral of $f(x, y)$—to show that (1) has the Fourier form and that $f(x, y)$ is the corresponding function. It will be convenient to use small letters with dashes for the double Fourier constants of $f(x, y)$, and large letters for those of $F(x, y)$; that is, for the coefficients in (2), so that the equations (6)–(9) still hold. Also, when either r or s is zero, the corresponding coefficients $A_{r,s}$, $B_{r,s}$, $C_{r,s}$, and $D_{r,s}$ are all zero. Integrating the double integrals which occur in these equations again by parts, we get

$$a_{r,s} = rsB_{r,s} = a'_{rs},$$

and similarly we recognize the identity of the other Fourier constants of $f(x, y)$ with the coefficients of the series (1). This proves the theorem.

It should be noticed that the theorem is equally true when the double trigonometrical series contains the terms at the beginning which were here omitted, since the theorem is true in one dimension. The function to which the integrated Fourier series then converges is still one of the indefinite integrals of the function corresponding to the Fourier series itself.

Hence also it follows, as in one dimension, that *a Fourier series remains a Fourier series if any finite number of terms be omitted.*

11. We now pass to a theorem concerning the mode of convergence to zero of the coefficients of a function of the type considered, hypothecating, however, continuity not only at the origin, but also elsewhere.

THEOREM.—*The double Fourier series of an even-even continuous function of bounded variation which is a double integral in every fundamental rectangle whose sides do not contain any point of either axis has the property that, if* $a_{m,n}$ *is the typical coefficient of the series,* $mna_{m,n}$ *has the unique double limit zero.*

We have to consider the integral

$$mn \int_0^\pi \int_0^\pi F(u, v) \cos mu \cos nv \, du \, dv.$$

We first remark that, if the zero limits of integration be replaced by a convenient small positive quantity e, the required property of the corresponding expression will follow by repeated integration by parts, since at the upper limits of integration $\sin mu$ and $\sin nv$ vanish. In fact, we thus get

$$F(e, e) \sin me \sin ne + \int_e^\pi F_u(u, e) \sin mu \sin ne \, du + \int_e^\pi F_v(e, v) \sin me \sin nu \, dv$$

$$+ \int_e^\pi \int_e^\pi F_{uv}(u, v) \sin mu \sin nv \, du \, dv.$$

Here all the terms but the first have the unique double limit zero, by the extended theorem of Riemann-Lebesgue, and the first term is as small as we please, the smallness depending only on our choice of e, provided, as in the preceding article, we have so arranged that $F(u, v)$ is zero at the origin.

Hence we have only to consider the sum of the three integrals (I), (II), and (III), where the limits of integration are $(0, 0\,;\ e, e)$, $(e, 0\,;\ \pi, e)$, $(0, e\,;\ e, \pi)$.

Let us first consider (I), viz.,

$$\int_0^e \int_0^e F(u, v) \cos mu \cos nu \, m\,du\,n\,dv.$$

We split up the region of integration into four rectangles meeting at the point π/m, π/n.

To the integral corresponding to the part farthest from the origin we apply the theorem of repeated integration by parts.

We thus get an expression which is numerically less than a linear function of values of F and of its total variation inside the rectangle $(0, 0; e, e)$. This is as small as we please, and depends only on our choice of e.

The integrals over the remaining three rectangles we treat in the same manner as in the work of § 9. The argument turns on the fact that either $\cos mu$, or $\cos nu$, is positive for the range of values considered. This enables us to assert that the corresponding integrals are certainly less than a linear expression, such as that just described. We use, of course, the extended Second Theorem of the Mean, and remark that the integrals over the simple domains are now of the type

$$\iint \cos mu \cos nv \, m\,du\,n\,dv,$$

and by Lemma 2 of § 7 are numerically not greater than 8, since the limits of integration with respect to u lie between 0 and π/m.

We next consider (II), viz.,

$$\int_e^\pi m\,du \int_0^e F(u, v) \cos mu \cos nv \, n\,dv.$$

We break up the rectangle of integration into two parts $(e, 0; \pi, \pi/n)$ and $(e, \pi/n; \pi, e)$. By the extended form of the Second Theorem of the Mean and one of the lemmas of § 7, the integral over the first of these rectangles is not greater than eight times a quantity which vanishes with e. To the integral over the second rectangle we apply the theorem of repeated integration by parts, and get $\sin me \sin mu$ vanishes when $u = \pi$ and $\sin nv$ when $v = \pi/n$,

$$-F(e, e) \sin me \sin ne + \int_e^\pi dv \int_{\pi/n}^e F_{uv}(u, v) \sin mu \sin nv \, du$$

$$-\int_e^\pi [F_u(u, e) \sin ne - F_u(u, \pi/n)] \sin mu \, du$$

$$+\int_{\pi/n}^e F_v(e, v) \sin me \sin nv \, dv.$$

The first two terms are evidently as small as we please. If in the third term, we put the sines both unity and write for F_{uv} its modulus, by which means we shall increase the numerical value of the integral, we get the variation of $F(u, v)$ in the rectangle considered, which, by the first lemma of § 4, is as small as we please, since F is continuous and of bounded variation, and this rectangle has an area as small as we please.

In the third term we may replace $F_u(u, e) \sin ne$ by $F_u(u, 0)$ without altering the double limit, since the part left out and the part introduced both correspond to portions of the integral which have, by the ordinary theorem of Riemann-Lebesgue, the unique limit zero.

Now the function $F(u, v) - F(u, 0)$ is zero along the axis of v, and hence, by § 4, the total variation with respect to u of $F(u, v) - F(u, 0)$ in the interval $0 \leqslant u \leqslant \pi$ is a continuous function of v, and therefore has, as v approaches zero in any manner, the unique limit zero. But the third term so modified is, as is evident if we take the integrand positively and put $\sin mu$ equal to unity, less than the total variation of $F(u, \pi/n) - F(u, 0)$ in the interval $0 \leqslant u \leqslant \pi$. Hence, as π/n is less than e, the third term we are considering has a value which may be made as small as we please by sufficiently diminishing e.

Finally, consider the fourth term. This is clearly numerically less than the total variation of the function $F(e, v)$ in the interval $(0, e)$, and is therefore as small as we please. Thus, (II) and similarly (III), is as small as we please.

Summing up all the results we have obtained, we find that the double limits of our expression are certainly as small as we please, and therefore uniquely zero.

12. THEOREM.—*If $F(u, v)$ is an even-even continuous function of bounded variation which is a double integral in every fundamental rectangle not containing any point of either axis, then*

$$n\{\tfrac{1}{2}a_{0,n} + a_{1,n} + \ldots + a_{m,n}\}$$

has the unique double limit zero, when m and n increase indefinitely, $a_{r,s}$ denoting the typical double Fourier constant of $F(u, v)$.

As before, we may suppose, without loss of generality, that the function has the value zero at the origin.

Bearing in mind the expressions for the coefficients, the summation in question is equal to

$$n \int_0^{\frac{1}{2}\pi} du \int_0^{\frac{1}{2}\pi} F(u, v) \sin(2m+1)u \operatorname{cosec} u \cos nv \, dv.$$

Since u cosec u has the same properties as F, we shall have proved that this has the unique double limit zero if we prove that

$$J = n \int_0^{\frac{1}{2}\pi} du \int_0^{\frac{1}{2}\pi} F(u, v) \frac{\sin mu}{u} \cos nv \, dv,$$

where m is odd and n is even, has, for all functions F of the type considered, the unique double limit zero.

If we replace the lower limit of integration with respect to u by e we may prove, as follows, that the expression so obtained has the unique double limit zero.

Dividing the limits of integration with respect to v into two parts at $v = \pi/n$ and absorbing the factor $1/u$ into $F(u, v)$, we have first, by the Extended Second Theorem of the Mean,

$$\left| \int_e^{\frac{1}{2}\pi} du \int_0^{\pi/n} F(u, v) \sin mu \cos nv \, n \, dv \right| \leqslant \frac{2}{m} [V + |F(e, 0)|],$$

where V is the variation of $F(u, v)$ over the rectangle of integration.

Since, by hypothesis, $F(u, v)$ has bounded variation, this integral has the unique double limit zero. Secondly, integrating by parts over the other rectangle, in which $F(u, v)$ is a double integral, we have, since $\sin nv$ is zero at both limits of integration and $\cos mu$ at one of them,

$$\int_e^{\frac{1}{2}\pi} du \int_{\pi/n}^{\frac{1}{2}\pi} F(u, v) \sin mu \cos nv \, n \, dv$$

$$= \int_e^{\frac{1}{2}\pi} du \int_{\pi/n}^{\frac{1}{2}\pi} F_{uv}(u, v)(-\cos mu) \sin nu \, dv + \int_{\pi/n}^{\frac{1}{2}\pi} F_v(e, v)(-\cos me) \sin nv \, dv,$$

which is numerically

$$\leqslant \frac{1}{m} \int_e^{\frac{1}{2}\pi} du \int_{\pi/n}^{\frac{1}{2}\pi} |F_{uv}(u, v)| \, dv + \frac{1}{m} \int_{\pi/n}^{\frac{1}{2}\pi} |F_v(e, v)| \, dv.$$

Since these integrals represent variations of $F(u, v)$ with respect to (u, v) and v respectively, it follows that this double integral also has the unique double limit zero. Hence also the statement made above is true, namely, we may replace the upper limit of integration with respect to u by e without altering the double limits of J.

We have therefore merely to consider the integral obtained from J by writing e for the upper limit of integration with respect to u. Splitting the corresponding region up into two rectangles $(0, 0 ; e, e)$ and $(0, e; e, \frac{1}{2}\pi)$, let us consider the corresponding parts of our integral separately, denoting them for convenience by (I) and (II). First consider (I). We now

subdivide the square $(0, 0; e, e)$ into three rectangles $(0, 0; \pi/m, e)$, $(\pi/m, 0; e, \pi/n)$, $(\pi/m, \pi/n; e, e)$.

In discussing the integral over the first of these three rectangles we use the extended form of the Second Theorem of the Mean and Lemma 4, bearing in mind that $(\sin mu)/u$ is positive in the region of integration, while F has in that region bounded variation of value as small as we please. Thus the corresponding integral has a value as small as we please. We deal similarly with the second rectangle, using now the corollary to Lemma 4, since $\cos nv$ is positive in half the range of integration with respect to v and negative in the other half. In the case of the third rectangle we use the extended theorem of integration by parts, and so get a linear expression involving terms containing F only and single and double integrals whose integrands contain F_{uv}, F_u, and F_v as factors. The other factors are, with the notation of § 9, $H(mu)$ and $\sin nv$, or particular values of these functions. By an argument precisely similar to that used in the article referred to we are thus able to shew that the expression is numerically less than another linear expression containing only the modulus of F and its total variation in the small rectangle $(0, 0; e, e)$. Thus this third integral can be made as small as we please by choosing e sufficiently small.

We have next to consider (II), viz.,

$$n \int_0^e du \int_e^{\frac{1}{2}\pi} F(u, v) \cos nv \frac{\sin mu}{u} dv.$$

We split up the interval of integration of u into two parts, $(0, \pi/m)$ and $(\pi/m, e)$, and so express the double integral as the sum of two. Consider the first of these; to this we apply the extended Second Theorem of the Mean, and bear in mind that, since $F(u, v)$ is at the point $(0, e)$ as small as we please, and that, since it has bounded variation, its total variation over the thin region of integration may be made as small as we please, by taking e sufficiently small. The corresponding portion of the whole integral considered has, accordingly, a value as small as we please, using one of the lemmas of § 7. It remains to consider

$$n \int_{\pi/m}^e du \int_e^{\frac{1}{2}\pi} F(u, v) \cos nv \frac{\sin mu}{u} dv.$$

Here $F(u, v)$ is a double integral in the region considered, and we may therefore employ the extended theorem of integration by parts. Bearing

in mind that m is odd and n is even, this gives us

$$2F(e, e)\,H(me)\sin ne + F(\pi/m, e)\,H(\pi)\sin ne$$
$$+\int_{\pi/m}^{e} du \int_{e}^{\frac{1}{2}\pi} F_{uv}(u, v)\sin nv\,H(mu)\,dv$$
$$+\int_{\pi/m}^{e} F_u(u, e)\sin ne\,H(u)\,du$$
$$-\int_{e}^{\frac{1}{2}\pi} \{F_v(e, v)\,H(me) - F_v(\pi/m, v)\,H(\pi)\}\sin nv\,dv.$$

The first two terms are evidently as small as we please. The third term is numerically less than the total variation of $F(u, v)$ over the small rectangle constituted by its region of integration, multiplied by a constant, and is therefore, to a constant factor *près*, numerically less than the total variation over the rectangle $(0, e;\ e, \frac{1}{2}\pi)$, and can therefore be made as small as we please, by choosing e sufficiently small. The fourth term is less than a constant multiple of the total variation of $F(u, e)$ in the interval $(0, e)$, and may also therefore be made as small as we please. The last integral breaks up naturally into two parts, of which the first obviously has the unique limit zero by the theorem of Riemann-Lebesgue.

Applying the same theorem a second time, we see that we may introduce into the integrand $F_v(0, v)H(\pi)$ instead of $F_v(e, v)H(me)$. Hence the last integral is, to a constant factor *près*, numerically less than the total variation with respect to v of the function $F(\pi/m, v) - F(0, v)$ in the interval $(e, \frac{1}{2}\pi)$. But this is the variation with respect to v of the function $F(u, v) - F(0, v)$ on the ordinate $u = \pi/m$ and inside the rectangle $(0, e;\ e, \frac{1}{2}\pi)$, since π/m is less than e. But $F(u, v) - F(0, v)$ is zero along the axis of v; therefore its variation, by the conclusion of § 4, being a continuous function of u, is as small as we please. Summing up the results obtained and applying the usual argument, we see that our theorem is true.

13. Let $f(u, v)$ be a summable function of (u, v), rendered if necessary periodic, and for convenience supposed such that its double Fourier series has no terms independent of u or v, so that we may write

$$f(u, v) \sim \sum_{r=1}\sum_{s=1} (a_{r,s}\cos ru\cos sv + b_{r,s}\sin ru\sin sv$$
$$+ c_{r,s}\cos ru\sin sv + d_{r,s}\sin ru\cos sv),$$
$$\sim \sum_{r=1}\sum_{s=1} (a, b, c, d, u, v)_{r,s}.$$

Therefore, as has been shown in § 10, denoting a suitable indefinite double integral of $f(u, v)$ by $F(u, v)$,

$$F(u, v) = \frac{1}{rs} \sum_{r=1} \sum_{s=1} (a_{r,s} \sin ru \sin sv + b_{r,s} \cos ru \cos sv - c_{r,s} \sin ru \cos sv - d_{r,s} \cos ru \sin sv),$$

and therefore

$$[F(x+2u, y+2v) - F(x-2u, y+2v) - F(x+2u, y-2v) + F(x-2u, y-2v)] = \frac{4}{rs} \sum_{r=1} \sum_{s=1} (a, b, c, d, x, y)_{r,s} \sin 2ru \sin 2rv.$$

Hence, if we multiply the left-hand side by $\sin 2ru \sin 2sv$ and integrate with respect to u and v over the rectangle of periods, the result will be

$$\frac{4\pi^2}{rs} (a, b, c, d, x, y)_{r,s}.$$

Writing

$$g(u, v) = \frac{[F(x+2u, y+2v) - F(x-2u, y+2v) - F(x+2u, y-2v) + F(x+2u, y+2v)]}{16 \sin u \sin v},$$

$g(u, v)$ is an even-even function which changes sign when we add or subtract π from one of the variables u and v. Hence we may write

$$g(u, v) \sim \sum_{r=1} \sum_{s=1} A_{r,s} \cos(2r-1)u \cos(2s-1)v.$$

We have therefore

$$A_{r,s} - A_{r,s+1} - A_{r+1,s} + A_{r+1,s+1}$$

$$= \frac{1}{\pi^2} \int_{-\pi}^{\pi} dv \int_{-\pi}^{\pi} g(u, v) \{\cos(2r-1)u \cos(2s-1)v - \cos(2r+1)u \cos(2s-1)v - \cos(2r-1)u \cos(2s+1)v + \cos(2r+1)u \cos(2s+1)v\} du$$

$$= \frac{4}{\pi^2} \int_{-\pi}^{\pi} dv \int_{-\pi}^{\pi} g(u, v)(\sin u \sin v \sin 2ru \sin 2sv)\, du$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} dv \int_{-\pi}^{\pi} [F(x+2u, y+2v) - F(x-2u, y+2v) - F(x+2u, y-2v) + F(x-2u, y-2v)] \sin 2ru \sin 2sv\, du$$

$$= \frac{1}{rs} (a, b, c, d, x, y)_{r,s}. \qquad (a)$$

Hence, taking into account the missing terms in the double Fourier series of $f(u, v)$, and denoting by $s_{m,n}$ the partial summation

$$\sum_{r=1}^{m-1}\sum_{s=1}^{n-1}(a, b, c, d, x, y)_{r,s},$$

and by $S_{m,n}$ the partial summation

$$\sum_{s=1}^{n}\sum_{r=1}^{m}A_{r,s},$$

we have

$$\begin{aligned}
s_{m,n} &= \sum_{r=1}^{m-1}\sum_{s=1}^{n-1}(a, b, c, d, x, y)_{rs} \\
&= \sum_{r=1}^{m-1} r \sum_{s=1}^{n-1} s\,(A_{r,s}-A_{r,s+1}-A_{r+1,s}+A_{r+1,s+1}) \\
&= \sum_{r=1}^{m-1} r\left(\sum_{s=1}^{n-1} s(A_{r,s}-A_{r,s+1})\right)-\sum_{s=1}^{n-1} s\,[(A_{r+1,s}-A_{r+1,s+1})] \\
&= \sum_{r=1}^{m-1} r\,[(A_{r,1}+A_{r,2}+\ldots+A_{r,n}-nA_{r,n})] \\
&\qquad -[(A_{r+1,1}+A_{r+1,2}+\ldots+A_{r+1,n}-nA_{r+1,n})] \\
&= \sum_{r=1}^{m-1}[r(A_{r,1}-A_{r+1,1})+r(A_{r,2}-A_{r+1,2})+\ldots+r(A_{r,n}-A_{r+1,n}) \\
&\qquad -rn(A_{r,n}-A_{r+1,n})] \\
&= \sum_{s=1}^{n}\sum_{r=1}^{m-1} r(A_{r,s}-A_{r+1,s})-n\sum_{r=1}^{m-1} r(A_{r,n}-A_{r+1,n}) \\
&= \sum_{s=1}^{n}(A_{1,s}+A_{2,s}+\ldots+A_{m,s}-mA_{m,s}) \\
&\qquad -n(A_{1,n}+A_{2,n}+\ldots+A_{m,n}-mA_{m,n}) \\
&= \sum_{s=1}^{n}\sum_{r=1}^{m}A_{r,s}-m\sum_{s=1}^{n}A_{m,s}-n\sum_{r=1}^{m}A_{r,n}+mnA_{m,n} \qquad \text{(I)} \\
&= S_{m,n}-m\,(S_{m,n}-S_{m-1,n})-n\,(S_{m,n}-S_{m,n-1}) \\
&\qquad +mn(S_{m,n}-S_{m-1,n}-S_{m,n-1}+S_{m-1,n-1}) \\
&= (m-1)(n-1)\,S_{m,n}-m\,(n-1)\,S_{m-1,n}-n\,(m-1)\,S_{m,n-1} \\
&\qquad +mn\,S_{m-1,n-1}.
\end{aligned}$$

14. We have now to calculate the double summation of $s_{m,n}$ from 1 to m and 1 to n. The term $S_{m,n}$ will, of course, occur only once, and each

term of the type $S_{m,t}$, where t is less than m, only twice, so that its coefficient in the final sum will be

$$(m-1)(t-1)-(m-1)(t+1) = -2(m-1).$$

Similarly the coefficient of each term of the type $S_{r,n}$ is $-2(n-1)$. On the other hand, the remaining terms, whose indices are respectively $(m-r)$ and $(n-t)$, less than m and less than n, have for coefficient in the final expression

$$\begin{aligned}(m-r-1)(n-t-1)-(m-r+1)(n-t-1)-(n-t+1)(m-r-1)&\\ +(2-r+1)(n-t+1)&\\ = -2(n-t-1)+2(n-t+1) = 4,&\end{aligned}$$

this formula holding also for the earlier terms, for which $m-r-1$ or $n-t-1$ may be zero. Thus we get the formula

$$\begin{aligned}\sum_{r=1}^{m}\sum_{t=1}^{n} s_{r,t} &= (m-1)(n-1)S_{m,n}-2(n-1)(S_{1,n}+\ldots+S_{m+1,n})\\ &\quad -2(m-1)(S_{m,1}+\ldots+S_{m,n-1})\\ &\quad +4\sum_{r=1}^{m-1}\sum_{t=1}^{n-1} S_{r,t}\\ &= (m+1)(n+1)S_{m,n}-2(1+n)(S_{1,n}+\ldots+S_{m,n})\\ &\quad -2(1+m)(S_{m,1}+\ldots+S_{m,n})+4\sum_{r=1}^{m}\sum_{t=1}^{m} S_{r,t}.\end{aligned}$$

Hence, dividing by $(m+1)(n+1)$, and denoting the Cesàro summations which occur on the left and right respectively by $t_{m,n}$ and $T_{m,n}$, we get the formula

$$\begin{aligned}t_{m,n} = 4T_{m,n}+S_{m,n}-&\frac{2}{m+1}\{S_{1,n}+S_{2,n}+\ldots+S_{m,n}\}\\ &-\frac{2}{n+1}\{S_{m,1}+S_{m,2}+\ldots+S_{m,n}\},\end{aligned}\qquad \text{(II)}$$

which may be written, with the notation of § 8,

$$[S_{C-C}]_{m,n} = [S_{C-C}]_{m,n}+S_{m,n}-2[S_{C-O}]_{m,n}-2[S_{O-C}]_{n,n},$$

these subscripts indicating partial summations.

15. Now

$$\sum_{t=1}^{n}\left(t\sum_{r=1}^{m} A_{r,t}\right) = \sum_{t=1}^{n} t(S_{m,t}-S_{m,t-1}) = -(S_{m,1}+S_{m,2}+\ldots+S_{m,n})+nS_{m,n}.$$

Hence the last term in (II) may be written

$$\frac{2}{n+1}\sum_{t=1}^{n}\left(t\sum_{r=1}^{m}A_{r,t}\right)-\frac{2n}{n+1}S_{m,n}.$$

A similar transformation being made for the preceding term, the equation (II) takes the following form

$$t_{m,n}=4T_{m,n}-\left(3-\frac{2}{m+1}-\frac{2}{n+1}\right)S_{m,n}+\frac{2}{m+1}\sum_{r=1}^{m}r\sum_{s=1}^{n}A_{r,s}$$
$$+\frac{2}{n+1}\sum_{s=1}^{n}s\sum_{r=1}^{m}A_{r,s}. \qquad \text{(II}a\text{)}$$

It should be here noticed that the last two terms on the right are the arithmetic means of $(m+1)$ quantities, of which the first is zero, and which themselves appear in the equation (I). Thus, denoting $m\sum_{s=1}^{n}A_{m,s}$ by $p_{m,n}$ and $n\sum_{r=1}^{m}A_{r,n}$ by $q_{m,n}$, the equation (I) is

$$s_{m,n}=S_{m,n}-p_{m,n}-q_{m,n}+mnA_{m,n}, \qquad \text{(I}b\text{)}$$

and the equation (II) is

$$t_{m,n}=4T_{m,n}-\left(3-\frac{2}{m+1}-\frac{2}{n+1}\right)S_{m,n}+\frac{2}{m+1}\sum_{r=1}^{m}p_{r,n}+\frac{2}{n+1}\sum_{s=1}^{n}q_{m,s}. \qquad \text{(II}b\text{)}$$

16. From the last two equations we at once deduce the following theorem :—

THEOREM.—*If*

$$g(u,v)=\tfrac{1}{4}\operatorname{cosec}u\operatorname{cosec}v\int_{-u}^{u}du\int_{-v}^{v}f(x+2u,\,y+2v)\,dv$$

be such a summable function of (u, v) *that the functions* $p_{m,n}$, $q_{m,n}$, *and* mnA_{mn} *of its double Fourier constants* $A_{m,n}$ *have the unique double limit zero, where* $p_{m,n}$ *is* m *times the sum of the first* n *terms of the* m*-th column, and* $q_{m,n}$ *is* n *times the sum of the first* m *terms of the* n*-th row of the array formed by the double Fourier constants, then, if it is known that either the double Fourier series of* $f(x, y)$, *or that of* $g(u, v)$, *at* $u=0$, $v=0$ *converges doubly in any Cesàro manner, it follows that both series converge doubly in the ordinary manner and have the same sum.*

In fact, performing any Cesàro process on both sides of (Ib), and then letting m and n increase indefinitely, it is clear that if one of the two

series converges in any manner, so does the other, and the sums are the same. Performing then that Cesàro process on (II*b*), which makes the left-hand side the partial Cesàro summation of the order which is known to converge, and, proceeding to the double limit, we see that the Cesàro summation of the series of quantities $A_{m,n}$ of index one lower than before converges. Hence, by repeated application of this process, it appears that the series of quantities $A_{m,n}$ converges in the ordinary way, and therefore by (I*b*) the same is true of the Fourier series of $f(x, y)$, and the sums are the same.

17. There is another way of proving the preceding theorem and at the same time of throwing light on the subject. We have, in fact, the following theorem in the theory of double series of constant terms.

THEOREM.—*If a double series of constant terms $a_{m,n}$ is such that each of the expressions*

$$p_{m,n} = m \sum_{t=1}^{n} a_{m,t}, \qquad q_{m,n} = n \sum_{r=1}^{m} a_{r,n}$$

satisfies the condition of finitude and has the unique double limit zero as m and n increase indefinitely, then the double series cannot converge in any double Cesàro, or Cesàro-ordinary, or ordinary-Cesàro way without converging in the ordinary way doubly. Further, if only one of these two relations holds, say the former, and it be known that the series converges when summed in the double Cesàro way [indices (c, c')], where, if c or c' is zero, the corresponding summation is ordinary, then the series converges doubly in the ordinary-Cesàro way [indices (0, c')]. If the second relation only holds, it converges in the Cesàro-ordinary way [indices (c, 0)].*

In fact, $$p_{m,n} = m(s_{m,n} - s_{m-1,n});$$

and therefore

$$\sum_{r=1}^{m} p_{r,n} = \sum_{r=1}^{m} r(s_{r,n} - s_{r-1,n}) = (m+1)\, s_{m,n} - \sum_{r=1}^{m} s_{r,n},$$

whence, dividing both sides by $(m+1)$, we get

$$\frac{1}{m+1}\left(\sum_{r=1}^{m} p_{r,n}\right) = s_{m,n} - \frac{1}{m+1}\left(\sum_{r=1}^{m} s_{r,n}\right) = s_{m,n} - [s_{C-0}]_{m,n}. \qquad (1)$$

* See for this term Bromwich and Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 163 and 173. As a referee kindly points out, this condition of finitude, or some corresponding condition, is here unavoidable.

Similarly

$$\frac{1}{n+1}\left(\sum_{t=1}^{n} q_{m,t}\right) = s_{m,n} - \frac{1}{n+1}\left(\sum_{t=1}^{n} s_{m,t}\right) = s_{m,n} - [s_{O-C}]_{m,n}. \qquad (2)$$

Summing the equation (1) for all values of n from 1 to a certain value, and dividing by $(n+1)$, and then adding to (2), we get the equation

$$\frac{\sum_{t=1}^{n}\sum_{r=1}^{m} p_{r,t}}{(n+1)(m+1)} + \frac{\sum_{t=1}^{n} q_{m,t}}{(n+1)} = s_{m,n} - \frac{\sum_{t=1}^{n}\sum_{r=1}^{m} s_{r,t}}{(n+1)(m+1)} = s_{m,n} - [s_{C-C}]_{m,n}. \qquad (3)$$

Now, since $p_{r,t}$ and $q_{m,t}$ both have the double limit zero, the left-hand sides of (1), (2), and (3) have the unique limit zero, and therefore if the series converges in the double Cesàro manner [indices (1, 1)], or in the Cesàro-ordinary or ordinary-Cesàro manner [indices (1, 0) and (0, 1)], it converges in the ordinary manner, since the limit of one of the terms on the right is unique, and therefore the other term converges also to the same unique limit.

Again repeating the Cesàro processes any number of times on the last equation, the left-hand side will always have the unique double limit zero. Hence, by the same argument, if any double Cesàro sum exists, the Cesàro sum of indices one lower also exists, and therefore one with at least one index unity exists, and therefore, by what has been proved, the series converges ordinarily.

So far we have supposed both of the given relations to hold. If we assume that only one of them holds, the corresponding result stated in the enunciation follows by inspection of the equation (1), or (2), according as it is the first or the second of the given quantities that converges to zero, and by inspection of the equations resulting when we operate on the equation by any Cesàro process.

18. Closely connected with the theorem just proved we have the following, which constitutes another generalization of the known result for one dimension :—

THEOREM. — *If a double series of constant terms a_{mn} is such that $mna_{m,n}$ satisfies the condition of finitude, and has the unique double limit zero, then, if the series converges in the Cesàro-ordinary and ordinary-Cesàro ways, it converges doubly in the ordinary way. For*

$$mna_{m,n} = mn\,(s_{m-1,n-1} - s_{m-1,n} - s_{m,n-1} + s_{m,n}).$$

Comparing this equation with the equation (a) of § 13, which is of the same form, we see that we can, by the analysis of that article, transform it into the same form as equation (1) of that article; that is

$$\sum_{r=1}^{m}\sum_{t=1}^{n}(rta_{r,t}) = \sum_{r=1}^{m}\sum_{t=1}^{n}s_{r,t} - (m+1)\sum_{t-1}^{n}s_{m,t} - (n+1)\sum_{r=1}^{m}s_{r,n} + (m+1)(n+1)s_{m,n}.$$

Dividing by $(m+1)(n+1)$, this becomes

$$\frac{\sum_{r=1}^{m}\sum_{t=1}^{n}(rta_{r,t})}{(m+1)(n+1)} = \frac{\sum_{r=1}^{m}\sum_{t=1}^{n}s_{r,t}}{(m+1)(n+1)} - \frac{\sum_{t=1}^{n}s_{m,t}}{n+1} - \frac{\sum_{r=1}^{m}s_{r,n}}{m+1} + s_{m,n}$$

$$= [s_{C-C}]_{m,n} - [s_{O-C}]_{m,n} - [s_{C-O}]_{m,n} + s_{m,n}. \qquad (3)$$

Since, when $mna_{m,n}$ has the unique double limit zero, the same is true of the left-hand side of (3), this proves the theorem.

19. Returning to the equation obtained at the end of § 13,

$$s_{m,n} = (m-1)(n-1)S_{m,n} - m(n-1)S_{m-1,n} - n(m-1)S_{m,n-1} + mnS_{m-1,n-1},$$

and summing with respect to m, keeping n constant, we get

$$\sum_{r=1}^{m}s_{r,n} = (m-1)(n-1)S_{m,n} - n(m-1)S_{m,n-1} - 2(n-1)\sum_{r=1}^{m-1}S_{r,n} + 2n\sum_{r=1}^{m-1}S_{r-1,n-1}$$

$$= (m+1)(n-1)S_{m,n} - n(m+1)S_{m,n-1} + 2\sum_{r=1}^{m}S_{r,n} - 2n\sum_{r=1}^{m}(S_{r,n} - S_{r,n-1}).$$

Hence we have the following formula:

$$\sum_{r=1}^{m}s_{r,n}/(m+1) = 2\sum_{r=1}^{m}S_{r,n}/(m+1) - S_{m,n} + n(S_{m,n} - S_{m,n-1}) - 2n\sum_{r=1}^{m}(S_{r,n} - S_{r,n-1})/(m+1). \qquad \text{(III)}$$

We notice that the last term but one on the right is $q_{m,n}$, the ordinary sum of m terms of the n-th row of the array of the $A_{r,t}$'s, each multiplied by n, and the last term is twice the Cesàro partial summation, say $q'_{m,n}$, of the same quantities; thus, the double limits of $q'_{m,n}$ lie between the same

bounds as those of $q_{m,n}$. Thus, if $q_{m,n}$ has the unique double limit zero, so has $q'_{m,n}$, and we have the following theorem :—

THEOREM.—*If, with the notation of the theorem of* § 17, $q_{m,n}$ *has the unique double limit zero, and one of the two series converges in the Cesàro-ordinary way, so does the other. If, on the other hand,* $p_{m,n}$ *has the unique double limit zero, and one of the two series converges in the ordinary-Cesàro way, so does the other.*

20. From equation (I) of § 13 we at once have the following theorem :—

THEOREM.—*If*

$$g(u, v) = \tfrac{1}{4} \operatorname{cosec} u \operatorname{cosec} v \int_{-u}^{u} du \int_{-v}^{v} f(x+2u, y+2v)\, dv$$

is a function of bounded variation in the closed rectangle* $0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant \pi$, *the double Fourier series of* $f(x, y)$ *converges doubly to the value* $g(0, 0)$.

For in this case $g(u, v)$ is such a function as was contemplated in §§ 9, 11, and 13. Hence its double Fourier series converges in the ordinary way at the origin to $g(0, 0)$, and the last three terms on the right-hand side of the equation (I) of § 13 vanish. Thus, letting n and m increase without limit in that equation, the required result follows immediately.

21. It will now be convenient to consider another auxiliary function, supposed to be summable, and evidently even-even,

$$g(u, v) = \tfrac{1}{4} \operatorname{cosec} v \int_{0}^{v} [f(x+2u, y+2v) + f(x+2u, y-2v) + f(x-2u, y+2v) + f(x-2u, y-2v)]\, dv.$$

Clearly $$g(u, v+\pi) = -g(u, v),$$

and $$g(u+\pi, v) = g(u, v).$$

Thus we may write

$$g(u, v) \sim \sum_{r=1} \sum_{s=1} A_{r,s} \cos 2ru \cos (2s-1)v,$$

---

* The expression given is meaningless when u or v is zero. It is, of course, implied that the values of $g(u, v)$ can be so assigned on the axes that the condition is satisfied.

where, moreover,

$$A_{r,s}-A_{r,s+1}=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}g(u,v)\cos 2ru\,\{\cos(2s-1)v-\cos(2s+1)v\}\,du\,dv$$

$$=\frac{2}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}g(u,v)\sin v\cos 2ru\sin 2sv\,du\,dv$$

$$=\frac{1}{s}(a,\,b,\,c,\,d,\,x,\,y)_{r,s}.$$

To see this we merely have to remark that, by one-dimensional theory,

$$\sin v\,.\,g(u,v)=\sum_{r=1}\sum_{s=1}(a,\,b,\,c,\,d,\,x,\,y)_{r,s}\cos 2ru\sin 2sv/2s.$$

Adding up all similar equations for all values of r and s up to $m-1$ and $n-1$, both inclusive, and using the notation hitherto adopted, we have

$$\sum_{s=1}^{n}(a,\,b,\,c,\,d,\,x,\,y)_{r,s}=\sum_{s=1}^{n}s\,(A_{r,s}-A_{r,s+1})$$

$$=A_{r,1}+A_{r,2}+\ldots+A_{r,n}-nA_{r,n}$$

$$=\sum_{s=1}^{n}A_{r,s}-nA_{r,n}.$$

Summing both sides of this equation in the Cesàro manner with respect to r from 1 to m, we get an equation which we may write

$$[s_{\mathrm{C-O}}]_{m,n}=[S_{\mathrm{C-O}}]_{m,n}-n\sigma_{m,n}, \qquad \text{(IV)}$$

where the left-hand side is the partial Cesàro-ordinary summation of the Fourier series of $f(x,\,y)$, and the first term of the right is the corresponding partial summation of the double Fourier series of $g(u,\,v)$ at $u=0,\,v=0$, while $\sigma_{m,n}$ denotes the m-th Cesàro partial summation of the elements of the n-th row of the array formed by the double Fourier constants of $g(u,\,v)$.

Now

$$[S_{\mathrm{C-C}}]_{m,n}-[S_{\mathrm{C-O}}]_{m,n}=\frac{1}{(m+1)(n+1)}\left(\sum_{r=1}^{m}\sum_{s=1}^{n}S_{r,t}-(n+1)\sum_{r=1}^{m}S_{r,n}\right)$$

$$=\frac{1}{(m+1)(n+1)}\sum_{r=1}^{m}(S_{r,1}+S_{r,2}+\ldots+S_{r,n-1}-nS_{r,n})$$

$$=\frac{1}{(m+1)(n+1)}\sum_{r=1}^{m}\sum_{t=1}^{n}t(-S_{r,t}+S_{r,t-1})$$

$$=-\frac{1}{n+1}\sum_{t=1}^{n}n\sigma_{r,t}.$$

Hence, summing the equation (IV) with respect to n and dividing by $(n+1)$, we get the equation

$$[s_{C-C}]_{m,n} = 2[S_{C-C}]_{m,n} - [S_{C-O}]_{m,n}. \qquad (V)$$

22. The following theorem in summation of series results immediately from the preceding analysis :—

THEOREM.—*If n times the Cesàro m-th partial summation of the n-th row of a double series of constant terms has the unique double limit zero, as m and n approach infinity in any manner, then, if the series converges doubly when summed in the Cesàro way, it will converge doubly when the rows are summed in the Cesàro and the columns in the ordinary way. More generally, if the series converge when summed in any Cesàro manner [indices (p, q)], where $p \geqslant 1$, it will converge in the Cesàro manner [indices $(p, 0)$], where $p \geqslant 1$, i.e. when summed with respect to m in the Cesàro-ordinary manner.**

For, by an equation just obtained, and with the same notation,

$$-\frac{1}{n+1}\left(\sum_{t=1}^{n} n\sigma_{m,t}\right) = \frac{1}{(m+1)(n+1)}\left(\sum_{r=1}^{m}\sum_{t=1}^{n} S_{r,t} - (n+1)\sum_{r=1}^{m} S_{r,n}\right)$$

$$= [S_{C-C}]_{m,n} - [S_{C-O}]_{m,n},$$

from which the first of the required results at once follows. The more general result follows, if we operate on this equation in a suitable Cesàro manner.

23. On the equation (V) of § 21 we may perform Cesaro summations of any indices with respect to m and n respectively. The types of summa-

* It should be remarked that this and the theorems of §§ 17, 18 are the generalisations for the simple case only of Hardy's theorem for single series. The interesting question as to whether our theorems still hold when the quantities here hypothecated to converge to zero merely oscillate boundedly is not directly relevant to the present paper, and has not therefore been considered here. It would appear, however, that the extension in the general case presents no difficulty. Thus Hardy's reasoning, for example, in § 3 of his paper, *loc. cit.*, applies almost as it stands, provided only we bear in mind that, when the partial sum of a single series involving the parameter n has an unique double limit, $|s_{m+p,n} - s_{m,n}|$ must have the unique double limit zero, with similar slight modifications of well known principles used by Hardy. The generalisation to multiple series are also easily realised, if we picture to ourselves as geometrical points, lines, planes, S_3's, and the like, the various algebraical entities concerned. It should be added that other interesting theorems of Hardy for single series appear to be similarly generalisable.

tion on the left and in the first term on the left will be the same, and in the last term on the right the index of the summation with respect to n will be one lower, or, if the former index was unity, the summation with respect to n will be ordinary.

The theory, from the present point of view, is closely analogous to the theory of one-dimensional Fourier series in my paper already quoted. We have at once the following theorem :—

THEOREM.—*If the double Fourier series of*

$$g(u, v) = \tfrac{1}{4}\operatorname{cosec} v \int_0^v [f(x+2u, y+2v)+f(x+2u, y-2v)+f(x-2u, y+2v) \\ +f(x-2u, y-2v)]\, dv$$

converges in the Cesàro-ordinary way at the point $u = 0$, $v = 0$, then the double Fourier series of $f(x, y)$ converges in the double Cesàro manner [indices (1, 1)]. More generally, if the former series converges in the double Cesàro manner [indices $(p, q-1)$], where, if $q = 1$, the latter mode of summation is ordinary with respect to n, then the latter series converges in the double Cesàro manner [indices (p, q)]. In all cases the sums are the same.

24. Again, using both (IV) and § 22, we have the following theorem :—

THEOREM.—*If the function $g(u, v)$, defined in the preceding enunciation, is such that n times the Cesàro m-th partial summation of the n-th row of the array of its double Fourier constants has the unique double limit zero as m and n increase indefinitely, then the double Fourier series of $f(x, y)$ and that of $g(u, v)$ at $u = 0$, $v = 0$ both converge in the Cesàro-ordinary manner and have the same sum, provided the latter series is known to converge when summed in some double Cesàro manner, or Cesàro-ordinary manner of any indices greater than or equal to unity.*

25. Analogy with one-dimensional theory suggests such a theorem as the following :—

THEOREM.—*If an even-even function $f(u, v)$ has bounded variation with respect to v and is, except along $v = 0$, the integral with respect to v of a function which, in a suitably closed neighbourhood of $u = 0$, is continuous with respect to u and bounded with respect to (u, v), then its*

double Fourier series converges doubly to the value of the function at the origin, supposing this point to be a point of continuity of $f(u, v)$ and supposing also the summation to be taken with respect to u in the Cesàro and with respect to v in the ordinary way.

We may, as usual, without loss of generality, suppose the value of the function at the origin to be zero. By the formula (2) of § 8 our theorem will have been proved if we show that

$$\frac{1}{m}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} f(u, v)\sin^2 mu \operatorname{cosec}^2 u \sin nv \operatorname{cosec} v\, du\, dv$$

—where, moreover, we may, as usual, replace $\operatorname{cosec}^2 u$ by $1/u^2$, and $\operatorname{cosec} v$ by $1/v$—has the unique double limit zero as the odd integers m and n increase without limit.

If we replace the lower limits of integration each by e, the required result at once follows by the extended theorem of Riemann-Lebesgue. The remaining region of integration may be split up into four rectangles,

$$(0, 0\,;\ e, \pi/n),\ (0, \pi/n\,;\ e, e),\ (0, e\,;\ e, \tfrac{1}{2}\pi),\ \text{and}\ (e, 0\,;\ \tfrac{1}{2}\pi, e):$$

denote the corresponding expressions by (I), (II), (III), and (IV).

In considering (I), we may conveniently employ the First Theorem of the Mean, as the integrand is positive, apart from the single factor $f(u, v)$. Since f is zero at the origin, and is continuous there, the upper bound of its modulus is as small as we please in the rectangle considered, the smallness depending only on our choice of e. Hence (I) is as small as we please, since by § 7, Cor., the integral multiplying it is numerically $\leqslant \frac{\pi^2}{2}$.

Next consider (II). This may be written

$$\int_0^{me} du\, \frac{\sin^2 u}{u^2}\int_{\pi/n}^{e} f(u/m, v)\,\frac{\sin nv}{v}\, dv$$

$$= \int_0^{me}\frac{\sin^2 u}{u^2}\left[f(u/m, v)\, H(nv)\right]_{v=\pi/n}^{v=e} du - \int_0^{me} du\,\frac{\sin^2 u}{u^2}\int_{\pi/n}^{e}\frac{d}{dv} f(u/m, v) H(nv)\, dv.$$

Now $f(u/m, v)$ is as small as we please, both when $v = e$ and when $v = \pi/n$, for all values of u/m from 0 to e; also $|H(nv)| \leqslant \pi$. Therefore the first term is as small as we please.

Again, the second term, since the integrand of the integral with respect

to v is less than some fixed finite quantity K, is less than

$$Ke\int_0^\infty \frac{\sin^2 u}{u^2}\,du,$$

and is therefore as small as we please.

We next consider (III). We may now suppose the factor $1/v$ absorbed into f. Writing then u for mu, our integral takes the form

$$\int_0^{me} du\,\frac{\sin^2 u}{u^2}\int_e^{\frac{1}{2}\pi} f(u/m,\ v)\,\sin nv\,dv.$$

Applying the Second Theorem of the Mean in its general form to the integration with respect to v, it follows that this expression is numerically less than the sum of two terms, one of the form

$$\frac{2}{n}\int_0^{me}\frac{\sin^2 u}{u^2}\,|f(u/m,\ e)|\,du,$$

and the other of the form

$$\frac{2}{n}\int_0^{me}\frac{\sin^2 u}{u^2}\,V(u/m,\ e)\,du,$$

where $V(u/m,\ e)$ stands for the total variation of $f(u/m,\ v)$ with respect to v in the interval

$$e \leqslant u \leqslant \frac{\pi}{2}.$$

These expressions are increased if we replace the upper limit of integration me by ∞, and the bounded quantities $|f|$ and V by any quantity greater than their upper bounds. Hence both these expressions have the unique double limit zero, and therefore so has the expression (III).

It remains to consider (IV). Here we may absorb $1/u^2$ into $f(u,\ v)$, and so get

$$\frac{1}{m}\int_e^{\frac{1}{2}\pi} du\int_0^e \sin^2 mu\,\frac{\sin nv}{v}\,f(u,v)\,dv.$$

Let us apply to the integration with respect to v the Second Theorem of the Mean in its general form. We shall then see that this expression is numerically less than the sum of two terms, one of the form

$$\frac{\pi}{m}\int_e^{\frac{1}{2}\pi}\sin^2 mu\,|f(u,0)|\,du,$$

and the other of the form

$$\frac{\pi}{m}\int_e^{\frac{1}{2}\pi} \sin^2 mu\, V(u, e)\, du.$$

If we write unity for $\sin^2 mu$ in these integrands, and bear in mind that the other factors are bounded, it becomes evident that each of these expressions has the unique double limit zero. Hence (IV) also has the unique double limit zero.

Summing up all our results so far, and applying the usual argument, it follows that the theorem is true.

26. The following theorem, like the preceding, is suggested by the equation (IV) of § 21.

THEOREM.—*If an even-even function $f(u, v)$ has bounded variation with respect to v, and is, except along $v = 0$, the integral with respect to v of a function which, in a suitable neighbourhood of $u = 0$, is continuous with respect to u and bounded with respect to (u, v), then the Cesàro summation of the first m elements of the n-th row of the array formed by the double Fourier constants of $f(u, v)$, when multiplied by n, has the unique double limit zero, as m and n increase indefinitely, that is,*

$$\operatorname*{Lt}_{(m\to\infty,\, n\to\infty)} \left\{ \frac{n}{m+1}\left[ma_{1,n}+(m-1)a_{2,n}+\ldots+a_{m,n}\right]\right\} = 0.$$

COR.—*The Cesàro sum of the first m coefficients of the n-th row of a double Fourier series converges to zero, when $m \to \infty$, $n \to \infty$, provided the function of the series is an even-odd bounded function of (x, y) and continuous with respect to x.*

It is clear from § 8 that it will be sufficient to prove that

$$\frac{n}{m}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} f(u, v)\,\frac{\sin^2 mu}{\sin^2 u}\cos nv\, du\, dv,$$

where n is even, has the unique double limit zero when $m \to \infty$, $n \to \infty$. Moreover, we may at once remark that we may, without loss of generality, suppose the function to vanish all along the axes of v and of u, since, if we write $f(0, v)$ instead of $f(u, v)$ in the double integral, it reduces to a constant multiple of

$$n\int_0^{\frac{1}{2}\pi} f(0, v)\cos nv\, dv,$$

which, by a known theorem, has the unique limit zero as n increases

without limit,* while, if we write $f(u, 0)$ instead of $f(u, v)$, it vanishes identically, since n is even.

We break the rectangle of integration up into four parts,

$$\left(e, e; \frac{\pi}{2}, \frac{\pi}{2}\right), \quad \left(e, 0; \frac{\pi}{2}, e\right), \quad \left(0, \frac{\pi}{n}, e, \frac{\pi}{2}\right), \quad \left(0, 0; e, \frac{\pi}{n}\right),$$

and denote the corresponding double integrals by (I), (II), (III), (IV).

First consider (I), and let us integrate by parts with respect to v. Bearing in mind that n is even, we thus get

$$-\frac{1}{m}\int_e^{\frac{1}{2}\pi} f_v(u, e) \sin ne \frac{\sin^2 mu}{\sin^2 u}\, du - \frac{1}{m}\int_e^{\frac{1}{2}\pi}\int_e^{\frac{1}{2}\pi} f_v(u, v) \sin nv \frac{\sin^2 mu}{\sin^2 u}\, du\, dv,$$

both terms of which vanish by the theorem of Riemann-Lebesgue and its extension respectively.

Next consider (II). Here we apply the Second Theorem of the Mean for the integration with respect to v. By this theorem the inside integral, which is

$$\int_0^e f(u, v) \cos nv\, n\, dv,$$

is numerically less than twice the total variation with respect to v of the function $f(u, v)$ in the interval $0 \leqslant v \leqslant e$, say $V(u, e)$. Hence our expression (II) is numerically less than

$$\frac{2}{m}\int_e^{\frac{1}{2}\pi} V(u, e) \operatorname{cosec}^2 u\, du,$$

and therefore, since the integrand is bounded, has the unique limit zero as m and n increase indefinitely.

Next consider (III). First replace $\operatorname{cosec}^2 u$ by $1/u^2$; this we may do, since $u^2 \operatorname{cosec}^2 u f(u, v)$ has the same properties as $f_v(u, v)$. We then integrate by parts with respect to v and so get

$$-\frac{1}{m}\int_0^e \int_{\pi/n}^{\frac{1}{2}\pi} f_v(u, v) \frac{\sin^2 mu}{u^2} \sin nv\, dv.$$

We next split up the rectangle of integration into several parts, and con-

* W. H. Young, "On the Convergence of a Fourier Series and of its Allied Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 10, § 2, p. 256.

sider each separately, viz., into the parts

$$\left(0, e; \frac{\pi}{m}, \frac{\pi}{2}\right), \quad \left(\frac{\pi}{m}, e; e, \frac{\pi}{2}\right), \quad \left(0, \frac{\pi}{n}; e, e\right),$$

and denote the corresponding double integrals by (a), (b), and (c). Thus

$$(a) = \frac{1}{m}\int_0^{\pi/m} du \int_e^{\frac{1}{2}\pi} f_v(u, v)\,\frac{\sin^2 mu}{u^2}\sin nv\,dv$$

$$= \int_0^{\pi} du \int_e^{\frac{1}{2}\pi} f_v(u/m, v)\,\frac{\sin^2 u}{u^2}\sin nv\,dv.$$

Now f has bounded variation with respect to v in the whole rectangle of integration, and hence, if we put unity for $\sin nv$ and write for f_v its modulus, and then perform the integration with respect to v, the result will be less than a finite quantity independent of m and n. Hence, in determining the double limits of our expression, we may perform the process of going to the limit inside in the integral with respect to u, since the integrand is a bounded function of (m, n, u). Thus, to prove that the expression (a) we are considering has the unique double limit zero, it will be sufficient to prove that

$$\int_e^{\frac{1}{2}\pi} f_v(u/m, v)\sin nv\,dv$$

has, for each fixed value of u, the unique double limit zero.

Now the integrand of this integral as m and n vary traces out an oscillating succession of functions of (u, v), formed by the product of two successions, of which one, that traced out by $\sin nv$, is a bounded succession, while the other is a bounded sequence, since df/dv is supposed to be a continuous function of u and a bounded function of (u, v). Moreover, the integrals of these successions converge, the former to zero and the latter to $f(0, v)$, which is also zero. Finally, we remark that the sequence

$$f_v(u/m, v),$$

being bounded, satisfies the condition that its integral over a set of points of content E has the unique double limit zero, as $E \to 0$, and $m \to \infty$. Hence, by a known theorem,* the integral of the product converges to zero.

Next consider (b). We get, in precisely the same way,

$$\int_{\pi}^{me} du \int_e^{\frac{1}{2}\pi} f_v(u/m, v)\,\frac{\sin^2 u}{u^2}\sin nv\,dv.$$

* W. H. Young, "On Successions of Integrals and Fourier Series," § 19, p. 76 of present volume.

Now, if we replace the lower limit of integration with respect to u by A and introduce into the integrand $\log u$ and $1/(\log u)$ as factors and then utilise the fact that $1/(\log u)$ is a monotone decreasing function of u for the values of u considered, the double integral in its new form is, by the Second Theorem of the Mean,

$$= \frac{1}{\log A} \int_A^{A'} du \frac{\sin^2 u}{u^2} \log u \int_e^{\frac{1}{2}\pi} f_v(u/m, v) \sin nv\, dv ;$$

and therefore, numerically,

$$\leqslant \frac{B}{\log A} \int_A^{\infty} \frac{\sin^2 u}{u^2} \log u\, du,$$

where B is a finite constant. Therefore, by suitably choosing A, this can be made as small as we please, whence it will appear, by the usual argument, that, for practical purposes in considering the double limits of (b), we may replace me by a fixed quantity A.

We have now merely to repeat the argument of case (a), which, indeed, applies as it stands, π and A having now taken the places of 0 and π respectively in the integration with respect to u.

We now consider (c), namely,

$$\frac{1}{m} \int_0^e du \int_{\pi n}^e f_v(u, v) \frac{\sin^2 mu}{u^2} \sin nv\, dv,$$

or, what is the same thing,

$$\int_0^{me} du \frac{\sin^2 u}{u^2} \int_{\pi/n}^e f_v(u/m, v) \sin nv\, dv.$$

The integrand of the inside integral is bounded by hypothesis. Let K be any quantity greater than its upper bound. Then we see that (c) is certainly less than

$$eK \int_0^{\infty} \frac{\sin^2 u}{u^2} du ;$$

and is therefore as small as we please.

Finally, consider (IV), namely, the integral

$$\frac{1}{m} \int_0^e du \int_0^{\pi/n} f(u, v) \frac{\sin^2 mu}{\sin^2 u} \cos nv\, n\, dv.$$

Write $|f(u, v)|$ for $f(u, v)$; then the integrand of this double integral is positive as long as v is less than $\pi/2n$, and negative for the remainder of the interval of values of v. Hence we may apply the First Theorem of the Mean to each of the corresponding portions of the double integral.

We thus show that the whole double integral is certainly less numerically than a suitable constant multiple of the upper bound of $f(u, v)$ in the square $(0, 0; e, e)$. Hence (IV) may be made as small as we please by diminishing e.

Adding up the four integrals (I), (II), (III), and (IV), we see that the original expression has all its limits numerically as small as we please; therefore it has the unique double limit zero.

27. From the two preceding theorems, together with the equation (IV) of § 21, we have at once the following theorem:—

THEOREM.—*If in the rectangle* $(0 \leqslant u, \leqslant \pi, 0 \leqslant v \leqslant \pi) f(x \pm 2u, y \pm 2v)$ *is a bounded function of* (u, v) *and a continuous function of* u, *and*

$$g(u, v) = \tfrac{1}{4} \operatorname{cosec} v \int_0^v [f(x+2u, y+2v) + f(x-2u, y+2v) + f(x+2u, y-2v) + f(x-2u, y-2v)]\, dv$$

is a continuous function of (u, v) *which has with respect to* v *a total variation which is bounded, then the double Fourier series of* $f(x, y)$ *converges doubly to* $g(0, 0)$ *when summed in the Cesàro way with respect to* x *and the ordinary way with respect to* y.

In fact, by the last theorem the quantity denoted in the later part of § 20 by $\sigma_{m, n}$ has, when multiplied by n, the unique double limit zero.

Hence, proceeding to the limit and remembering that, by the theorem of § 25, the Cesàro-ordinary summation of the double Fourier series of $g(u, v)$ at the origin has $g(0, 0)$ for value, the theorem at once follows.

28. That simpler tests exist for the convergence of the double Fourier series corresponding to the known simpler tests for a function of a single variable may be shown by means of the general test given in § 20. We have only to generalize for two dimensions the remarks of de la Vallée Poussin * on this head.

Thus, *if* $f(x, y)$ *is a function of bounded variation in* (x, y) *in a crucial* † *neighbourhood of the point of continuity considered, the double*

* De la Vallée Poussin, "Un nouveau cas de convergence des séries de Fourier," 1911, *Rend. di Pal.*, XXXI.

† In dealing with functions of more than one variable, we are often concerned with the behaviour of a function in the neighbourhood of certain lines. In particular this is the case in the neighbourhood of the axial cross through the point on which our attention is concen-

Fourier series of $f(x, y)$ will converge to $f(x, y)$ at the point considered, since $g(u, v)$, with the notation of § 16, *has then the properties required in that article. Similarly, by* § 27, *if $f(x, y)$ is in a crucial neighbourhood of the point of continuity considered bounded with respect to (x, y), continuous with respect to x, and has a variation with respect to y which is bounded, the double Fourier series of $f(x, y)$ converges at the point considered to the value of the function there, provided the summation be performed in the Cesàro manner with respect to x and in the ordinary manner with respect to y.*

29. Completing the trio of these theorems for double series we have the following theorem, which is proved as simply as the corresponding theorem in one dimension:—

THEOREM.—*If $f(x, y)$ is at the point (x, y), and in a crucial neighbourhood of that point, a bounded function of (x, y), and is continuous at the point itself, then the double Fourier series of $f(x, y)$ converges at that point to the value of the function there if the summation be performed in the double Cesàro manner.*

Using the notation of § 8, and denoting by $h(u, v)$ the function $g(u, v) - f(u, v)$, we have for the Cesàro-Cesàro partial summation the expression

$$f(x, y) + \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{1}{mn} h(u, v) \sin^2 mu \operatorname{cosec}^2 u \sin^2 nv \operatorname{cosec}^2 v \, du dv.$$

We have therefore only to prove that this double integral has the unique double limit zero when m and n increase indefinitely.

First replace the lower limits of integration by e, where the point (e, e) lies within the given crucial neighbourhood. We get something numerically

$$\leqslant \frac{1}{mn} \operatorname{cosec}^4 e \int_e^{\frac{1}{2}\pi} \int_e^{\frac{1}{2}\pi} | h(u, v) | \, du \, dv,$$

which has certainly the unique double limit zero when m and n increase indefinitely.

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trated. Where such a neighbourhood has itself the form of a cross of calibre e, where e is as small as is convenient, I shall use the term "a crucial neighbourhood of the point (a, b)," that is the part of the region considered which lies between the lines

$$x = a \pm \tfrac{1}{2}e, \qquad y = b \pm \tfrac{1}{2}e.$$

In the present theorem the region considered is the square of side π with the point (x, y) as centre.

We have now the integral over the part of the crucial neighbourhood in the rectangle $(0, 0 ; \frac{1}{2}\pi, \frac{1}{2}\pi)$. This consists of the three rectangles $(0, e ; e, \frac{1}{2}\pi)$, $(e, 0 ; \frac{1}{2}\pi, \frac{1}{2}\pi)$, and $(0, 0 ; e, e)$.

The integral over the first of these rectangles is numerically

$$\leqslant \frac{B \operatorname{cosec}^2 e}{mn} \int_0^e du \int_e^{\frac{1}{2}\pi} \sin^2 mu \operatorname{cosec}^2 u \, dv$$

$$\leqslant \frac{B \operatorname{cosec}^2 e}{mn} \int_e^{\frac{1}{2}\pi} dv \int_0^e \sin^2 mu \, du \leqslant \frac{B \operatorname{cosec}^2 e}{n} \int_e^{\frac{1}{2}\pi} \frac{\pi}{2} \, dv,$$

using the First Theorem of the Mean, and denoting by B the upper bound of the absolute values of $h(u, v)$ in the given crucial neighbourhood; therefore this expression has the unique double limit zero.

Similarly the integral over the second of the rectangles vanishes in the limit. Finally, the integral over the rectangle $(0, 0 ; e, e)$ is, by the First Theorem of the Mean, numerically

$$\leqslant M \int_0^e \int_0^e \frac{1}{mn} \sin^2 mu \, u^2 \sin^2 nv \, v^2 \, du \, dv,$$

where M, being the upper bound of $h(u, v)$ in the small square of integration, may be made as small as we please by sufficiently decreasing e. By § 7, therefore, this term is as small as we please. Thus, the whole integral has all its limits as small as we please and has therefore the unique double limit zero, which proves the theorem.

30. Again, if, for example, we suppose

$$\frac{1}{uv} h(u, v) = \frac{1}{uv} [f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) + f(x-u, y-v) - 4f(x, y)]$$

to be summable, we can easily shew that the function $g(u, v)$ of § 13 has bounded variation. To do this we only need to prove that the mixed differential coefficient of $g(u, v)$ taken positively can be integrated over a square $(0, 0 ; e, e)$. Now

$$g(u, v) = \tfrac{1}{4} \operatorname{cosec} u \operatorname{cosec} v \int_0^{2u} \int_0^{2v} h(u, v) \, du \, dv.$$

This function has evidently bounded variation if it has so when we change $\operatorname{cosec} u$ into $1/u$ and $\operatorname{cosec} v$ into $1/v$. Denoting the function so modified

by $F(u, v)$, we have

$$4\frac{d^2F}{du\,dv} = \frac{h(u,v)}{uv} - \frac{1}{u^2v}\int_0^u h(u,v)\,du - \frac{1}{uv^2}\int_0^v h(u,v)\,dv + \frac{1}{u^2v^2}\int_0^u\int_0^v h(u,v)\,du\,dv.$$

Now
$$\int_0^e\int_0^e \frac{du\,dv}{u^2v}\int_0^u |\,h(w,v)\,|\,dw = \int_0^e\int_0^e \frac{du\,dv}{u^2v}\int_0^1 |\,h(ut,v)\,|\,u\,dt$$
$$= \int_0^1 dt\int_0^e\int_0^e |\,h(ut,v)\,|\,\frac{1}{uv}\,du\,dv$$
$$= \int_0^1 dt\int_0^{et} du\int_0^e |\,h(u,v)\,|\,\frac{1}{uv}\,dv$$
$$\leqslant \int_0^e du\int_0^e \frac{|\,h(u,v)\,|}{uv}\,dv.$$

In precisely the same way

$$\int_0^e\int_0^e \frac{du\,dv}{uv^2}\int_0^v |\,h(w,v)\,|\,dw = \int_0^e du\int_0^e \frac{|\,h(u,v)\,|}{uv}\,dv.$$

Again,

$$\int_0^e\int_0^e \frac{du\,dv}{u^2v^2}\int_0^u\int_0^v |\,h(u,v)\,|\,du\,dv = \int_0^e\int_0^e \frac{du\,dv}{uv}\int_0^1\int_0^1 |\,h(ut_1,vt_2)\,|\,dt_1\,dt_2$$
$$= \int_0^1\int_0^1 dt_1\,dt_2\int_0^e\int_0^e \frac{1}{uv}\,|\,h(ut_1,vt_2)\,|\,du\,dv$$
$$= \int_0^1\int_0^1 dt_1\,dt_2\int_0^{et_1}\int_0^{et_2} \frac{1}{uv}\,|\,h(ut_1,ut_2)\,|\,du\,dv$$
$$\leqslant \int_0^e\int_0^e \frac{|\,h(u,v)\,|}{uv}\,du\,dv.$$

Hence, adding $\int_0^e\int_0^e \frac{d^2F}{du\,dv}\,du\,dv \leqslant \int_0^e\int_0^e \frac{|\,h(u,v)\,|}{uv}\,du\,dv.$

A fortiori, if (a, b) and (c, d) are any two points in the completely open square $(0, 0\,;\ e, e)$

$$\int_a^c du\int_b^d \frac{d^2F}{du\,dv}\,dv \leqslant \int_0^e\int_0^e \frac{h(u,v)}{uv}\,du\,dv.$$

But, since the rectangle $(a, b\,;\ c, d)$ contains no point of the axis, $g(u, v)$ is an integral in it, and therefore the integral on the left is the variation of $F(u, v)$ over the rectangle $(a, b\,;\ c, d)$. Thus the function $F(u, v)$ has

bounded variation in the square $(0, 0 : e, e)$. Taking $e = \pi$, this shews that the condition of § 20 is satisfied, and therefore the condition given at the beginning of this article is sufficient to ensure the double convergence of the double Fourier series of $f(x, y)$ to the value $g(0, 0)$.

31. In the present paper we have confined our attention to the Fourier series of $f(x, y)$. The extension to any number of variables is immediate. I content myself therefore with referring the reader to the introduction where these generalisations are given, so far as the two main theorems of the paper are concerned. It should also be remarked that similar results to those here obtained hold for the various allied series of a multiple Fourier series. I hope to deal with these in a subsequent communication. The reader should, however, have little difficulty in constructing the required line of argument with my earlier paper in these *Proceedings* and that in the *Münchener Bericht* in his hands.

NOTE.—I take this opportunity of calling the attention of my readers to the inadvertent omission in the last line of p. 61 of my paper on "Successions of Functions and Fourier Series," in the present volume of the *Proceedings*, of the condition that $\int_a^b |f'(x)|^{1+p} dx$ should be a bounded function of n. It is tacitly assumed in the proof on p. 63 and is expressly stated in the enunciations of the corresponding theorems.

INVARIANTS OF LINEAR DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO RULED SURFACES IN FIVE-DIMENSIONAL SPACE

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IN Wilczynski's *Projective Differential Geometry of Curves and Ruled Surfaces*, the general theory of non-developable ruled surfaces in ordinary space is put into relation with the theory of a system of two linear homogeneous differential equations of the second order. In this paper the same notion is extended to five-dimensional space. It is shown that the projective differential properties of every ruled surface in five dimensions, which does not belong to a certain exceptional class, can be obtained by studying a system of two linear homogeneous differential equations of the third order.

The author takes pleasure in acknowledging his indebtedness to Professor Wilczynski for helpful advice and general guidance in the preparation of this paper.

1. *Transformations of the General System.*

Let us consider the system of linear homogeneous differential equations

$$y_i^{(m)} + \sum_{l=0}^{m-1} \sum_{k=1}^{n} \binom{m}{l} p_{ikl} y_k^{(l)} = 0 \quad (i = 1, 2, \ldots, n), \tag{1}$$

where

$$y_k^{(l)} = \frac{d^l y_k}{dx^l},$$

and where p_{ikl} are functions of the independent variable x. Wilczynski* has shown that the most general point transformation which converts this system into another of the same form is given by the equations

$$x = f(\xi), \qquad y_\kappa = \sum_{\lambda=1}^{n} a_{\kappa\lambda}(\xi)\eta_\lambda \quad (\kappa = 1, 2, \ldots, n), \tag{2}$$

* Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, 1906, Chap. I.

where $\alpha_{\kappa\lambda}$ and f are arbitrary functions of ξ, and where the determinant

$$|\alpha_{\kappa\lambda}| \quad (\kappa, \lambda = 1, 2, \ldots, n)$$

does not vanish identically.

A function of the coefficients of (1) and of their derivatives which has the same value for (1) as for any system derived from (1) by transformations of the form (2) is called an absolute *invariant.* If the function contains also the dependent variables and their derivatives, it is called a *covariant.* Functions which remain invariant under all transformations of the form (2) for which the independent variable remains unaltered, so that only the dependent variables are transformed, are called *semi-invariants* and *semi-covariants.*

The transformations (2) form an infinite continuous group which, according to Lie's terminology, is defined by differential equations. Lie* has shown that the invariants of such a group may always be determined as the solutions of a complete system of partial differential equations which is obtained by equating to zero the symbol of the most general infinitesimal transformation of the group.

These same considerations apply to the transformations

$$y_\kappa = \sum_{\lambda=1}^{n} \alpha_{\kappa\lambda}(\xi)\eta_\lambda \tag{3}$$

of the dependent variables alone. They form a sub-group of the infinite group defined by (2).

The transformations (3) convert equations (1) into a new system whose coefficients $\pi_{\lambda\mu\nu}$ are expressed in terms of the old coefficients by means of the equations†

$$\Delta\Pi_{\lambda\mu\nu} = \sum_{i=1}^{n} A_{i\lambda}\left[\alpha_{i\mu}^{(m-\nu)} + \sum_{\kappa=1}^{n}\sum_{\tau=0}^{m-i-\nu}\binom{m-\nu}{\tau} p_{i,\kappa,\nu+\tau}\,\alpha_{\kappa\mu}^{(\tau)}\right] \tag{4}$$

$$(\lambda, \mu = 1, 2, \ldots, n;\ \nu = 0, 1, \ldots, m-1),$$

where $A_{i\lambda}$ is the minor of $\alpha_{i\lambda}$ in the determinant

$$\Delta = |\alpha_{i\lambda}| \quad (i, \lambda = 1, 2, \ldots, n).$$

The most general infinitesimal transformation of the form (3) will be obtained by putting

$$\alpha_{ii} = 1 + \phi_{ii}(x)\delta t, \qquad \alpha_{i\kappa} = \phi_{i\kappa}(x)\delta t \quad (i \neq \kappa,\ i, \kappa = 1, 2, \ldots, n),$$

* Lie, *Mathematische Annalen*, 1884, Vol. XXIV, p. 537.

† Wilczynski, *loc. cit.*, p. 93. The difference between our formula (4) and that of Wilczynski arises from the change of notation involved in introducing binomial coefficients in (1).

where δt is an infinitesimal and the $\phi_{i\kappa}$'s are arbitrary functions of x. If we substitute these values of $\alpha_{i\kappa}$ in (4) and denote the infinitesimal difference $\pi_{\lambda\mu\nu} - p_{\lambda\mu\nu}$ by $\delta p_{\lambda\mu\nu}$, we find for the infinitesimal transformations of $p_{\lambda\mu\nu}$,

$$(5)\quad \frac{\delta p_{\lambda\mu\nu}}{\delta t} = \sum_{\kappa=1}^{n} (\phi_{\kappa\mu} p_{\lambda\kappa\nu} - \phi_{\lambda\kappa} p_{\kappa\mu\nu}) + \sum_{\kappa=1}^{n} \sum_{\tau=1}^{m-1-\nu} \binom{m-\nu}{\tau} \phi_{\kappa\mu}^{(\tau)} p_{\lambda,\,\kappa,\,\nu+\tau} + \phi_{\lambda\mu}^{(m-\nu)}$$

$$(\lambda, \mu = 1, 2, \ldots, n;\ \nu = 0, 1, 2, \ldots, m-1).$$

The infinitesimal transformations of the derivatives of $p_{\lambda\mu\nu}$ can be obtained directly from these equations by differentiation.

2. *Calculation of the Seminvariants for* $m = 3$, $n = 2$.

Let us now confine ourselves to the special case of a system of two linear homogeneous differential equations of the third order. We shall assume the equations to be written in the form

$$(A)\quad \begin{cases} y''' + 3p_{11}y'' + 3p_{12}z'' + 3q_{11}y' + 3q_{12}z' + r_{11}y + r_{12}z = 0, \\ z''' + 3p_{21}y'' + 3p_{22}z'' + 3q_{21}y' + 3q_{22}z' + r_{21}y + r_{22}z = 0. \end{cases}$$

By comparing these equations with the general equations (1), we see that we have put

$$p_{\lambda\mu 2} = p_{\lambda\mu}, \quad p_{\lambda\mu 1} = q_{\lambda\mu}, \quad p_{\lambda\mu 0} = r_{\lambda\mu}.$$

The general equations (5) give for the infinitesimal transformations of the coefficients of (A) the results

$$(6)\quad \begin{cases} \dfrac{\delta p_{\lambda\mu}}{\delta t} = \sum\limits_{\kappa=1}^{2} (\phi_{\kappa\mu} p_{\lambda\kappa} - \phi_{\lambda\kappa} p_{\kappa\mu}) + \phi'_{\lambda\mu}, \\ \dfrac{\delta q_{\lambda\mu}}{\delta t} = \sum\limits_{\kappa=1}^{2} (\phi_{\kappa\mu} q_{\lambda\kappa} - \phi_{\lambda\kappa} q_{\kappa\mu} + 2\phi'_{\kappa\mu} p_{\lambda\kappa}) + \phi''_{\lambda\mu}, \\ \dfrac{\delta r_{\lambda\mu}}{\delta t} = \sum\limits_{\kappa=1}^{2} (\phi_{\kappa\mu} r_{\lambda\kappa} - \phi_{\lambda\kappa} r_{\kappa\mu} + 3\phi'_{\kappa\mu} q_{\lambda\kappa} + 3\phi''_{\kappa\mu} p_{\lambda\kappa}) + \phi'''_{\lambda\mu}. \end{cases}$$

We shall proceed immediately to the calculation of the seminvariants of the system (A). Let f be any seminvariant depending only on the arguments $p_{\lambda\mu}$, $p'_{\lambda\mu}$, $q_{\lambda\mu}$. The infinitesimal transformations of the dependent variables give to f the increment

$$\delta f = \sum_{\lambda\mu} \left(\frac{\partial f}{\partial p_{\lambda\mu}} \delta p_{\lambda\mu} + \frac{\partial f}{\partial p'_{\lambda\mu}} \delta p'_{\lambda\mu} + \frac{\partial f}{\partial q_{\lambda\mu}} \delta q_{\lambda\mu} \right).$$

This expression must vanish for all values of the arbitrary functions ϕ_{rs},

ϕ'_{rs}, and ϕ''_{rs}. Consequently the coefficients of these functions, when equated to zero, give a system of partial differential equations of which f must be a solution. Moreover, Lie's theory of infinite groups tells us that this is a complete system, and that any solution of it is a seminvariant.

Writing out this system, we find

(7)

$$\begin{cases} (a) \quad \dfrac{\partial f}{\partial p'_{rs}} + \dfrac{\partial f}{\partial q_{rs}} = 0, \\ (b) \quad \dfrac{\partial f}{\partial p_{rs}} + \sum_{\lambda=1}^{2} \left(p_{\lambda r} \dfrac{\partial f}{\partial p'_{\lambda s}} - p_{s\lambda} \dfrac{\partial f}{\partial p'_{r\lambda}} + 2 p_{\lambda r} \dfrac{\partial f}{\partial q_{\lambda s}} \right) = 0, \\ (c) \quad \sum_{\lambda=1}^{2} \left(p_{\lambda r} \dfrac{\partial f}{\partial p_{\lambda s}} - p_{s\lambda} \dfrac{\partial f}{\partial p_{r\lambda}} + p'_{\lambda r} \dfrac{\partial f}{\partial p'_{\lambda s}} - p'_{s\lambda} \dfrac{\partial f}{\partial p'_{r\lambda}} + q_{\lambda r} \dfrac{\partial f}{\partial q_{\lambda s}} - q_{s\lambda} \dfrac{\partial f}{\partial q_{r\lambda}} \right) = 0 \end{cases}$$

$(r, s = 1, 2).$

It contains 12 variables and 12 equations, but only 10 of the equations are independent. One relation between them is evident. For, the left member of the equation of (7)(*c*), obtained by putting $r = s = 1$, is, except for its sign, the same as that obtained by putting $r = s = 2$. This relation is especially important as it persists in the equations for the seminvariants containing higher derivatives of $p_{\lambda\mu}$, $q_{\lambda\mu}$, $r_{\lambda\mu}$. The second relation between equations (7) does not so persist.

The general theory of partial differential equations tells us that the system (7) has two independent solutions. We shall proceed to find them.

Equations (*a*) evidently have the four solutions

$$p_{rs} - q_{rs}.$$

Equations (*b*) show that $p'_{rs} - q_{rs}$, and p_{rs} can occur only in the four combinations

(8) $$u_{i\kappa} = p'_{i\kappa} - q_{i\kappa} + \sum_{j=1}^{2} p_{ij} p_{j\kappa} \quad (i, \kappa = 1, 2).$$

Finally, equations (*c*) are found to have the two independent solutions

(9) $$I = u_{11} + u_{22}, \qquad J = u_{11} u_{22} - u_{12} u_{21}.$$

These are the two seminvariants involving only the arguments p_{rs}, p'_{rs}, q_{rs}.

Let us next find the seminvariants involving also p''_{rs}, q'_{rs}, r_{rs}. The

system of differential equations for these seminvariants contains 24 variables and 16 equations. Only 15 of the equations are independent, as there is a relation here similar to the first one in the preceding case. The second relation does not persist. Therefore, there are just 9 independent seminvariants in this case.

By the method employed above we find that 12 of the equations are satisfied by the independent quantities

$$(10)\quad \begin{cases} u_{i\kappa},\ v_{i\kappa} = u'_{i\kappa} + \sum_{j=1}^{2} (p_{ij}u_{j\kappa} - p_{j\kappa}u_{ij}), \\ t_{i\kappa} = p''_{i\kappa} - 3q'_{i\kappa} + 2r_{i\kappa} - \sum_{j=1}^{2} (p_{ij}p'_{j\kappa} - 3p_{j\kappa}p_{ij} + 2p_{j\kappa}q_{ij} - 3p_{ij}u_{j\kappa} - p_{j\kappa}u_{ij}) \\ \qquad\qquad (i,\ \kappa = 1,\ 2). \end{cases}$$

The remaining three independent equations are satisfied by the following 9 combinations of the quantities (10),

$$(11)\quad \begin{cases} I = u_{11} + u_{22}, & J = u_{11}u_{22} - u_{12}u_{21}, \\ I' = v_{11} + v_{22}, & G = v_{11}v_{22} - v_{12}v_{21}, \\ H = t_{11} + t_{22}, & K = t_{11}t_{22} - t_{12}t_{21}, \\ J' = u_{11}v_{22} + u_{22}v_{11} - u_{12}v_{21} - u_{21}v_{12}, & \\ L = u_{11}t_{22} + u_{22}t_{11} - u_{12}t_{21} - u_{21}t_{12}, & \\ M = v_{11}t_{22} + v_{22}t_{11} - v_{12}t_{21} - v_{21}t_{12}. & \end{cases}$$

That these 9 seminvariants are independent is easily seen if we put $p_{ik} = 0$ and notice that the functional determinant

$$\frac{\partial(I, I', J, J', G, H, K, L, M)}{\partial(q_{11}, q_{12}, q_{21}, q_{22}, q'_{11}, q'_{12}, q'_{21}, r_{11}, r_{12})} = 32r_{21}(q'_{21}r_{12} - q'_{12}r_{21}) \begin{vmatrix} q_{11}-q_{22}, & q_{12}, & q_{21} \\ q'_{11}-q'_{22}, & q'_{12}, & q'_{21} \\ r_{11}-r_{22}, & r_{12}, & r_{21} \end{vmatrix}$$

does not vanish identically.

In order to determine the seminvariants involving the further variables p'''_{rs}, q'''_{rs}, r'_{rs}, we must solve a system of differential equations containing 4 more equations and 12 more variables than the system in the preceding case. The one relation between the equations maintains itself, so that we have just 8 new seminvariants. Now 7 of these are I'', J'', G', H', K', L', M'. We shall find the eighth without actually integrating the equations.

The quantities u_{ik}, v_{ik}, t_{ik} are cogredient under a transformation of the dependent variables. This fact is made evident by the infinitesimal

transformations of these quantities

$$(12)\qquad \begin{cases} \dfrac{\delta u_{i\kappa}}{\delta t} = \sum_{j=1}^{2} (u_{ij}\phi_{j\kappa} - u_{j\kappa}\phi_{ij}), \\ \dfrac{\delta v_{i\kappa}}{\delta t} = \sum_{j=1}^{2} (v_{ij}\phi_{j\kappa} - v_{j\kappa}\phi_{ij}), \\ \dfrac{\delta t_{i\kappa}}{\delta t} = \sum_{j=1}^{2} (t_{ij}\phi_{j\kappa} - t_{j\kappa}\phi_{ij}). \end{cases}$$

Now we have found in (11) that certain combinations of the u_{ik}'s and the p_{ik}'s are seminvariants. If we replace u_{ik} by v_{ik} or by t_{ik} in these combinations, we must still have seminvariants. In particular, let us form the quantities w_{ik} from the v_{ik}'s in the same way that the latter are formed from the u_{ik}'s, so that we have

$$(13)\qquad w_{ik} = v'_{ik} + \sum_{j=1}^{2} (p_{ij}v_{jk} - p_{jk}v_{ij}).$$

Then the combination

$$(14)\qquad N = w_{11}w_{22} - w_{12}w_{21}$$

is obtained by replacing u_{ik} by v_{ik} in G, and is, therefore, a seminvariant. That it is independent of the seminvariants (11) and their first derivatives can be proved in the same way that the seminvariants (11) were proved independent.

We have now found 17 independent seminvariants involving only p_{ik}, p'_{ik}, p''_{ik}, p'''_{ik}, q_{ik}, q'_{ik}, q''_{ik}, r_{ik}, r'_{ik}. Any other seminvariant depending on these quantities alone must be expressible in terms of the seminvariants which we have found.

The system of equations for the seminvariants involving also the next higher derivatives of p_{ik}, q_{ik}, r_{ik} contains 4 more independent equations and 12 more variables than the system in the last case. Therefore there are 8 more seminvariants. These are evidently the next higher derivatives of I, J, G, H, K, L, M, N. If we proceed in this way, we find that *all the seminvariants of the system* (A) *are functions of* I, J, G, H, K, L, M, N *and of their derivatives.*

Let us write the transformation (3) of the dependent variables in the form

$$y = a\bar{y} + \beta\bar{z}, \qquad z = \gamma\bar{y} + \delta\bar{z}, \qquad a\delta - \beta\gamma = \Delta.$$

If we make this transformation and denote the coefficients of the new

system by $\bar{p}_{ik}$, $\bar{q}_{ik}$, $\bar{r}_{ik}$, we find

$$(15)\qquad \begin{cases} \Delta\bar{p}_{11} = \alpha'\delta - \gamma'\beta + p_{11}\alpha\delta + p_{12}\gamma\delta - p_{21}\alpha\beta - p_{22}\gamma\beta, \\ \Delta\bar{p}_{12} = \beta'\delta - \delta'\beta + p_{11}\beta\delta + p_{12}\delta^2 - p_{21}\beta^2 - p_{22}\delta\beta. \end{cases}$$

$$(16)\qquad \begin{cases} \Delta\bar{q}_{11} = \alpha''\delta - \gamma''\beta + 2p_{11}\alpha'\delta + 2p_{12}\gamma'\delta - 2p_{21}\alpha'\beta - 2p_{22}\gamma'\beta \\ \qquad\qquad + q_{11}\alpha\delta + q_{12}\gamma\delta - q_{21}\alpha\beta - q_{22}\gamma\beta, \\ \Delta\bar{q}_{12} = \beta''\delta - \delta''\beta + 2p_{11}\beta'\delta + 2p_{12}\delta'\delta - 2p_{21}\beta'\beta - 2p_{22}\delta'\beta \\ \qquad\qquad + q_{11}\beta\delta + q_{12}\delta^2 - q_{21}\beta^2 - q_{22}\delta\beta, \end{cases}$$

$$(17)\qquad \begin{cases} \Delta\bar{r}_{11} = \alpha'''\delta - \gamma'''\beta + 3p_{11}\alpha''\delta + 3p_{12}\gamma''\delta - 3p_{21}\alpha''\beta - 3p_{22}\gamma''\beta \\ \qquad\qquad + 3q_{11}\alpha'\delta + 3q_{12}\gamma'\delta - 3q_{21}\alpha'\beta - 3q_{22}\gamma'\beta \\ \qquad\qquad + r_{11}\alpha\delta + r_{12}\gamma\delta - r_{21}\alpha\beta - r_{22}\gamma\beta, \\ \Delta\bar{r}_{12} = \beta'''\delta - \delta'''\beta + 3p_{11}\beta''\delta + 3p_{12}\delta''\delta - 3p_{21}\beta''\beta - 3p_{22}\delta''\beta \\ \qquad\qquad + 3q_{11}\beta'\delta + 3q_{12}\delta'\delta - 3q_{21}\beta'\beta - 3q_{22}\delta'\beta \\ \qquad\qquad + r_{11}\beta\delta + r_{12}\delta^2 - r_{21}\beta^2 - r_{22}\delta\beta. \end{cases}$$

The values of $\bar{p}_{21}$ and $\bar{p}_{22}$ can be obtained from the values of $\bar{p}_{12}$ and $\bar{p}_{11}$ respectively by interchanging simultaneously the quantities α and δ, β and γ, and the subscripts 1 and 2. By the same changes we can derive $\bar{q}_{21}$ and $\bar{q}_{22}$ from (16), and $\bar{r}_{12}$ and $\bar{r}_{22}$ from (17).

From these equations and equations (8) we find

$$(18)\qquad \begin{cases} \Delta\bar{u}_{11} = u_{11}\alpha\delta + u_{12}\gamma\delta - u_{21}\alpha\beta - u_{22}\gamma\beta, \\ \Delta\bar{u}_{12} = u_{11}\beta\delta + u_{12}\delta^2 - u_{21}\beta^2 - u_{22}\delta\beta, \\ \Delta\bar{u}_{21} = -u_{11}\alpha\gamma - u_{12}\gamma^2 + u_{21}\alpha^2 + u_{22}\alpha\gamma, \\ \Delta\bar{u}_{22} = -u_{11}\beta\gamma - u_{12}\gamma\delta + u_{21}\alpha\beta + u_{22}\alpha\delta. \end{cases}$$

Since v_{ik}, t_{ik}, w_{ik} are cogredient with u_{ik}, the equations for $\bar{v}_{ik}$, $\bar{t}_{ik}$, $\bar{w}_{ik}$ are of the same form as (18).

3. *Calculation of the Invariants.*

Evidently the invariants of system (A) are such functions of the seminvariants as remain unchanged after an arbitrary transformation of the independent variable. Therefore, in order to calculate the invariants it is necessary to find the effect of such a transformation on the seminvariants.

If we make the transformation

$$\bar{\xi} = \bar{\xi}(x),$$

and denote the coefficients of the new system by $\bar{p}_{ik}$, $\bar{q}_{ik}$, $\bar{r}_{ik}$, we find

$$(19)\quad \begin{cases} \bar{p}_{11} = \dfrac{1}{\bar{\xi}'}(p_{11}+\eta), & \bar{p}_{12} = \dfrac{1}{\bar{\xi}'}\,p_{12}, \\ \bar{p}_{21} = \dfrac{1}{\bar{\xi}'}\,p_{21}, & \bar{p}_{22} = \dfrac{1}{\bar{\xi}'}(p_{22}+\eta), \\ \bar{q}_{11} = \dfrac{1}{(\bar{\xi}')^2}\left(q_{11}+p_{11}\eta+\dfrac{\eta'+\eta^2}{3}\right), & \bar{q}_{12} = \dfrac{1}{(\bar{\xi}')^2}(q_{12}+p_{12}\eta), \\ \bar{q}_{21} = \dfrac{1}{(\bar{\xi}')^2}(q_{21}+p_{21}\eta), & \bar{q}_{22} = \dfrac{1}{(\bar{\xi}')^2}\left(q_{22}+p_{22}\eta+\dfrac{\eta'+\eta^2}{3}\right), \\ & \bar{r}_{ik} = \dfrac{1}{(\bar{\xi}')^3}\,r_{ik}, \end{cases}$$

where

$$\eta = \frac{\bar{\xi}''}{\bar{\xi}'}.$$

In these equations the expression for $\bar{p}_{11}$ and $\bar{p}_{12}$ are of the same form as the expressions for $\bar{p}_{22}$ and $\bar{p}_{21}$ respectively. The same is true for $\bar{q}_{ik}$ and $\bar{r}_{ik}$, and is, therefore, true also for $\bar{u}_{ik}$, $\bar{v}_{ik}$, $\bar{t}_{ik}$, $\bar{w}_{ik}$. Consequently, we need to indicate only the effect of the transformation on those quantities whose subscripts are 11 and 12.

From (8) and the formula obtained by differentiating $\bar{p}_{ik}$ with respect to $\bar{\xi}$, we obtain

$$(20)\qquad \bar{u}_{11} = \frac{1}{(\bar{\xi}')^2}(u_{11}+\tfrac{2}{3}\mu), \qquad \bar{u}_{12} = \frac{1}{(\bar{\xi}')^2}\,u_{12},$$

where

$$\mu = \eta' - \tfrac{1}{2}\eta^2.$$

From (20), we find

$$(21)\qquad \bar{I} = \frac{1}{(\bar{\xi}')^2}(I+\tfrac{4}{3}\mu), \qquad \bar{J} = \frac{1}{(\bar{\xi}')^4}(J+\tfrac{2}{3}\mu I+\tfrac{4}{9}\mu^2).$$

These equations give at once the relative invariant of weight 4,

$$(22)\qquad \Theta_4 = I^2 - 4J.*$$

* For the notion of weight, see Wilczynski, *loc. cit.*, p. 106.

From equations (10) we find

$$(23)\qquad \bar{v}_{11} = \frac{1}{(\bar{\xi}')^3}(v_{11} - 2u_{11}\eta + \tfrac{2}{3}\mu' - \tfrac{4}{3}\mu\eta), \quad \bar{v}_{12} = \frac{1}{(\bar{\xi}')^3}(v_{12} - 2u_{12}\eta),$$

and

$$(24)\qquad \bar{t}_{11} = \frac{1}{(\bar{\xi}')^3} t_{11}, \quad \bar{t}_{12} = \frac{1}{(\bar{\xi}')^3} t_{12}.$$

The last equations show at once that H and K are relative invariants of weight 3 and 6 respectively. We shall denote them by Θ_3 and Θ_6.

Finally, from (13), we find

$$(25)\qquad \begin{cases} \bar{w}_{11} = \dfrac{1}{(\bar{\xi}')^4}(w_{11} + 5u_{11}\eta^2 - 2u_{11}\mu - 5v_{11}\eta + \tfrac{2}{3}\mu'' - \tfrac{10}{3}\mu'\eta - \tfrac{4}{3}\mu^2 + \tfrac{10}{3}\mu\eta^2), \\ \bar{w}_{12} = \dfrac{1}{(\bar{\xi}')^4}(w_{12} + 5u_{12}\eta^2 - 2u_{12}\mu - 5v_{12}\eta). \end{cases}$$

In computing the invariants we shall need the infinitesimal transformations of the quantities whose finite transformations we have found. The most general infinitesimal transformation of the independent variable is found by putting

$$(26)\qquad \bar{\xi}(x) = x + \phi(x)\,\delta t, \quad \delta x = \phi(x)\,\delta t,$$

where $\phi(x)$ is an arbitrary function and δt an infinitesimal. For this value of $\bar{\xi}(x)$ we find

$$\bar{\xi}' = 1 + \phi'(x)\,\delta t, \quad \eta = \phi''(x)\,\delta t, \quad \mu = \phi'''(x)\,\delta t, \quad \mu' = \phi^{(4)}\delta t, \quad \ldots.$$

Substituting these values in (19), we have

$$(27)\qquad \begin{cases} \delta p_{11} = (\phi'' - \phi' p_{11})\,\delta t, & \delta p_{12} = -\phi' p_{12}\,\delta t, \\ \delta q_{11} = (\tfrac{1}{3}\phi''' + \phi'' p_{11} - 2\phi' q_{11})\,\delta t, & \delta q_{12} = (\phi'' p_{12} - 2\phi' q_{12})\,\delta t, \\ \delta r_{ik} = -3\phi' r_{ik}\,\delta t. \end{cases}$$

From (20), we find

$$(28)\qquad \delta u_{11} = (\tfrac{2}{3}\phi''' - 2\phi' u_{11})\,\delta t, \quad \delta u_{12} = -2\phi' u_{12}\,\delta t,$$

whence

$$(29)\qquad \delta I = (\tfrac{4}{3}\phi''' - 2\phi' I)\,\delta t, \quad \delta J = (\tfrac{2}{3}\phi''' I - 4\phi' J)\,\delta t.$$

Again, from (23),

$$(30)\quad \delta v_{11} = (\tfrac{2}{3}\phi^{(4)} - 2\phi'' u_{11} - 3\phi' v_{11})\,\delta t, \quad \delta v_{12} = (-2\phi'' u_{12} - 3\phi' v_{12})\,\delta t,$$

whence

$$\delta G = (\tfrac{2}{3}\phi^{(4)} I' - 2\phi'' J' - 6\phi' G)\delta t. \tag{31}$$

Further

$$\delta t_{ik} = -3\phi' t_{ik}\delta t, \tag{32}$$

so that we have

$$\begin{cases} \delta L = (\tfrac{2}{3}\phi''' H - 5\phi' L)\delta t, \\ \delta M = (\tfrac{2}{3}\phi^{(4)} H - 2\phi'' L - 6\phi' M)\delta t. \end{cases} \tag{33}$$

Finally, from (25),

$$\begin{cases} \delta w_{11} = (\tfrac{2}{3}\phi^{(5)} - 2\phi''' u_{11} - 5\phi'' v_{11} - 4\phi' w_{11})\delta t, \\ \delta w_{12} = (-2\phi''' u_{12} - 5\phi'' v_{12} - 4\phi' w_{12})\delta t, \end{cases} \tag{34}$$

whence

$$\delta N = (\tfrac{2}{3}\phi^{(5)} I'' - 2\phi''' J'' + 4\phi''' G - 5\phi'' G' - 8\phi' N)\delta t. \tag{35}$$

By applying the formula

$$\delta\left(\frac{df}{dx}\right) = \frac{d}{dx}(\delta f) - \phi'\frac{df}{dx}\delta t,^*$$

we can find the infinitesimal transformations of the derivatives of the seminvariants. In particular we have

$$\begin{cases} \delta I' = (\tfrac{4}{3}\phi^{(4)} - 2\phi'' I - 3\phi' I')\delta t, \\ \delta J' = (\tfrac{2}{3}\phi^{(4)} I + \tfrac{2}{3}\phi''' I' - 4\phi'' J - 5\phi' J')\delta t. \end{cases} \tag{36}$$

We have found three invariants, Θ_3, Θ_4, and Θ_6, of weight 3, 4 and 6 respectively. We can easily find one of weight 5. We have

$$\frac{\delta(IH)}{\delta t} = \tfrac{4}{3}\phi''' H - 5\phi' IH.$$

From this expression and (33), we see at once that

$$\Theta_5 = IH - 2L \tag{37}$$

is such an invariant.

In a similar manner we can obtain an invariant of weight 10. We have

$$\frac{\delta(4G - I'^2)}{\delta t} = 4\phi''(II' - 2J') - 6\phi'(4G - I'^2),$$

$$\frac{\delta(II' - 2J')}{\delta t} = -2\phi''(I^2 - 4J) - 5\phi'(II' - 2J').$$

---

* Wilczynski, *loc. cit.*, p. 106.

By the elimination of ϕ'' from these equations, we obtain the invariant

$$\Theta_{10} = (I^2-4J)(4G-I'^2)+(II'-2J')^2. \tag{38}$$

The invariants which we have found thus far involve only the quantities p_{ik}, p'_{ik}, p''_{ik}, q_{ik}, q'_{ik}, r_{ik}. Let us now find all the independent invariants depending on these quantities alone. Such invariants must be functions of the seminvariants

$$I,\ I',\ J,\ J',\ G,\ H,\ K,\ L,\ M. \tag{39}$$

We can find the system of partial differential equations for the invariants depending on the quantities (39) in the same way that we have found the systems for the seminvariants. This system will be found to contain 4 independent equations, and has, therefore, 5 solutions. Hence there are 6 such relative invariants, We have found 5, and by integrating the equations we easily find another of weight 10,

$$\Im_{10} = (I'H-2M)(I^2-4J)-(IH-2L)(II'-2J'). \tag{40}$$

Thus we have found the following six relative invariants

$$\Theta_3,\ \Theta_4,\ \Theta_5,\ \Theta_6,\ \Theta_{10},\ \Im_{10}. \tag{41}$$

That they are independent can be easily verified by considering them in the above order and noticing that each contains a seminvariant not in a preceding one.

We can prove by direct substitution that

$$\Theta_8 = \begin{vmatrix} u_{11}-u_{22}, & u_{12}, & u_{21} \\ t_{11}-t_{22}, & t_{12}, & t_{21} \\ v_{11}-v_{22}, & v_{12}, & v_{21} \end{vmatrix} \tag{42}$$

is also an invariant. It depends on the same quantities as the invariants (41), and therefore must be expressible in terms of them. We shall find this expression later, and we shall see that Θ_8 is not a rational function of the invariants (41).

We can easily find a number of invariants involving the next higher derivatives of p_{ik}, q_{ik}, r_{ik}. The *Jacobian* of any two invariants Θ_μ and Θ_ν is an invariant of weight $\mu+\nu+1$. By the Jacobian of Θ_μ and Θ_ν we mean the factor $\mu\Theta_\mu\Theta'_\nu-\nu\Theta_\nu\Theta'_\mu$ which occurs in the numerator of the derivative of the absolute invariant $\Theta^\mu_\nu/\Theta^\nu_\mu$. We have the following

Jacobians

$$(43)\qquad \begin{cases} \Im_8 = 4\Theta_4\Theta_3' - 3\Theta_3\,\Theta_4', \\ \Phi_{10} = 4\Theta_4\Theta_5' - 5\Theta_5\,\Theta_4', \\ \Theta_{11} = 2\Theta_4\Theta_6' - 3\Theta_6\,\Theta_4', \\ \Theta_{15} = 2\Theta_4\Theta_{10}' - 5\Theta_{10}\Theta_4', \\ \Im_{15} = 2\Theta_4\,\Im_{10}' - 5\Im_{10}\Theta_4'. \end{cases}$$

From any invariant Θ_m we can always deduce another of weight $2m+2$,

$$\Theta_{m\cdot 1} = 2m\Theta_m''\Theta_m - (2m+1)(\Theta_m')^2 + \tfrac{3}{2}m^2 I\Theta_m^2,$$

which is called the *quadriderivative** of Θ_m. In particular we have

$$(44)\qquad \Theta_{4\cdot 1} = 8\Theta_4''\Theta_4 - 9\,(\Theta_4')^2 + 24I\Theta_4^2.$$

We have further the invariant

$$(45)\qquad \Theta_9 = \begin{vmatrix} u_{11}-u_{22}, & u_{12}, & u_{21} \\ v_{11}-v_{22}, & v_{12}, & v_{21} \\ w_{11}-w_{22}, & w_{12}, & w_{21} \end{vmatrix},$$

similar to Θ_8 and containing the seminvariant N. If the invariants (41), (43), (44), (45) are considered in the proper order, it is easily seen that they are independent.

The system of differential equations for the invariants depending on the seminvariants (39), and their first derivatives shows that there are just 13 independent relative invariants of this kind. We have found all of them.

4. *The Complete System of Invariants.*

Before proving that we have found a complete system of invariants, it will be necessary to show how the system (A) can be transformed so that it will have certain special properties. Let us make the transformation

$$y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta, \qquad \alpha\delta - \beta\gamma \neq 0.$$

The system (A) goes into another of the same form whose terms in η'' and

* Wilczynski, *loc. cit.*, p. 112.

ζ'' will vanish if (α, γ) and (β, δ) are chosen as two pairs of solutions of the system of equations

$$\left\{\begin{aligned} \rho' &= -(p_{11}\rho + p_{12}\sigma), \\ \sigma' &= -(p_{21}\rho + p_{22}\sigma). \end{aligned}\right. \tag{46}$$

Since $\alpha\delta - \beta\gamma \neq 0$, (α, γ) and (β, δ) must be two independent systems of solutions. As two such systems always exist, the system (A) can always be converted into another for which $p_{ik} = 0$. We shall call this the *semi-canonical form of the system* (A).

Let us now assume that the system is given in its semi-canonical form. The general transformation

$$\bar{\xi} = \bar{\xi}(x), \quad y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta,$$

converts it into a system which is also in the semi-canonical form, pro vided that

$$\alpha\bar{\xi}'' + \alpha'\bar{\xi}' = \beta\bar{\xi}'' + \beta'\bar{\xi}' = \gamma\bar{\xi}'' + \gamma'\bar{\xi}' = \delta\bar{\xi}'' + \delta'\bar{\xi}' = 0.$$

Thus α, β, γ, δ must have the values

$$\alpha = \frac{a}{\bar{\xi}'}, \quad \beta = \frac{b}{\bar{\xi}'}, \quad \gamma = \frac{c}{\bar{\xi}'}, \quad \delta = \frac{d}{\bar{\xi}'},$$

where a, b, c, d are constants whose determinant $ad - bc$ does not vanish. Therefore, *the most general transformation which leaves the semi-canonical form invariant is given by the equations*

$$\bar{\xi} = \bar{\xi}(x), \quad \eta = (ay + bz)\bar{\xi}', \quad \zeta = (cy + dz)\bar{\xi}', \tag{47}$$

where $\bar{\xi}(x)$ is an arbitrary function, and a, b, c, d are arbitrary constants.

We can reduce the system (A) to another special form which will be found very useful. Let us choose the coefficients α, β, γ, δ of the general transformation of the dependent variables in such a way that β/δ and α/γ are the two roots of the equation

$$-u_{21}\lambda^2 + (u_{11} - u_{22})\lambda + u_{12} = 0. \tag{48}$$

Then equations (18) show that $\bar{u}_{12} = \bar{u}_{21} = 0$. Since $\alpha\delta - \beta\gamma$ must not vanish, the roots of (48) must be distinct. This means that the transformation is not possible if

$$(u_{11} - u_{22})^2 + 4u_{12}u_{21} = \Theta_4 = 0.$$

In exactly the same way we can reduce the system (A) to another for which

$$\bar{t}_{12} = \bar{t}_{21} = 0,$$

if $$(t_{11}-t_{22})^2+4t_{12}t_{21} = H^2-4K = \Theta_3^2-4\Theta_6 \neq 0.$$

Let us assume that system (A) has been reduced to a form for which either $u_{12} = u_{21} = 0$ or $t_{12} = t_{21} = 0$. Equations (18), (20), and (24) show that these conditions will be left invariant by a transformation of the form

$$y = \alpha\eta, \quad z = \delta\zeta, \quad \xi = \xi(x),$$

where α, δ and ξ are arbitrary functions of x. After such a transformation has been made, the new system will have its coefficients $\bar{p}_{11}$ and $\bar{p}_{22}$ equal to zero if α and δ satisfy the equations

$$\alpha'+p_{11}\alpha+\frac{\xi''}{\xi'}\alpha = 0, \quad \delta'+p_{22}\delta+\frac{\xi''}{\xi'}\delta = 0.$$

By solving these equations we see that a system for which either $u_{12} = u_{21} = 0$ or $t_{12} = t_{21} = 0$ is converted into another having also $p_{11} = p_{22} = 0$ by the transformation

$$y = \frac{a}{\xi'}e^{-\int p_{11}\,dx}\eta, \quad z = \frac{b}{\xi'}e^{-\int p_{22}\,dx}\zeta, \quad \xi = \xi(x),$$

where ξ is an arbitrary function, and a and b are arbitrary constants.

We can still select the arbitrary function $\xi(x)$ so as to make any of the non-vanishing invariants equal to unity. In particular, the equation

$$\bar{\Theta}_4 = \frac{1}{(\xi')^4}\Theta_4$$

shows that we shall have $$\Theta_4 = (u_{11}-u_{22})^2 = 1$$

if we put $$\xi = \int\sqrt{(u_{11}-u_{22})}\,dx+\text{const.}$$

We can now easily find the expression for Θ_8 in terms of the invariants (41). Let us assume that the system (A) has been transformed into one for which

$$u_{12} = u_{21} = 0, \quad p_{11} = p_{22} = 0, \quad u_{11}-u_{22} = 1.$$

A relation between the invariants will not be disturbed. Under these conditions we have

$$\Theta_8 = t_{12}p_{21}+t_{21}p_{12},$$
$$\Theta_5 = t_{11}-t_{22},$$
$$\Theta_3^2-4\Theta_6 = 4t_{12}t_{21}+(t_{11}-t_{22})^2,$$
$$\Theta_{10} = 4p_{12}p_{21},$$
$$\Im_{10} = 2(t_{12}p_{21}-t_{21}p_{12}),$$

whence $$4\Theta_3^2\Theta_4 = \Theta_{10}(\Theta_3^2\Theta_4 - 4\Theta_6\Theta_4 - \Theta_5^2) + \mathfrak{I}_{10}^2.$$

In deducing this equation we have introduced $\Theta_4 = 1$ in such a way as to make the whole expression isobaric.

Let us now assume that the eight independent invariants

$$\Theta_3,\ \Theta_4,\ \Theta_5,\ \Theta_8,\ \Theta_9,\ \Theta_{4\cdot 1},\ \Theta_{10},\ \mathfrak{I}_{10} \tag{49}$$

are given as functions of x. We shall show that a system of equations of the form (A) having these invariants is determined uniquely except for transformations of the form

$$y = \alpha\eta + \beta\xi, \qquad z = \gamma\eta + \delta\xi.$$

For the present we shall assume that Θ_4 and Θ_{10} are both different from zero.

Suppose that

$$\left\{\begin{aligned} &y''' + 3p_{11}y'' + 3p_{12}z'' + 3q_{11}y' + 3q_{12}z' + r_{11}y + r_{12}z = 0,\\ &z''' + 3p_{21}y'' + 3p_{22}z'' + 3q_{21}y' + 3q_{22}z' + r_{21}y + r_{22}z = 0, \end{aligned}\right. \tag{50}$$

is a system whose coefficients are unknown, but whose invariants (49) are the arbitrarily chosen functions of x. Since we are taking Θ_4 different from zero, we can transform (50) into a system satisfying the conditions

$$u_{12} = u_{21} = p_{11} = p_{22} = 0. \tag{51}$$

Then the invariants (49) become

$$\left\{\begin{aligned} \Theta_4 &= (u_{11} - u_{22})^2,\\ \Theta_{4\cdot 1} &= 8\Theta_4\Theta_4'' - 9(\Theta_4')^2 + 24(u_{11} + u_{22})\Theta_4^2,\\ \Theta_{10} &= 4p_{12}p_{21}(u_{11} - u_{22})^4,\\ \Theta_9 &= (u_{11} - u_{22})^3(p_{12}'p_{21} - p_{21}'p_{12}),\\ \Theta_3 &= 3(u_{11}' + u_{22}') - 6(p_{12}'p_{21} + p_{21}'p_{12}) + 2(r_{11} + r_{22}),\\ \Theta_5 &= (u_{11} - u_{22})\{3(u_{11}' - u_{22}') + 2(p_{12}'p_{21} - p_{21}'p_{12}) + 2(r_{11} - r_{22})\},\\ \mathfrak{I}_{10} &= 2(u_{11} - u_{22})^3[\{2p_{21}'' - 3p_{21}u_{11} - p_{21}u_{22} + 2p_{21}q_{22} - 2r_{21}\}p_{12}\\ &\qquad - \{2p_{12}'' - p_{12}u_{11} - 3p_{12}u_{22} + 2p_{12}q_{11} - 2r_{12}\}p_{21}],\\ \Theta_8 &= -(u_{11} - u_{22})^2[\{2p_{21}'' - 3p_{21}u_{11} - p_{21}u_{22} + 2p_{21}q_{22} - 2r_{21}\}p_{12}\\ &\qquad + \{2p_{12}'' - p_{12}u_{11} - 3p_{12}u_{22} + 2p_{12}q_{11} - 2r_{12}\}p_{21}]. \end{aligned}\right. \tag{52}$$

These expressions together with the conditions (51) enable us to express the coefficients of (50) in terms of the invariants (49). Thus we find

$$
(53)\quad \left\{\begin{aligned}
&p_{11}=0, \qquad p_{22}=0,\\
&p_{12}=\epsilon' C\frac{\sqrt{\Theta_{10}}}{2\Theta_4}\,e^{2\epsilon\int\sqrt{\Theta_4}\,(\Theta_9/\Theta_{10})\,dx},\\
&p_{21}=\epsilon'\frac{1}{C}\frac{\sqrt{\Theta_{10}}}{2\Theta_4}\,e^{-2\epsilon\int\sqrt{\Theta_4}\,(\Theta_9/\Theta_{10})\,dx},\\
&q_{12}=p'_{12}, \qquad q_{21}=p'_{21},\\
&48q_{11}=\frac{1}{\Theta_4^2}\{12\Theta_{10}-\Theta_{4\cdot 1}+8\Theta_4\Theta''_4-9(\Theta'_4)^2\}-24\epsilon\sqrt{\Theta_4},\\
&48q_{22}=\frac{1}{\Theta_4^2}\{12\Theta_{10}-\Theta_{4\cdot 1}+8\Theta_4\Theta''_4-9(\Theta'_4)^2\}+24\epsilon\sqrt{\Theta_4},\\
&4r_{11}=\Theta_3+\frac{\Theta_5}{\epsilon\sqrt{\Theta_4}}+8p_{12}p'_{21}+4p'_{12}p_{21}-6u'_{11},\\
&4r_{22}=\Theta_3-\frac{\Theta_5}{\epsilon\sqrt{\Theta_4}}+8p'_{12}p_{21}+4p_{12}p'_{21}-6u'_{22},\\
&4r_{12}=\frac{\Theta_8}{\Theta_4 p_{21}}+\frac{\Theta_{10}}{2\epsilon\Theta_4^{\frac{3}{2}}p_{21}}+4p''_{12}+4p_{12}q_{11}-2u_{11}p_{12}-6u_{22}p_{12},\\
&4r_{21}=\frac{\Theta_8}{\Theta_4 p_{12}}-\frac{\Theta_{10}}{2\epsilon\Theta_4^{\frac{3}{2}}p_{12}}+4p''_{21}+4p_{21}q_{22}-6u_{11}p_{21}-2u_{22}p_{21},
\end{aligned}\right.
$$

where C is an arbitrary constant and $\epsilon=\pm 1$ and $\epsilon'=\pm 1$. The method of expressing q_{12}, q_{21}, and r_{ik} in terms of the invariants directly is obvious.

We can easily remove the arbitrary constant and the ambiguous signs. The transformation

$$y=k\eta, \qquad z=l\zeta,$$

where

$$\frac{k}{l}=\epsilon' C$$

will convert a system whose coefficients are given by (53) with C equal to any constant, and ϵ' either $+1$ or -1 into the system whose coefficients are also given by (53) with $C=1$, $\epsilon'=1$. Again, the transformation

$$y=\xi, \qquad z=\eta,$$

will convert the system whose coefficients are given by (53) with $\epsilon=-1$

into the system whose coefficients are given by the same equations with $\epsilon = 1$.

We have now proved the following fundamental theorem. *If the invariants* (49) *are given as arbitrary functions of* x, *but so that* Θ_4 *and* Θ_{10} *are different from zero, they determine a system of differential equations of the form* (A) *uniquely except for transformations of the dependent variables of the form*

$$y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta,$$

where α, β, γ, δ *are arbitrary functions of* x.

We shall see later that the system is not completely determined if $\Theta_4 = 0$ or $\Theta_{10} = 0$, so that the theorem really breaks down in these cases.

The invariants (49) *together with those formed from them by repetitions of the Jacobian process give a functionally complete system of invariants of system* (A). We shall now show this. As we have just seen, the coefficients of the system can be expressed in terms of the invariants Θ_3, Θ_4, ..., ϑ_{10} of the system, their derivatives and certain exponentials. Since the seminvariants (11) and (14) are functions of the coefficients, they can also be expressed in terms of the invariants Θ_3, Θ_4, ..., ϑ_{10}, their derivatives and certain exponentials. However, by direct substitution, we find that the exponential factors cancel out in the seminvariants, and that the seminvariants are rational algebraic functions of the invariants and their derivatives alone. This fact tells us that any invariant can be expressed as a function of Θ_3, Θ_4, ..., ϑ_{10} and of their derivatives. If we write out the system of partial differential equations for the invariants depending on Θ_3, Θ_4, ..., ϑ_{10} and their first derivatives, we find that it contains 2 independent equations and 16 variables. Hence there are 15 relative invariants depending on these quantities. The 8 invariants (49) together with the 7 obtained by combining Θ_4 with each of the others by the Jacobian process give these 15 invariants.

The same reasoning tells us that there are 7 more invariants depending on the invariants (49) and their first and second derivatives. These can be obtained by combining Θ_4 once more with the other 7 of (49) by means of the Jacobian process. It is evident that we can continue in this way and obtain the invariants depending on the invariants (49) and their derivatives up to any order by repeated applications of the Jacobian process.

5. *The Covariants.*

It is easy to verify directly that

$$E_1 = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz \tag{54}$$

is a semi-covariant of weight 2. Since v_{ik}, t_{ik}, w_{ik} are cogredient with u_{ik} for transformations of the dependent variables, we must have the three additional semi-covariants

$$(55)\qquad \begin{cases} E_2 = t_{12}z^2 - t_{21}y^2 + (t_{11} - t_{22})yz, \\ E_3 = v_{12}z^2 - v_{21}y^2 + (v_{11} - v_{22})yz, \\ E_4 = w_{12}z^2 - w_{21}y^2 + (w_{11} - w_{22})yz. \end{cases}$$

Making the transformation

$$y = \alpha\bar{y} + \beta\bar{z}, \qquad z = \gamma\bar{y} + \delta\bar{z}, \qquad \Delta = \alpha\delta - \beta\gamma \neq 0,$$

we find $$E_1 = \Delta\bar{E}_1,$$

whence $$\frac{E_1'}{E_1} = \frac{\bar{E}_1'}{\bar{E}_1} + \frac{\Delta'}{\Delta}$$

Also we have, from (15), $p_{11} + p_{22} = \bar{p}_{11} + \bar{p}_{22} - \dfrac{\Delta'}{\Delta}$.

The last two equations show that

$$(56)\qquad F_1 = E_1' + (p_{11} + p_{22})E_1$$

is a semi-covariant. In the same way we can obtain three other semi-covariants from equations (55).

Let us put

$$(57)\qquad \rho = y' + p_{11}y + p_{12}z, \qquad \sigma = z' + p_{21}y + p_{22}z.$$

It is easily verified that ρ and σ are cogredient with y and z respectively for transformations of the dependent variables. Consequently we have a semi-covariant

$$(58)\qquad P_1 = z\rho - y\sigma.$$

Again, let us put

$$(59)\qquad \mu = \rho' + p_{11}\rho + p_{12}\sigma, \qquad \nu = \sigma' + p_{21}\rho + p_{22}\sigma.$$

Since μ and ν are of the same form in ρ and σ as are ρ and σ in y and z, they are cogredient with ρ and σ, and also with y and z. Consequently we have the two further semi-covariants

$$60\qquad \begin{cases} P_2 = z\mu - y\nu, \\ P_3 = \sigma\mu - \rho\nu. \end{cases}$$

We need not consider semi-covariants containing any higher derivatives of y and z than the second. For, derivatives of the third and higher

order can be removed by means of the original equations (A). The system of differential equations for the semi-covariants involving the arguments p_{ik}, p'_{ik}, p''_{ik}, q_{ik}, q'_{ik}, r_{ik}, and y, y', y'', z, z', z'' is the same as the system for the seminvariants depending on the first twenty-four quantities with the terms containing the partial derivatives with respect to the last six quantities added. This new system contains 16 independent equations and 30 variables. Hence there are 15 seminvariants and relative semi-covariants depending on the arguments mentioned. Since 9 of the 15 are seminvariants, 6 of them must be semi-covariants. We can take them to be the 6 independent semi-covariants

$$E_1,\ E_2,\ F_1,\ P_1,\ P_2,\ P_3.$$

The system of equations for the semi-covariants involving higher derivatives of p_{ik}, q_{ik}, r_{ik} shows that no new independent semi-covariants can occur.

The absolute covariants must be functions of the seminvariants and absolute semi-covariants. Hence the system of differential equations for the covariants will be the same as the system for the invariants, except that terms involving derivatives with respect to the five absolute semi-covariants must be added. Thus we see that there are five independent absolute, or six independent relative, covariants.

We can verify directly that

(61) $$E_1,\ E_2,\ F_1,\ P_1$$

are covariants. From the relations

$$\bar{P}_1 = \frac{1}{\xi'}\ P_1,$$

$$\bar{\Theta}_4 = \frac{1}{(\xi')^4}\,\Theta_4,$$

$$\bar{\Theta}'_4 = \frac{1}{(\xi')^5}\,(\Theta'_4 - 4\eta\Theta_4),$$

$$\bar{P}_2 = \frac{1}{(\xi')^2}\,(P_2 + \eta P_1),$$

we derive at once the covariant

(62) $$4P_2\Theta_4 + P_1\Theta'_4.$$

Finally, by integrating the system of differential equations, we obtain the sixth independent relative covariant

(63) $$(30I\Theta_4 + \Theta''_4)\,P_1 + 9\Theta'_4 P_2 + 36\Theta_4 P_3.$$

We have found the following independent covariants

$$(64)\qquad \begin{cases} C_1 = P_1, \quad C_2 = E_1, \quad C_3 = E_2, \\ K_3 = E_1' + (p_{11}+p_{22})E_1, \quad C_6 = 4P_2\Theta_4 + P_1\Theta_4', \\ C_7 = (30I\Theta_4 + \Theta_4'')P_1 + 9\Theta_4'P_2 + 36\Theta_4 P_3, \end{cases}$$

all of which are quadratic. The weight of each is indicated by the subscript.

6. *Geometrical Interpretation. Ruled Surfaces in 5 Dimensions.*

The general theory of differential equations teaches us that the system (A) possesses two *general* solutions y and z, *i.e.*, two solutions which are analytic in the vicinity of $x = x_0$ if the coefficients are analytic in that vicinity, and which can be made to satisfy also the conditions that y, z, y', z', y'', z'' shall assume arbitrarily prescribed values for $x = x_0$. Furthermore, we easily see that any six pairs of solutions (y_i, z_i), $(i = 1, 2, \ldots, 6)$, which satisfy the condition that the determinant

$$(65)\qquad D = D(y_k'' z_k'' y_k z_k' y_k z_k) = \begin{vmatrix} y_1'' & z_1'' & y_1' & z_1' & y_1 & z_1 \\ y_2'' & z_2'' & y_2' & z_2' & y_2 & z_2 \\ y_3'' & z_3'' & y_3 & z_3' & y_3 & z_3 \\ y_4'' & z_4'' & y_4' & z_4' & y_4 & z_4 \\ y_5'' & z_5'' & y_5' & z_5' & y_5 & z_5 \\ y_6'' & z_6'' & y_6' & z_6' & y_6 & z_6 \end{vmatrix}$$

shall not vanish for $x = x_0$, form a fundamental system of solutions of (A). By this statement we mean to say that the general solutions y and z can be expressed in the form

$$(66)\qquad y = \sum_{i=1}^{6} c_i y_i, \qquad z = \sum_{i=1}^{6} c_i z_i,$$

where $c_1, c_2, \ldots, c_6$ are arbitrary constants.

If six pairs of functions (y_i, z_i) whose determinant D does not vanish identically are given, we can determine a system of differential equations of the form (A) of which these functions are a fundamental system of solutions. It is only necessary to solve for p_{ik}, q_{ik}, r_{ik} in the twelve equations

$$(67)\qquad \begin{cases} y_i''' + 3p_{11}y_i'' + 3p_{12}z_i'' + 3q_{11}y_i' + 3q_{12}z_i' + r_{11}y_i + r_{12}z_i = 0, \\ z_i''' + 3p_{21}y_i'' + 3p_{22}z_i'' + 3q_{21}y_i' + 3q_{22}z_i' + r_{21}y_i + r_{22}z_i = 0, \end{cases}$$

$$(i = 1, 2, \ldots, 6).$$

We find for p_{ik} the values

$$(68)\qquad \begin{cases} Dp_{11} = -\frac{1}{3}D(y_i'' z_i'' y_i' z_i' y_i z_i), \\ Dp_{12} = -\frac{1}{3}D(y_i'' y_i''' y_i' z_i' y_i z_i), \\ Dp_{21} = -\frac{1}{3}D(z_i''' z_i'' y_i' z_i' y_i z_i), \\ Dp_{22} = -\frac{1}{3}D(y_i'' z_i''' y_i' z_i' y_i z_i). \end{cases}$$

The values of q_{ik} and r_{ik} are given by similar expressions.

From equations (68) is obtained at once the relation

$$(69)\qquad D = Ce^{-3\int (p_{11}+p_{22})\,dx},$$

where C is a non-vanishing constant.

If the general solutions of system (A) undergo the transformation

$$(70)\qquad \eta = \alpha y + \beta z, \qquad \xi = \gamma y + \delta z,$$

where α, β, γ, δ are arbitrary functions of x, then η and ξ will be the general solutions of a new system of equations of the same form as (A). If we put

$$(71)\qquad \eta_i = \alpha y_i + \beta z_i, \qquad \xi_i = \gamma y_i + \delta z_i \qquad (i = 1, 2, \ldots, 6),$$

the six pairs of functions (η_i, ξ_i) will form a fundamental system of solutions of the new system. The general solutions will be

$$(72)\qquad \eta = \sum_{i=1}^{6} c_i \eta_i, \qquad \xi = \sum_{i=1}^{6} c_i \xi_i.$$

From these considerations it is evident that six pairs of solutions which form a fundamental system are transformed cogrediently with each other and with the pair of general solutions.

Let us now assume that (A) has been integrated, so that we have $(y_1, y_2, y_3, y_4, y_5, y_6)$ and $(z_1, z_2, z_3, z_4, z_5, z_6)$ expressed as functions of x. We can interpret these two sets of quantities as the homogeneous coordinates of two points P_y and P_z in five-dimensional space. Then, as x varies these two points describe two curves C_y and C_z, which we shall call integral curves. The points on these two curves can be put into a definite correspondence by regarding those as corresponding points which belong to the same value of x.

However, this correspondence must not be such that the determinant D defined by equation (65) is zero. This is equivalent to saying that it must not be possible to find six functions of x, λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, such

that the equations

$$\lambda_1 y_i+\lambda_2 y_i'+\lambda_3 y_i''+\lambda_4 z_i+\lambda_5 z_i'+\lambda_6 z_i'' = 0 \qquad (i = 1, 2, \ldots, 6), \tag{73}$$

are satisfied. These equations can be written in the form

$$\lambda_1 y_i+\lambda_2 y_i'+\lambda_3 y_i'' = -(\lambda_4 z_i+\lambda_5 z_i'+\lambda_6 z_i'').$$

The left member represents a point in the osculating plane* of C_y at P_y, while the right member represents a point in the osculating plane of C_z at P_z. If equations (73) hold, these two points coincide for all values of x. Therefore, *the necessary and sufficient condition that C_y and C_z may be the integral curves of a system of form* (A) *is that the osculating planes to these two curves at corresponding points shall not in general intersect, i.e., the osculating planes at corresponding points must not in general lie in the same space of four dimensions.*

Let us join corresponding points P_y and P_z by a straight line L_{yz}. As x takes on all values, the line L_{yz} will generate a ruled surface. Now the transformation (71) converts P_y and P_z into two points P_η and P_ζ on the line L_{yz}. Since α, β, γ, δ are arbitrary functions of x, we can convert the curves C_y and C_z into any two curves C_η and C_ζ which intersect every generator. It is evident that the lines joining corresponding points on C_η and C_ζ remain generators of the surface. Moreover, the most general transformation of the independent variable simply changes the parametric representation of the curves C_y and C_z. Thus we have the following theorem :—

To every fundamental system of solutions of a system of differential equations of the form (A) *there corresponds a ruled surface in five-dimensional space. This ruled surface is unchanged by transformations of the form*

$$\eta = \alpha y+\beta z, \qquad \zeta = \gamma y+\delta z, \qquad \xi = f(x),$$

where α, β, γ, δ and f are arbitrary functions of x.

However, this integrating ruled surface must not be such that the osculating planes of C_y and C_z at corresponding points intersect. This means that three consecutive generators of the surface must not lie in the same space of four dimensions. It is evident at once that the surface

* For the definition of an osculating space of n dimensions of C_y and the proof that the coordinates of n independent points of it are given by y_i, y_i', ..., $y_i^{(n)}$, see Wilczynski, *loc. cit.*, p. 54.

cannot have two distinct sets of generators. In fact, it is impossible to construct a straight line intersecting three consecutive generators. A ruled surface in five dimensions for which any three consecutive generators lie in the same space of four dimensions, we shall call a *pseudo-developable*. These pseudo-developables are excluded from our theory.

There may be particular values of x which make the determinant D equal to zero. Along the generator determined by such a value of x the osculating planes to the two integral curves intersect. We shall call a generator of this kind a *pseudo-torsal generator*.

Now we have seen that

$$D = Ce^{-3\int (p_{11}+p_{22})\,dx}.$$

For a value of x which gives a pseudo-torsal generator, one or both of the quantities $\int p_{11}dx$, $\int p_{22}dx$ must be infinite. Therefore, for such a value of x the transformation

$$y = \eta e^{-\int p_{11}\,dx}, \qquad z = \zeta e^{-\int p_{22}\,dx},$$

which gives a system of form (A) with $p_{11} = p_{22} = 0$, is not permissible.

The integrating surface of (A) has been defined by a particular fundamental system of solutions. But any other fundamental system $(\bar{y}_k, \bar{z}_k)$ is given by the expressions

$$\bar{y}_k = \sum_{i=1}^{6} c_{ki}y_i, \qquad \bar{z}_k = \sum_{i=1}^{6} c_{ki}z_i \quad (k = 1, 2, \ldots, 6).$$

Therefore, any projective transformation of the original surface is also an integrating surface of (A).

Suppose now that the integrating surfaces of two systems of the form (A) coincide. Any pair of curves on the surface can be transformed into any other pair by (70), and the independent variables are expressible in terms of each other since to each generator there corresponds a value of each variable. As the surface has but one set of generators, it is evident that the two systems of form (A) can be transformed into each other.

If we have given any ruled surface in five-dimensional space which is not a pseudo-developable, it is easy to find a system of form (A) which defines it. It is only necessary to trace upon the surface two curves which are not generators and to express the coordinates of their points as functions of a parameter x in such a way that points given by the same value of x lie on the same generator. The coefficients of the required system are at once given by equations (68) and the corresponding equations for q_{ik} and r_{ik}.

We have now shown that *there belongs to every system of form* (A) *a ruled surface in five-dimensional space which is not a pseudo-developable, and, conversely, every such surface may be defined by a system of form* (A). Therefore, the projective properties of the integrating ruled surface can be obtained by studying the system (A). Every invariant equation or system of equations of (A) expresses a projective property of the integrating surface, and, conversely, every projective property of the surface can be expressed by an invariant equation, or system of equations.

It has been seen that, if the invariants $\Theta_3, \Theta_4, \ldots, \Im_{10}$ are given as functions of x, provided that Θ_4 and Θ_{10} are not equal to zero, a system of equations of form (A) having these functions for its invariants is determined uniquely except for transformations of the dependent variables. This fundamental theorem can now be stated in the following form.

If the invariants $\Theta_3, \Theta_4, \ldots, \Im_{10}$ *are given as functions of* x, *provided that* Θ_4 *and* Θ_{10} *are not equal to zero, they determine a ruled surface in five-dimensional space uniquely except for projective transformations.*

7. *Dualistic Interpretation.*

Let us consider a ruled surface S which is defined by a system of form (A). Just as with surfaces in ordinary space, a line is said to be tangent to S at a point P if it is tangent at P to some curve on S. We shall now define the tangent spaces of two, three, and four dimensions. Throughout this paper any linear space of r dimensions will be called an *r-space*, and any r-dimensional continuum, which is not necessarily linear, will be called an *r-spread.*

The tangent line at P_y to an integral curve C_y together with the generator through P_y determine a 2-space, or plane. Every line tangent to S at P_y lies in this plane. We shall now show this. The coordinates of a point $P_{\bar{y}}$ of any curve $C_{\bar{y}}$ on S are given by

$$\bar{y}_k = \alpha y_k + \beta z_k \quad (k = 1, 2, 3, \ldots, 6),$$

where y_k and z_k are the coordinates of the points on the same generator of the two integral curves, and α and β are arbitrary functions of x. The tangent line to $C_{\bar{y}}$ at P_y is determined by the points $P_{\bar{y}}$ and $P'_{\bar{y}}$, where $P'_{\bar{y}}$ is the point whose coordinates are

$$\bar{y}'_k = \alpha y'_k + \beta z'_k + \alpha' y_k + \beta' z_k.$$

If $P_{\bar{y}}$ is made to coincide with P_y, where P_y lies on the generator corresponding to the value $x = x_0$, we must have $\beta = 0$ for $x = x_0$. The line

tangent to $C_{\bar{y}}$ at $P_{\bar{y}} = P_y$ is now determined by two points whose coordinates are given by

$$\bar{y}_k = \alpha y_k, \qquad \bar{y}'_k = \alpha y'_k + \alpha' y_k + \beta' z_k.$$

But the plane defined above contains the points P_y, P'_y, P_z, and contains, therefore, the line tangent to $C_{\bar{y}}$ at P_y. Thus, there is at each point of S a plane containing all the tangent lines at the point. We shall call it the *tangent plane at the point.*

The tangent planes at all points of a generator L_{yz} are evidently contained in a 3-space determined by L_{yz} and its next consecutive generator. We shall speak of this 3-space as the *tangent 3-space along the generator* L_{yz}.

The osculating plane at P of any curve on the surface through P, together with the tangent 3-space along the generator through P, determine a 4-space, and this 4-space contains the osculating planes at P of all the curves through P. This fact can be proved by an extension of the method used in proving that the tangent plane at a point contains all the tangent lines at the point. We shall call this 4-space the *tangent 4-space at P.*

The principle of duality teaches us that, in 5-dimensional space, to a point corresponds a 4-space, to a line corresponds a 3-space, and to a plane corresponds again a plane.

We wish to think of our ruled surface not merely as a point locus, but as capable of generation by means of lines, planes, 3-spaces, and 4-spaces as well. Moreover, we wish to do this in such a way that the combined point, line, 3-space, and 4-space locus may be converted into another one of the same kind by a dualistic transformation. In order to do this let us associate with any point P of S the tangent 4-space at P. If P_y and P_z are two corresponding points of the integral curves C_y and C_z, the tangent 4-space at P_y is determined by the points P_y, P'_y, P''_y, P_z, P'_z. Hence, in accordance with Grassmann's definition, its coordinates u_k are proportional to the minors of z''_k in D. Likewise the coordinates v_k of the tangent 4-space at P_z are proportional to the minors of y''_k in D. We shall indicate these facts by writing

$$u = |\, y''y'z'yz \,|, \qquad v = |\, z''y'z'yz \,|.$$

The notation in similar cases will explain itself.

In order to attain the self-dual character which we wish our combined point, line, 3-space, 4-space locus to have, the line determined by y and z must correspond to the 3-space common to u and v, *i.e.*, to the 3-space

$|y'z'yz|$, which is the tangent 3-space of S along L_{yz}. Also, the plane determined by y', y, z must correspond to the plane common to u', u, v. From the definition of u we have

$$u' = -|y''z''y'yz| - 3p_{11}u - 3p_{12}v, \tag{74}$$

so that the tangent plane corresponds to the plane $|y'yz|$, *i.e.*, to itself. Furthermore, L_{yz} is the line common to u', v', u, v, and P_y is the point common to u'', u', v', u, v. Thus we have established between the elements of the ruled surface a self-dual one-to-one correspondence in which each point of the surface corresponds to its tangent 4-space, each generator to its tangent 3-space, and each tangent plane to itself.

It has already been noted that the coordinates u_k and v_k of the tangent 4-spaces at P_y and P_z respectively are proportional to certain minors in D. Let us now make the transformation

$$y_k = a\bar{y}_k + \beta\bar{z}_k, \qquad z_k = \gamma\bar{y}_k + \delta\bar{z}_k \qquad (k = 1, 2, \ldots, 6). \tag{75}$$

By some easy calculations it is proved that u_k *and* v_k *are transformed according to the equations*

$$u_k = \Delta^2(a\bar{u}_k + \beta\bar{v}_k), \qquad v_k = \Delta^2(\gamma\bar{u}_k + \delta\bar{v}_k) \qquad (k = 1, 2, \ldots, 6), \tag{76}$$

where

$$\Delta = a\delta - \beta\gamma.$$

This gives a direct generalization of Chasles' correlation for ordinary ruled surfaces.

By direct calculation the value of D after the transformation (75) is found to be

$$D = \Delta^3\bar{D}. \tag{77}$$

Consequently, the quantities U_k and V_k, where

$$U_k = \frac{u_k}{D^{\frac{2}{3}}}, \qquad V_k = \frac{v_k}{D^{\frac{2}{3}}}, \tag{78}$$

are absolutely cogredient with y_k and z_k.

The value of the determinant formed from u_k and v_k in the same way as D was formed from y_k and z_k, is D^5, and it cannot, therefore, vanish identically. Hence we can regard u_k and v_k as a fundamental system of solutions of a pair of equations of the same form as (A). These equations are easily found by differentiating u_k and v_k three times and making the necessary eliminations.

However, we shall have more occasion to use the similar system for

which U_k and V_k are a fundamental system of solutions. Since we have, from (69), the relation

$$D = Ce^{-3\int(p_{11}+p_{22})dx},$$

the new system may be obtained by making the transformation

$$u = Ue^{-2\int(p_{11}+p_{22})dx}, \qquad v = Ve^{-2\int(p_{11}+p_{22})dx}. \tag{79}$$

It is found to be

$$\begin{cases} U'''+3p_{11}U''+3p_{12}V''+3(q_{11}+u_{11}-u_{22})U'+3(q_{12}+2u_{12})V' \\ \quad +\{r_{11}+3p_{11}(u_{11}-u_{22})+6p_{21}u_{12}+\tfrac{3}{2}(v_{11}-v_{22})-\tfrac{1}{2}(t_{11}+t_{22})\}U \\ \quad +\{r_{12}+3p_{12}(u_{11}-u_{22})+6p_{22}u_{12}+3v_{12}\}V=0, \\ V'''+3p_{21}U''+3p_{22}V''+3(q_{21}+2u_{21})U'+3(q_{22}-u_{11}+u_{22})V' \\ \quad +\{r_{21}-3p_{21}(u_{11}-u_{22})+6p_{11}u_{21}+3v_{21}\}U \\ \quad +\{r_{22}-3p_{22}(u_{11}-u_{22})+6p_{12}u_{21}-\tfrac{3}{2}(v_{11}-v_{22})-\tfrac{1}{2}(t_{11}+t_{22})\}V=0, \end{cases} \tag{80}$$

where u_{ik}, v_{ik}, t_{ik} are the quantities so designated previously. *We shall speak of the system* (80) *as the adjoint of* (A).

Not only is (80) the adjoint of (A), but also (A) is the adjoint of (80). To show this, let us form the quantities U_{ik}, V_{ik}, T_{ik}, W_{ik} from (80) in the same way that we have formed u_{ik}, v_{ik}, t_{ik}, w_{ik} from (A). We find

$$\begin{cases} U_{11}=u_{22}, & U_{12}=-u_{12}, & U_{21}=-u_{21}, & U_{22}=u_{11}, \\ V_{11}=v_{22}, & V_{12}=-v_{12}, & V_{21}=-v_{21}, & V_{22}=v_{11}, \\ T_{11}=-t_{22}, & T_{12}=t_{12}, & T_{21}=t_{21}, & T_{22}=-t_{11}, \\ W_{11}=w_{22}, & W_{12}=-w_{12}, & W_{21}=-w_{21}, & W_{22}=w_{11}. \end{cases} \tag{81}$$

If we denote the coefficients of (80) by P_{ik}, Q_{ik}, R_{ik}, we have

$$\begin{cases} P_{ik}=p_{ik}, \\ Q_{11}=q_{11}+u_{11}-u_{22}, \qquad Q_{12}=q_{12}+2u_{12}, \\ Q_{21}=q_{21}+2u_{21}, \qquad Q_{22}=q_{22}-u_{11}+u_{22}, \\ R_{11}=r_{11}+3p_{11}(u_{11}-u_{22})+6p_{21}u_{12}+\tfrac{3}{2}(v_{11}-v_{22})-\tfrac{1}{2}(t_{11}+t_{22}), \\ R_{12}=r_{12}+3p_{12}(u_{11}-u_{22})+6p_{22}u_{12}+3v_{12}, \\ R_{21}=r_{21}-3p_{21}(u_{11}-u_{22})+6p_{11}u_{21}+3v_{21}, \\ R_{22}=r_{22}-3p_{22}(u_{11}-u_{22})+6p_{12}u_{21}-\tfrac{3}{2}(v_{11}-v_{22})-\tfrac{1}{2}(t_{11}+t_{22}). \end{cases} \tag{82}$$

From these equations and equations (81) we have

$$
(83)\quad \begin{cases}
p_{ik} = P_{ik}, \\
q_{11} = Q_{11}+U_{11}-U_{22}, \qquad q_{12} = Q_{12}+2U_{12}, \\
q_{21} = Q_{21}+2U_{21}, \qquad q_{22} = Q_{22}-U_{11}+U_{22}, \\
r_{11} = R_{11}+3P_{11}(U_{11}-U_{22})+6P_{21}U_{12}+\tfrac{3}{2}(V_{11}-V_{22})-\tfrac{1}{2}(T_{11}+T_{22}), \\
r_{12} = R_{12}+3P_{12}(U_{11}-U_{22})+6P_{22}U_{12}+3V_{12}, \\
r_{21} = R_{21}-3P_{21}(U_{11}-U_{22})+6P_{11}U_{21}+3V_{21}, \\
r_{22} = R_{22}-3P_{22}(U_{11}-U_{22})+6P_{12}U_{21}-\tfrac{3}{2}(V_{11}-V_{22})-\tfrac{1}{2}(T_{11}+T_{22}).
\end{cases}
$$

These equations show that p_{ik}, q_{ik}, r_{ik} are formed from P_{ik}, Q_{ik}, R_{ik} in the same way that P_{ik}, Q_{ik}, R_{ik} are formed from p_{ik}, q_{ik}, r_{ik}, *i.e., the relation between equations* (A) *and* (80) *is a reciprocal one.*

Equations (82) show that *the adjoint of* (A) *coincides with* (A) *if, and only if,*

$$
(84)\qquad u_{11}-u_{22} = u_{12} = u_{21} = t_{11}+t_{22} = 0.
$$

Suppose that these conditions are satisfied. Then, since (y_k, z_k) is a fundamental system of solutions for the system (80) as well as the system (A), it must be possible to express U_k and V_k in terms of y_k and z_k by means of equations of the form

$$
(85)\qquad U_i = \sum_{k=1}^{6} c_{ik}y_k, \qquad V_i = \sum_{k=1}^{6} c_{ik}z_k,
$$

where c_{ik} are constants. From the definitions of U_k and V_k, we have

$$
(86)\qquad \Sigma U_i y_i = 0, \quad \Sigma V_i z_i = 0, \quad \Sigma U_i z_i = 0, \quad \Sigma V_i y_i = 0,
$$

where the summation is always from 1 to 6. If the values of U_i and V_i from (85) are substituted in (86), it becomes evident that *the system* (A) *and its adjoint coincide only if the entire surface S lies in a quadratic 4-spread.*

We shall now show the geometrical significance of the semi-canonical form. In addition to the relations (86) we have, from the definition of U_k and V_k,

$$
(87)\quad \begin{cases}
\Sigma U_i z_i'' = -D^{\frac{1}{2}}, & \Sigma V_i y_i'' = D^{\frac{1}{2}}, \\
\Sigma U_i y'' = 0, & \Sigma V_i z_i'' = 0, \\
\Sigma U_i z_i' = 0, & \Sigma V_i y_i' = 0, \\
\Sigma U_i y' = 0, & \Sigma V_i z_i' = 0.
\end{cases}
$$

If we multiply equations (67) by U_k and V_k successively, and add, we find by using the relations (86) and (87) that

$$(88)\qquad \begin{cases} \Sigma U_i y_i''' = \quad 3p_{12}D^{\frac{1}{3}}, & \Sigma U_i z_i''' = \quad 3p_{22}D^{\frac{1}{3}}, \\ \Sigma V_i y_i''' = -3p_{11}D^{\frac{1}{3}}, & \Sigma V_i z_i''' = -3p_{21}D^{\frac{1}{3}}. \end{cases}$$

Now we have seen that the system (A) can always be transformed into one for which $p_{12} = p_{21} = 0$. Let us assume that this transformation has been made. Then equations (86), (87), and (88) give the relations

$$(89)\qquad \begin{cases} \Sigma U_i y_i = 0, & \Sigma U_i y_i' = 0, & \Sigma U_i y_i'' = 0, & \Sigma U_i y_i''' = 0, \\ \Sigma V_i z_i = 0 & \Sigma V_i z_i' = 0, & \Sigma V_i z_i'' = 0, & \Sigma V_i z_i''' = 0. \end{cases}$$

Thus, the integral curve C_y is such that its osculating 3-space at P_y is contained in the tangent 4-space of the surface at P_y. In other words, the tangent 4-space at a point on C_y has four consecutive points in common with C_y. Likewise the tangent 4-space at a point on C_z has four consecutive points in common with C_z. Because of their close analogy to asymptotic curves on ordinary ruled surfaces, we shall call the curves C_y and C_z, when determined so that the conditions $p_{12} = p_{21} = 0$ are satisfied, *pseudo-asymptotic curves.*

The system for which $p_{12} = p_{21} = 0$ can be converted into another for which also $p_{11} = p_{22} = 0$ by the transformation

$$y = \eta e^{-\int p_{11}\,dx}, \qquad z = \zeta e^{-\int p_{22}\,dx}.$$

Such a transformation evidently has no geometrical significance. Hence, if the system (A) is in its semi-canonical form, the integral curves on S are pseudo-asymptotic curves.

The most general transformation leaving the semi-canonical form invariant is

$$\bar{\xi} = \xi(x), \qquad \bar{y} = \xi'(ay+bz), \qquad \bar{z} = \xi'(cy+dz),$$

where ξ is an arbitrary function, and a, b, c, d are arbitrary constants. Therefore, *there exists upon the ruled surface S a single infinity of pseudo-asymptotic curves. Moreover, the double ratio of the four points in which any generator intersects four fixed pseudo-asymptotic curves is constant.* This is an extension to ruled surfaces in 5-dimensional space of Serret's well known theorem for ruled surfaces in ordinary space.

If we differentiate equations (67), multiply the result by U_k and V_k successively, and add, we find

$$(90)\qquad \begin{cases} \Sigma U_i y_i^{(4)} = \quad 3D^{\frac{1}{3}}\{p_{12}'+q_{12}-3(p_{11}+p_{22})p_{12}\}, \\ \Sigma V_i z_i^{(4)} = -3D^{\frac{1}{3}}\{p_{21}'+q_{21}-3(p_{11}+p_{22})p_{21}\}. \end{cases}$$

If the conditions

$$p_{12} = p_{21} = u_{12} = u_{21} = 0 \tag{91}$$

are satisfied, we have in addition to (89) the relations

$$\Sigma U_i y_i^{(4)} = 0, \qquad \Sigma V_i z_i^{(4)} = 0. \tag{92}$$

Therefore, *under the conditions* (91) *the osculating* 4*-space at* P_y *of* C_y *coincides with the tangent* 4*-space at* P_y *of* S. We shall later find another interpretation for the conditions (91).

We may reinterpret the solutions U_k and V_k of the adjoint of (A) as the homogeneous coordinates of a point in 5-dimensional space. Then the adjoint system determines a second ruled surface dual to the original surface S.

Let us denote the invariants of the adjoint of (A) by $\bar{\Theta}_3, \bar{\Theta}_4, \ldots$. From (81), we find

$$\begin{cases} \bar{\Theta}_3 = -\Theta_3, & \bar{\Theta}_8 = \Theta_8, \\ \bar{\Theta}_4 = \Theta_4, & \bar{\Theta}_9 = -\Theta_9, \\ \bar{\Theta}_5 = -\Theta_5, & \bar{\Theta}_{10} = \Theta_{10}, \\ \bar{\Theta}_6 = \Theta_6, & \bar{\Im}_{10} = -\Im_{10}. \end{cases} \tag{93}$$

Therefore, (A) *and its adjoint can be transformed into each other if, and only if,*

$$\Theta_3 = \Theta_5 = \Theta_9 = \Im_{10} = 0. \tag{94}$$

When these conditions are satisfied, there exists a dualistic transformation which converts the ruled surface into itself generator for generator. We shall say that the surface S is *identically self-dual.*

If the conditions (84) are satisfied, there exists a dualistic transforma- which converts each point of the surface into its tangent 4-space and each tangent 4-space into its point of contact, or, as we might say, the surface is identically self-dual point for point.

8. *The Derivative Surfaces.*

Let us return to the consideration of equations (57). If we substitute

$$y = y_k, \quad z = z_k \quad (k = 1, 2, \ldots, 6),$$

in these expressions, we find

$$\left.\begin{aligned}\rho_k &= y_k' + p_{11}y_k + p_{12}z_k \\ \sigma_k &= z_k' + p_{21}y_k + p_{22}z_k\end{aligned}\right\} \quad (k = 1, 2, \ldots, 6). \tag{95}$$

Evidently the ρ_k's are the homogeneous coordinates of a point P_ρ in the tangent plane of S at P_y, while the σ_k's are the coordinates of a point P_σ in the tangent plane of S at P_z.

It has been shown that ρ and σ are cogredient with y and z for transformations of the form

$$y = \alpha\bar{y} + \beta\bar{z}, \qquad z = \gamma\bar{y} + \delta\bar{z}.$$

Therefore, when P_y and P_z are tranformed into two points $P_{\bar{y}}$ and $P_{\bar{z}}$ of the generator L_{yz}, P_ρ and P_σ are transformed into two points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ of $L_{\rho\sigma}$ which lie in the tangent planes of S at $P_{\bar{y}}$ and $P_{\bar{z}}$ respectively. Thus, by means of equations (95), there is made to correspond to each generator L_{yz} of S a line $L_{\rho\sigma}$ in the 3-space tangent to S along L_{yz}.

Let us now assume that (A) has been reduced to its semi-canonical form, so that the integral curves are pseudo-asymptotic curves. Then equations (95) become

$$\rho_k = y_k', \qquad \sigma_k = z_k'.$$

The points P_ρ and P_σ not only lie in the tangent planes of S at P_y and P_z respectively, but also on the tangent lines to C_y and C_z at P_y and P_z respectively. Thus, there is established between the points of L_{yz} and $L_{\rho\sigma}$ a one-to-one correspondence in which the point P' of $L_{\rho\sigma}$ corresponding to any point P of L_{yz} lies on the line t tangent at P to the pseudo-asymptotic curve through P. As P moves along L_{yz}, the line t describes a quadric surface Q in the 3-space tangent along L_{yz}. For t always intersects the generator L_{yz}, the next consecutive generator, and the line $L_{\rho\sigma}$.

Since a transformation $\qquad \bar{\xi} = \bar{\xi}(x)$

of the independent variable converts ρ and σ into $\bar{\rho}$ and $\bar{\sigma}$, where

$$\bar{\rho} = \frac{1}{\bar{\xi}'}(\rho + \eta y), \qquad \bar{\sigma} = \frac{1}{\bar{\xi}'}(\sigma + \eta z), \qquad \eta = \frac{\bar{\xi}''}{\bar{\xi}'},$$

the position of $L_{\rho\sigma}$ on Q is changed. In fact, by giving η the proper value, we can make $L_{\rho\sigma}$ coincide with any generator on Q of the same kind as $L_{\rho\sigma}$.

Let us now construct the line $L_{\rho\sigma}$ corresponding to each generator of S. We obtain a new ruled surface S', which we shall call *the first derivative of the first kind of S with respect to x*. When the independent variable

is transformed, S' changes unless $\eta = 0$ for the transformation, *i.e.*, unless the transformation is linear and integral of the form

$$\xi = ax + b,$$

where a and b are constants.

Equations (59) define two quantities μ and ν which are formed from ρ and σ in the same way as ρ and σ are formed from y and z.

We have, as before,

$$(96) \qquad \begin{cases} \mu_k = \rho'_k + p_{11}\rho_k + p_{12}\sigma_k, \\ \nu_k = \sigma'_k + p_{21}\rho_k + p_{22}\sigma_k. \end{cases}$$

It is clear that P_μ is a point in the tangent plane of S' at P_ρ, and P_ν is a point in the tangent plane of S' at P_σ. If we substitute the values of ρ_k and σ_k into (96), we have

$$(97) \qquad \begin{cases} \mu_k = y''_k + 2p_{11}y'_k + 2p_{12}z'_k + (q_{11}+u_{11})\,y_k + (q_{12}+u_{12})\,z_k, \\ \nu_k = z''_k + 2p_{21}y'_k + 2p_{22}z'_k + (q_{21}+u_{21})\,y_k + (q_{22}+u_{22})\,z_k. \end{cases}$$

These equations show that P_μ lies in the tangent 4-space of S at P_y and P_ν in the tangent 4-space at P_z. If equations (A) are reduced to their semi-canonical form, equations (96) and (97) become

$$\mu_k = \rho'_k, \qquad \nu_k = \sigma'_k,$$

and

$$\mu_k = y''_k, \qquad \nu_k = z''_k.$$

The point P_μ lies in the line tangent to C_ρ at P_ρ, and this tangent line, in turn, lies in the osculating plane of C_y at P_y. The point P_ν is similarly located.

Any transformation of the dependent variables transforms μ and ν cogrediently with ρ and σ and also cogrediently with y and z. Therefore, such a transformation changes P_μ and P_ν into two points P_μ and $P_{\bar{\nu}}$ of the line $L_{\mu\nu}$ at the same time that it transforms P_ρ and P_σ into two points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ of the line $L_{\rho\sigma}$. Thus, to each generator $L_{\rho\sigma}$ of S' there corresponds a line $L_{\mu\nu}$, and we have another ruled surface S'_1 associated with S'. We shall speak of this new surface as *the derivative of the second kind of S with respect to x*. It is evident that S'_1 changes when the independent variable is transformed.

If we omit the subscripts in (97) and differentiate, we find a pair of equations from which can be obtained by the elimination of y''', y'', z''

from the first and of z''', y'', z'' from the second, the equations

$$(98)\quad \begin{cases} \mu'+p_{11}\mu+p_{12}\nu = 3u_{11}y'+3u_{12}z'+(3u_{11}p_{11}+3u_{12}p_{21}+\frac{3}{2}v_{11}-\frac{1}{2}t_{11})\,y \\ \qquad\qquad +(3u_{11}p_{12}+3u_{12}p_{22}+\frac{3}{2}v_{12}-\frac{1}{2}t_{12})\,z, \\ \nu'+p_{21}\mu+p_{22}\nu = 3u_{21}y'+3u_{22}z'+(3u_{21}p_{11}+3u_{22}p_{21}+\frac{3}{2}v_{21}-\frac{1}{2}t_{21})\,y \\ \qquad\qquad +(3u_{21}p_{12}+3u_{22}p_{22}+\frac{3}{2}v_{22}-\frac{1}{2}t_{22})\,z. \end{cases}$$

It is easily verified that these equations are unchanged by a transformation of the independent variable. They give the points of intersection of the tangent 3-space along L_{yz} with the tangent planes of S_1' at P_μ and P_ν. They also show that the tangent planes of S at P_y and P_z meet the tangent planes of S_1' at P_μ and P_ν respectively if, and only if the conditions

$$u_{12}=u_{21}=0$$

are satisfied. The left members of equations (98) are formed from μ and ν in the same way that μ and ν are formed from ρ and σ. They may be regarded as defining a new ruled surface S_2' associated with S_1'. Then equations (98) show that the tangent planes of S at P_y and P_z contain the corresponding points of S_2' if, and only if, $u_{12}=u_{21}=0$.

It has been seen that it is always possible to reduce the system (A) to another for which $u_{12}=u_{21}=0$ provided that $\Theta_4\neq 0$. In fact, if we factorize the covariant E_1 and denote its factors by η and ζ, we have at once a transformation which converts (A) into the desired form. The factors are proportional to

$$(99)\quad \begin{cases} \eta = \dfrac{u_{11}-u_{22}-\sqrt{\Theta_4}}{2}\,y+u_{12}z, \\ \zeta = \dfrac{u_{11}-u_{22}+\sqrt{\Theta_4}}{2}\,y+u_{12}z. \end{cases}$$

Since the most general transformation of the dependent variables which preserves the conditions $u_{12}=u_{21}=0$ may be obtained from

$$y=a\eta, \qquad z=\delta\zeta,$$
$$y=\zeta, \qquad z=\eta,$$

the curves C_y and C_z are uniquely determined. Thus, *there exist upon S two curves, obtained by factorizing the covariant E_1, such that the tangent plane at any point of either of them always contains the corresponding point of the surface S_2'.* Of course these two curves, in general, are parts of the same irreducible curve which intersects each generator in two points. Equations (99) show that the two parts are distinct if $\Theta_4\neq 0$.

It should be noticed here that the factors of the covariant E_2 determine

a system of equations for which the conditions $t_{12} = t_{21} = 0$ are satisfied. *These conditions completely determine two curves on S which are distinct if*

$$\Theta_3^2 - 4\Theta_6 \neq 0.$$

In this paper we shall denote by C_u the curve determined by the conditions $u_{12} = u_{21} = 0$, and by C_t the curve determined by the conditions $t_{12} = t_{21} = 0$.

Suppose that a branch of C_u coincides with a pseudo-asymptotic curve. We can take it to be the integral curve C_y so that we have $u_{12} = p_{12} = 0$. Then the first equation of (98) becomes

$$\mu' + p_{11}\mu = 3u_{11}y' + (3u_{11}p_{11} + \tfrac{3}{2}v_{11} - \tfrac{1}{2}t_{11})\,y - \tfrac{1}{2}t_{12}z.$$

Hence the tangent plane to S at P_y meets the line tangent to C_μ at the point on S_1' corresponding to P_y.

The necessary and sufficient condition that a branch of C_u coincide with a pseudo-asymptotic curve is $\Theta_{10} = 0$, provided that $\Theta_4 \neq 0$. For, suppose the system (A) is in the form for which $u_{12} = u_{21} = 0$. Then we have

$$\Theta_{10} = (u_{11} - u_{22})^2\, p_{12}p_{21} = \Theta_4 p_{12}p_{21}.$$

Therefore, $\Theta_{10} = 0$ if, and only if, $p_{12} = 0$ or $p_{21} = 0$.

If the conditions $$p_{12} = u_{12} = t_{12} = 0$$

are satisfied, the first equation of (98) becomes

$$\mu' + p_{11}\mu = 3u_{11}y' + (3p_{11}u_{11} + \tfrac{3}{2}v_{11} - \tfrac{1}{2}t_{11})y.$$

Hence the tangent line to C_y at P_y meets the tangent line to C_μ at P_μ. Moreover, under these conditions the quantities q_{12} and r_{12} also vanish and the first equation of (A) becomes

$$y''' + 3p_{11}y'' + 3q_{11}y' + r_{11}y = 0.$$

As this equation has only three independent solutions, there must be three linear homogeneous relations between the six solutions $y_1, y_2, \ldots y_6$. In other words, *the curve C_y lies in a plane.*

Let us assume that the two branches of C_u do not coincide with each other and also that they do not both coincide with pseudo-asymptotic curves. Then the necessary and sufficient condition that a branch of C_u, a branch of C_t, and a pseudo-asymptotic curve coincide is

$$\Theta_{10} = \Im_{10} = 0.$$

For we can take the integral curves to be the two branches of C_u. Then we have

$$\Theta_{10} = \Theta_4 p_{12}\, p_{21},$$

$$\Im_{10} = 2\Theta_4^{\frac{3}{4}}(p_{21}\,t_{12} - p_{12}\,t_{21}).$$

It is evident that $\Theta_{10} = \Im_{10} = 0$ if, and only if,

$$p_{12} = t_{12} = 0 \quad \text{or} \quad p_{21} = t_{21} = 0.$$

We shall now show that *S is not determined by its invariants Θ_3, Θ_4, ... $\Im_{10}$ when $\Theta_{10} = 0$ or $\Theta_4 = 0$, i.e., when a branch of C_u coincides with a pseudo-asymptotic curve, or when the two branches of C_u coincide.* Let us first take $\Theta_{10} = 0$, $\Theta_4 \neq 0$. We can reduce the system (A) to one for which $p_{11} = p_{22} = u_{12} = u_{21} = 0$. Then we have either $p_{12} = 0$ or $p_{21} = 0$. We shall take $p_{12} = 0$, and then q_{12} must also vanish. The non-vanishing fundamental invariants are not sufficient for the determination of p_{21}, so that it may be taken as an arbitrary function of x. The most general transformation of the dependent variables which leaves all of these conditions unchanged contains only arbitrary constants, and cannot, therefore, remove an arbitrary function. Moreover, the arbitrary function cannot be removed by any definite choice of the independent variable.

Let us next assume that the two branches of C_u coincide, so that we have $\Theta_4 = 0$. We may reduce the system (A) to one for which

$$u_{12} = u_{11} - u_{22} = p_{11} = p_{22} = p_{21} = 0.$$

Moreover, we may choose the independent variable so as to make

$$u_{11} + u_{22} = 0.$$

The coefficients of the system when determined by the fundamental invariants which do not vanish under these conditions contain two arbitrary functions of x. Since the most general transformation leaving the above conditions invariant contains only arbitrary constants, the two arbitrary functions cannot be removed.

9. *The Differential Equations for S'.*

If we substitute for μ, μ', ν, ν' in (98) their values in terms of $\rho, \rho', \sigma, \sigma'$, and eliminate y' and z' by means of the equations defining ρ and σ, we have

$$(100) \quad \begin{cases} R = \rho'' + 2p_{11}\rho' + 2p_{12}\sigma' + (q_{11} - 2u_{11})\rho + (q_{12} - 2u_{12})\sigma = s_{11}y + s_{12}z, \\ S = \sigma'' + 2p_{21}\rho' + 2p_{22}\sigma' + (q_{21} - 2u_{21})\rho + (q_{22} - 2u_{22})\sigma = s_{21}y + s_{22}z, \end{cases}$$

where

$$(101) \qquad s_{ik} = \tfrac{1}{2}(3v_{ik} - t_{ik}).$$

Equations (100) show that *the necessary and sufficient condition that*

S' be a pseudo-developable is

$$F = s_{11}s_{22} - s_{12}s_{21} = 9G + K - 3M = 0. \tag{102}$$

For S' is a pseudo-developable if, and only if, the tangent 4-space at P_ρ coincides with the tangent 4-space at P_σ. If these two 4-spaces coincide, the point given by the first equation of (100) in which the tangent 4-space at P_ρ meets L_{yz} must coincide with the point given by the second equation of (100) in which the tangent 4-space at P_σ meets the line L_{yz}. These points coincide if $F = 0$. Moreover, if $F = 0$, y and z can be eliminated from (100) so as to give an equation of the form

$$\lambda_1\rho'' + \lambda_2\rho' + \lambda_3\rho + \lambda_4\sigma'' + \lambda_5\sigma' + \lambda_6\sigma = 0.$$

This is the condition that S' be a pseudo-developable. The differential equation which the independent variable making $F = 0$ must satisfy is of the second order in η. *There are, therefore, ∞^2 surfaces S' which are pseudo-developables.*

We shall now proceed to find the differential equations for S' under the assumption that F is not equal to zero. Let us denote by g_{ik} the quantities formed from the s_{ik}'s in the same way that the v_{ik}'s are formed from the u_{ik}'s, so that we have

$$g_{ik} = s'_{ik} + \sum_{j=1}^{2} (s_{jk}p_{ij} - s_{ij}p_{jk}). \tag{103}$$

If we differentiate R and S and eliminate y', y, z', z from the resulting equations, we find

$$\begin{cases} FR' - Fs_{11}\rho - Fs_{12}\sigma + \tau_{11}R + \tau_{12}S = 0, \\ FS' - Fs_{21}\rho - Fs_{22}\sigma + \tau_{21}R + \tau_{22}S = 0, \end{cases} \tag{104}$$

where

$$\begin{cases} \tau_{11} = Fp_{11} + s_{21}g_{12} - s_{22}g_{11}, \\ \tau_{12} = Fp_{12} - s_{11}g_{12} + s_{12}g_{11}, \\ \tau_{21} = Fp_{21} + s_{21}g_{22} - s_{22}g_{21}, \\ \tau_{22} = Fp_{22} - s_{11}g_{22} + s_{12}g_{21}. \end{cases} \tag{105}$$

Let us put

$$\begin{cases} s_{21}g_{12} - s_{22}g_{11} = 3F\lambda_{11}, \\ -s_{11}g_{12} + s_{12}g_{11} = 3F\lambda_{12}, \\ s_{21}g_{22} - s_{22}g_{21} = 3F\lambda_{21}, \\ -s_{11}g_{22} + s_{12}g_{21} = 3F\lambda_{22}, \end{cases} \tag{106}$$

whence

$$(107)\qquad \begin{cases} 3(s_{11}\lambda_{11}+s_{21}\lambda_{12}) = -g_{11}, \\ 3(s_{12}\lambda_{11}+s_{22}\lambda_{12}) = -g_{12}, \\ 3(s_{11}\lambda_{21}+s_{21}\lambda_{22}) = -g_{21}, \\ 3(s_{12}\lambda_{21}+s_{22}\lambda_{22}) = -g_{22}. \end{cases}$$

Now, if we introduce the values of R and S from (100), equations (104) become

$$(108)\qquad \begin{cases} \rho'''+3P_{11}\rho''+3P_{12}\sigma''+3Q_{11}\rho'+3Q_{12}\sigma'+R_{11}\rho+R_{12}\sigma = 0, \\ \sigma'''+3P_{21}\rho''+3P_{22}\sigma''+3Q_{21}\rho'+3Q_{22}\sigma'+R_{21}\rho+R_{22}\sigma = 0, \end{cases}$$

where

$$(109)\qquad \begin{cases} P_{ik} = p_{ik}+\lambda_{ik}, \\ Q_{ik} = q_{ik}+2\sum_{j=1}^{2} p_{jk}\lambda_{ij}, \\ R_{ik} = r_{ik}-3v_{ik}+3\sum_{j=1}^{2}(q_{jk}-2u_{jk})\lambda_{ij}. \end{cases}$$

Equations (108) constitute the system of form (A) which defines S'.

We can now determine S'', the derivative of the first kind of S', the second derivative of the first kind of S. To find two points on the generator of S'' corresponding to the generator $L_{\rho\sigma}$ of S', it is only necessary to form two quantities α and β from equations (108) in the same way as ρ and σ are formed from equations (A). The expressions for α and β are

$$(110)\qquad \begin{cases} \alpha = \rho'+P_{11}\rho+P_{12}\sigma = \rho'+(p_{11}+\lambda_{11})\rho+(p_{12}+\lambda_{12})\sigma, \\ \beta = \sigma'+P_{21}\rho+P_{22}\sigma = \sigma'+(p_{21}+\lambda_{21})\rho+(p_{22}+\lambda_{22})\sigma. \end{cases}$$

We have seen that two points on the generator of S_1' corresponding to $L_{\rho\sigma}$ are given by the quantities μ and ν where

$$\mu = \rho'+p_{11}\rho+p_{12}\sigma,$$
$$\nu = \sigma'+p_{21}\rho+p_{22}\sigma.$$

Clearly two corresponding generators of S'' and S_1' intersect if, and only if,

$$(111)\qquad \lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21} = 0.$$

Since F cannot be zero if we assume that S' is not a pseudo-developable, we find on substituting the values of the λ_{ik}'s from (106) that the condition (111) is equivalent to the condition

$$(112)\qquad g_{11}g_{22}-g_{12}g_{21} = 0.$$

The most general independent variable for which condition (112) is fulfilled is obtained by integrating a differential equation of the third order in η. Therefore, *there exist* ∞^3 *surfaces* S' *whose derivatives of the first kind have the property that their generators intersect the corresponding generators of the derivative of the second kind of* S.

Equations (110) also show that S'' and S_1' coincide if, and only if, $\lambda_{ik} = 0$. The conditions $\lambda_{ik} = 0$ are equivalent to $g_{ik} = 0$. If these conditions are satisfied for any pairs of curves on S, they are satisfied for every pair as long as the independent variable is not transformed.

Furthermore, without changing the independent variable, we can make $p_{ik} = 0$. Then also will $P_{ik} = 0$, so that the pseudo-asymptotic curves of S and S' will correspond to each other. The conditions $g_{ik} = 0$ are now equivalent to the conditions $r_{ik} = \text{const.}$ In other words, if the system (A) can be written in the form

$$\begin{cases} y'''+3q_{11}y'+3q_{12}z'+r_{11}y+r_{12}z = 0, \\ z'''+3q_{21}y'+3q_{22}z'+r_{21}y+r_{22}z = 0, \end{cases} \tag{113}$$

where the coefficients r_{ik} are constants, then the differential equations of S' are also in the semi-canonical form. The system for S' under these conditions is

$$\begin{cases} \rho'''+3q_{11}\rho'+3q_{12}\sigma'+(r_{11}+3q_{11}')\rho+(r_{12}+3q_{12}')\sigma = 0, \\ \sigma'''+3q_{21}\rho'+3q_{22}\sigma'+(r_{21}+3q_{21}')\rho+(r_{22}+3q_{22}')\sigma = 0. \end{cases} \tag{114}$$

If the q_{ik}'s are also constants, systems (113) and (114) are identical, and S and S' are projective transformations of each other.

10. *Significance of the Covariants* K_3 *and* C_6.

Let us consider the covariant

$$K_3 = E_1'+(p_{11}+p_{22})E_1, \tag{115}$$

where

$$E_1 = u_{12}z^2-u_{21}y^2+(u_{11}-u_{22})yz.$$

If we introduce this value of E_1 into K_3 and substitute for y' and z' their values obtained from equations (57), we have

$$\begin{aligned} K_3 = \quad & z\{2u_{12}\sigma+(u_{11}-u_{22})\rho+\tfrac{1}{2}(v_{11}-v_{22})y+v_{12}z\} \\ & -y\{2u_{21}\rho-(u_{11}-u_{22})\sigma-\tfrac{1}{2}(v_{11}-v_{22})z+v_{21}y\}. \end{aligned}$$

It is clear that K_3 determines a ruled surface Σ whose generator corre-

sponding to L_{yz} is the line joining the points P_γ and P_δ, where

$$(116) \qquad \begin{cases} \gamma = 2u_{12}\sigma + (u_{11}-u_{22})\,\rho + \frac{1}{2}(v_{11}-v_{22})y + v_{12}z, \\ \delta = 2u_{21}\rho - (u_{11}-u_{22})\sigma - \frac{1}{2}(v_{11}-v_{22})z + v_{21}y. \end{cases}$$

Since a transformation $\bar{\xi} = \bar{\xi}(x)$ of the independent variable converts γ and δ into $\bar{\gamma}$ and $\bar{\delta}$, where

$$\bar{\gamma} = \frac{1}{(\bar{\xi}')^3}\gamma, \qquad \bar{\delta} = \frac{1}{(\bar{\xi}')^3}\delta,$$

the surface Σ is left invariant, *i.e.*, it is uniquely determined by S and does not in any way depend on the independent variable. As the points γ and δ are found to be transformed cogrediently with y and z by a transformation of the dependent variables, there is a one-to-one correspondence between the points of a generator of S and the corresponding generator of Σ.

It is easy to find a construction for the surface Σ under the assumption that $\Theta_4 \neq 0$. We can take the integral curves on S to be the two branches of C_u, so that we have $u_{12} = u_{21} = 0$. By a proper choice of the independent variable we can make $u_{11}-u_{22} = 1$, and then we have $\Theta_4 = 1$. This particular selection of the independent variable determines a unique derivative surface of the first kind, which we shall call *the principal derivative surface of* S. After these reductions have been made, the expressions for γ and δ become

$$(117) \qquad \begin{cases} \gamma = \quad \rho - p_{12}z = \quad y' + p_{11}y, \\ \delta = -\sigma + p_{21}y = -z' - p_{22}z. \end{cases}$$

It is evident from these equations that P_γ is the point of intersection of the line $P_\rho P_z$ with the tangent line to C_y at P_y, while P_δ is the point of intersection of the line $P_\sigma P_y$ with the tangent line to C_z at P_z. The generator of the surface Σ is thus completely determined.

The covariant

$$(118) \qquad C_6 = 4P_2\Theta_4 + P_1\Theta_4' = 4(z\mu - y\nu)\Theta_4 + \Theta_4'(z\rho - y\sigma)$$

also determines a system of ruled surfaces associated with S. Of course, it is necessary to assume that Θ_4 does not vanish.

Now C_6 can be written in the form

$$C_6 = z(4\Theta_4\mu + \Theta_4'\rho) - y(4\Theta_4\nu + \Theta_4'\sigma).$$

Let us consider the ruled surface T whose generator is the line joining

P_λ and P_τ, where

$$\text{(119)} \qquad \begin{cases} \lambda = 4\Theta_4\mu + \Theta_4'\rho, \\ \tau = 4\Theta_4\nu + \Theta_4'\sigma. \end{cases}$$

The quantities λ and τ are cogredient with y and z respectively under a transformation of the dependent variables, but a transformation $\bar{\xi} = \bar{\xi}(x)$ of the independent variable converts λ and τ into $\bar{\lambda}$ and $\bar{\tau}$, where

$$\text{(120)} \qquad \begin{cases} \bar{\lambda} = \dfrac{1}{(\bar{\xi}')^6}\{\lambda + (4\eta'\Theta_4 + \eta\Theta_4' - 4\eta^2\Theta_4)y\}, \\ \bar{\tau} = \dfrac{1}{(\bar{\xi}')^6}\{\tau + (4\eta'\Theta_4 + \eta\Theta_4' - 4\eta^2\Theta_4)z\}. \end{cases}$$

As $P_{\bar{\lambda}}$ is a point on the line joining P_y and P_λ, and $P_{\bar{\tau}}$ is a point on the line joining P_z and P_τ, it is clear that the generator $L_{\lambda\tau}$ merely moves along two fixed lines L_1 and L_2 through P_y and P_z respectively when the independent variable is transformed.

We have seen that, when the independent variable is so chosen that $\Theta_4 = 1$, S' becomes the principal derivative surface. In that case P_λ coincides with P_μ, and P_τ with P_ν, so that T coincides with S_1'. Hence L_1 is the line joining P_y and the corresponding point of the derivative of the second kind associated with the principal derivative surface. The line L_2 is determined similarly. Now let us take two pseudo-asymptotic curves for the integral curves on S. The line L_1 and the points P_ρ and P_μ all lie in the osculating plane of C_y at P_y for any form of the independent variable. The intersection of L_1 and the line joining P_ρ and P_μ, which is the tangent line to C_ρ at P_ρ, is the point P_λ. Likewise the intersection of L_2 and the tangent to C_σ at P_σ is the point P_δ. The line joining P_λ and P_δ is the generator of T corresponding to L_{yz}.

ON SOME PROPERTIES OF GROUPS WHOSE ORDERS ARE POWERS OF PRIMES

By W. BURNSIDE.

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I. *On the Maximum Abelian Self-Conjugate Sub-Groups.*

1. The group of isomorphisms I of a given group G of finite order has a single conjugate set of sub-groups of order p^α, if p^α is the highest power of a prime p which divides the order of I. Any group, simply isomorphic with one of these sub-groups, may be called the *group of p-isomorphisms* of G. It is a group of definite type when G is given.

If G, of order N, when represented as a regular transitive permutation-group of the N symbols of its operations, is denoted by g; and if the group of p-isomorphisms of G, when represented as an intransitive permutation-group of the N symbols of the operations of G, is denoted by i; then $\{g, i\}$, or any group simply isomorphic with it, may be called the *p-holomorph* of G. It again is a group of definite type when G is given.

If G is a group of prime power order p^n, the number of its self-conjugate sub-groups of any given order is congruent to unity, mod p. Hence there must be at least one sub-group G_1, of index p, which is transformed into itself by every isomorphism of the group of p-isomorphisms of G. The same remark applies to G_1. There is, therefore, a series of groups

$$G,\ G_1,\ G_2,\ \ldots,\ G_{n-1},\ E,$$

each of index p in the preceding and each self-conjugate in G, such that every isomorphism of the group of p-isomorphisms of G changes each one of them into itself.

2. Mr. Hilton* has shewn that the order of the group of p-isomorphisms of an Abelian group of order p^n is equal to or a factor of $p^{\frac{1}{2}n(n-1)}$.

* Hilton: "On the Order of the Group of Isomorphisms of an Abelian Group," *Messenger of Mathematics*, 1909, p. 132.

If the Abelian group is of type (1, 1, ..., 1), it is well known that the order of its group of p-isomorphisms actually has the above value; and the same is true* for an Abelian group of type (2, 1, ..., 1). For groups of other types the order of the group of p-isomorphisms is less than this maximum value.

From this result it is easy to deduce an upper limit for the order of a group K of p-isomorphisms of an Abelian group G of order p^n, which transforms every operation of a sub-group C of order p^c into itself. Every operation of K gives an isomorphism of G/C and the identical isomorphism of C. If K_1 is the self-conjugate sub-group of K whose operations give the identical isomorphism of G/C, the order of K/K_1 is, by the previous result, a factor of $p^{\frac{1}{2}(n-c)(n-c-1)}$. The greatest possible order for K_1 is clearly $p^{(n-c)c}$. Hence the order of K is equal to or a factor of $p^{\frac{1}{2}(n-c)(n+c-1)}$.

3. Let G be any non-Abelian group of order p^n and let C, of order p^c, be the central of G. There is at least one self-conjugate Abelian sub-group of G, say C_1, containing C and of order p^{c+1}. To every operation of G/C_1 there corresponds an isomorphism of C_1, and those operations of G/C_1 which give the identical isomorphism of G/C_1 constitute a self-conjugate sub-group of G/C_1. This self-conjugate sub-group of G/C_1, unless it is E, necessarily contains a self-conjugate operation, other than E, of G/C_1. If P is an operation of G, which corresponds to this self-conjugate operation of G/C_1, then $\{P, C_1\}$, or C_2, is an Abelian self-conjugate sub-group of G. To every operation of G/C_2 there corresponds an isomorphism of C_2; and repeating the previous reasoning, if any operation of G/C_2, other than E, gives the identical isomorphism of G, there must be an Abelian self-conjugate sub-group of G containing C_2 and of greater order.

Continuing thus we must arrive at an Abelian self-conjugate sub-group A of G, such that to each operation of G/A there corresponds a distinct isomorphism of A. No sub-group of G, containing A and of greater order, can then be Abelian; and therefore A will be called a *maximum* Abelian self-conjugate sub-group of G. Such a sub-group is not necessarily unique. In fact, if P is an operation of G which is permutable with all its conjugates, the sub-group generated by P and its conjugates is an Abelian self-conjugate sub-group and must be contained in some maximum Abelian self-conjugate sub-group. If Q is

* Miller: "Isomorphisms of a Group whose Order is a Power of a Prime," *Transactions of the American Mathematical Society*, Vol. XII, p. 396.

another operation of G, permutable with all its conjugates but not permutable with P, there is another maximum Abelian self-conjugate sub-group containing Q and necessarily distinct from the preceding one.

As a simple illustration a non-Abelian group of order p^3 has $p+1$ distinct maximum Abelian self-conjugate sub-groups; while the group of order p^4 defined by

$$P^p = Q^p = R^p = S^p = P^{-1}Q^{-1}PQ = Q^{-1}R^{-1}QR = R^{-1}P^{-1}RP = S^{-1}P^{-1}SP = E,$$

$$R^{-1}S^{-1}RS = Q, \quad Q^{-1}S^{-1}QS = P,$$

has only one.

4. If A of order p^ν is a maximum Abelian self-conjugate sub-group of G of order p^n, whose central C is of order p^c, every operation of G/A gives a distinct isomorphism of A such that every operation of C is changed into itself. Now an upper limit to the order of a group of p-isomorphisms of A, with this condition, has been seen above to be $p^{\frac{1}{2}(\nu-c)(\nu+c-1)}$, and this is therefore an upper limit for the order of G/A. Hence*

$$n-\nu \leqslant \tfrac{1}{2}(\nu-c)(\nu+c-1).$$

It may be noticed that for given values of ν and c it is always possible to choose the types of A and C, so that the group of p-isomorphisms of A for which the operations of C are unchanged actually has the value $p^{\frac{1}{2}(\nu-c)(\nu+c-1)}$. The above relation may be expressed in the form

$$\nu \geqslant -\tfrac{1}{2}+\sqrt{2n+c^2-c+\tfrac{1}{4}}.$$

A group of order p^n whose central is of order p^c therefore contains an Abelian self-conjugate sub-group of order p^ν, where ν satisfies the above inequality.

5. I have shewn† that an irreducible group of linear substitutions of prime power order can always be expressed as a group of monomial substitutions. It is to be expected that the maximum Abelian self-conjugate sub-groups are intimately connected with this mode of expressing such a group, and it will be seen immediately that this is the case.

* In the *Messenger of Mathematics*, Vol. 36, p. 79, Prof. Miller has shewn that, if $n > \frac{1}{2}\nu(\nu-1)$, a group of order p^n necessarily contains an Abelian self-conjugate sub-group of order p^ν. He does not consider the effect of the order of the central on the orders of the Abelian sub-groups. The point of view here is quite different from that of Prof. Miller.

† *Theory of Groups* (2nd edition), pp. 352, 471.

To a group of monomial substitutions most of the phraseology used for permutation-groups may be immediately extended. Such groups may be described in quite a definite sense as transitive or intransitive, primitive or imprimitive. The word "degree" to denote the number of variables operated on by any group of linear substitutions is already in common use.

An irreducible group of monomial substitutions is necessarily transitive. An irreducible group of monomial substitutions G of prime power order necessarily has an Abelian self-conjugate sub-group H which does not permute the symbols; and G/H is simply isomorphic with the transitive permutation-group which arises when the multipliers are omitted in the substitutions of G.

6. An irreducible group of linear substitutions G of prime power order p^m is necessarily of prime power degree p^n. Let H be a reducible self-conjugate sub-group of G, and suppose G expressed so that H is completely reduced. Then the p^n symbols fall into p^{n_2} sets of p^{n_1} each, such that H transforms each set of p^{n_1} irreducibly among themselves. If the p^{n_2} irreducible representations of H which thus arise are not all distinct, they must (since H is self-conjugate) fall into $p^{n_2'}$ sets of $p^{n_2''}$ each $(n_2 = n_2' + n_2'')$, such that each set of $p^{n_2''}$ are equivalent to each other and distinct from all the rest. The p^n symbols then fall into $p^{n_2'}$ sets of $p^{n_1+n_2''}$, each such that every substitution of G replaces the symbols of one set of $p^{n_1+n_2''}$ either by linear functions of themselves, or by linear functions of the symbols of some other set.

Let K be the greatest sub-group of G which transforms the symbols of some one set of $p^{n_1+n_2''}$ among themselves. So far as this set of symbols is concerned, K may be expressed as a group of monomial substitutions, and, since G is irreducible, this group of monomial substitutions is transitive.

For K, when so expressed, the product of the first set of $p^{n_1+n_2''}$ symbols is an invariant (in general relative), and some power of the product of these symbols, which may be denoted by I_1, is an absolute invariant.

If S_i is a substitution of G which changes the symbols of the first set of $p^{n_1+n_2''}$ into linear functions of the symbols of the i-th set, and if S_i change I_1 into I_i, then $I_1 I_2 \ldots I_{p^{n_2'}}$ is an absolute invariant for G.

Hence the variables may be so chosen that G is a group of monomial substitutions, while H is completely reduced; and the $p^{n_2'}$ sets of $p^{n_1+n_2''}$ symbols are a set of imprimitive systems for G. Since the $p^{n_2'}$ sets of $p^{n_1+n_2''}$ symbols are a set of imprimitive systems for G, G must have a self-

conjugate sub-group, say H', which transforms the symbols of each system transitively among themselves.

Now H' contains H, and therefore H'/H is a self-conjugate sub-group of G/H. Hence H'/H must contain a self-conjugate sub-group, of order p, of G/H. Let H_1 be the corresponding sub-group of G. Then H_1 is a self-conjugate sub-group of G, of order pn_H (where n_H is the order of H), containing H and contained in H'.

Since the degree of each irreducible component of H is p^{n_1}, the degree of each irreducible component of H_1 is either p^{n_1} or p^{n_1+1}.

Suppose, if possible, that the latter were the case, and consider a single irreducible component of H_1. If χ_s is the characteristic of any substitution S of H_1 in this irreducible component, then

$$\sum_S \chi_S \chi_{S^{-1}} = n_{H_1} = pn_H,$$

where the summation is extended to all the substitutions of H_1. Since the p irreducible components of H in H_1 are all equivalent to each other

$$\sum_T \chi_T \chi_{T^{-1}} = p^2 n_H,$$

where the summation is extended to all the substitutions of H. This relation is obviously inconsistent with the previous one, and therefore the degree of H_1 is necessarily p^{n_1}. Hence :—

THEOREM.—If G is an irreducible group of linear substitutions of prime power order and H is any self-conjugate sub-group of G, then G can be so expressed as a group of monomial substitutions that H is completely reduced. If, when this is done, the irreducible components of H are not all distinct, the variables may be so chosen that, G remaining a group of monomial substitutions, a self-conjugate sub-group H_1 of G, containing H and of greater order, transforms the variables in the same reduced sets as H does.

7. If for H a maximum Abelian self-conjugate sub-group A be taken, G may be so expressed as a group of monomial substitutions that A is completely reduced. If the irreducible components of A were not all distinct, A would be contained in a greater self-conjugate sub-group which would transform the symbols in the same reduced sets as A. Since A does not permute the symbols, this greater sub-group would be Abelian, which is not the case. Hence :—

COROLLARY.—If G is an irreducible group of linear substitutions of prime power order, and A any maximum Abelian self-conjugate sub-group of G, then G can be so expressed as a group of monomial substitutions

that A is completely reduced, and the irreducible components of A are all distinct.

8. It may be of interest to give an example of this theorem as applied to a comparatively simple case. The p-holomorph of an Abelian group of type (2, 1) is a group of order p^6 with a central of order p. Taking the Abelian group in the abstract form

$$P^{p^2} = Q^p = E, \qquad PQ = QP,$$

the two isomorphisms $\begin{pmatrix} P & Q \\ PQ & Q \end{pmatrix}$ and $\begin{pmatrix} P & Q \\ P & P^pQ \end{pmatrix}$

generate its group of p-isomorphisms, which is a non-Abelian group of order p^5. Denoting these two isomorphisms by I_1 and I_2, then $I_1^{-1}I_2^{-1}I_1I_2$ is the self-conjugate isomorphism

$$\begin{pmatrix} P & Q \\ P^{1-p} & Q \end{pmatrix}.$$

The p-holomorph of the Abelian group contains a sub-group simply isomorphic with the group of p-isomorphisms of the Abelian group, and having no operation but E in common with the latter.

Hence the p-holomorph may be defined by

$$P^{p^2} = Q^p = E, \qquad PQ = QP,$$

$$I_1^p = I_2^p = I_3^p = E, \qquad I_1^{-1}I_2^{-1}I_1I_2 = I_3, \qquad I_1^{-1}I_3^{-1}I_1I_3 = I_2^{-1}I_3^{-1}I_2I_3 = E,$$

$$I_1^{-1}PI_1 = PQ, \qquad I_1^{-1}QI_1 = Q,$$

$$I_2^{-1}PI_2 = P, \qquad I_2^{-1}QI_2 = P^pQ,$$

$$I_3^{-1}PI_3 = P^{1-p}, \qquad I_3^{-1}QI_3 = Q.$$

These relations are obviously not all independent of each other, but it is convenient to have them all expressed with the object of constructing concrete representations of the group.

Three maximum Abelian self-conjugate sub-groups at once present themselves. These are $\{P, Q\}$, $\{P^p, I_2, I_3\}$, and $\{P^p, Q, I_1, I_3\}$. Denote

$$(E + \eta^{-1}P + \eta^{-2}P^2 + \ldots + \eta^{-p^2+1}P^{p^2-1})\{I_1, Q\},$$

where η is an assigned primitive p^2-th root of unity, by $x_{0,0}$. Then, by post-multiplication, p^2 symbols arise from $x_{0,0}$; and we may write

$$x_{0,0}I_2^\alpha I_3^\beta = x_{\alpha,\beta},$$

where the suffixes are to be reduced (mod p). Now

$$x_{\alpha,\beta}I_1 = x_{0,0}I_1^{-1}I_2^\alpha I_1 I_3^\beta = x_{0,0}I_2^\alpha I_3^{\beta-\alpha} = x_{\alpha,\beta-\alpha},$$

$$x_{\alpha,\beta}I_2 = x_{0,0}I_2^{\alpha+1}I_2^{-1}I_3^\beta I_2 = x_{\alpha+1,\beta},$$

$$x_{\alpha,\beta}I_3 = x_{0,0}I_2^\alpha I_3^{\beta+1} = x_{\alpha,\beta+1},$$

$$x_{\alpha,\beta}P = x_{0,0}PI_2^\alpha P^{-1}I_3^\beta P = \eta x_{0,0}I_2^\alpha I_3^\beta P^{\beta p} = \eta^{1+\beta p}x_{\alpha,\beta},$$

$$x_{\alpha,\beta}Q = x_{0,0}Q^{-1}I_2^\alpha QI_3^\beta = x_{0,0}P^{-\alpha p}I_2^\alpha I_3^\beta = \eta^{-\alpha p}x_{\alpha,\beta}.$$

The group is thus represented as a group of monomial substitutions on p^2 symbols, for which the maximum Abelian self-conjugate sub-group $\{P, Q\}$ appears in completely reduced form. That this group of linear substitutions is simply isomorphic with the abstract group is clear since the substitutions corresponding to the self-conjugate operations of the abstract group are different from identity. The abstract group has no Abelian sub-group of index p, and

therefore no group of linear substitutions in p symbols can be simply isomorphic with it. The group of monomial substitutions is therefore irreducible.

Take now new variables defined by

$$y_{s,t} = \sum_{\alpha,\beta} \omega^{-s\alpha - t\beta} x_{\alpha,\beta},$$

where $\omega = \eta^p$. Then

$$y_{s,t} I_1 = \sum_{\alpha,\beta} \omega^{-s\alpha - t\beta} x_{\alpha,\beta-\alpha} = y_{s+t,t},$$

$$y_{s,t} I_2 = \sum_{\alpha,\beta} \omega^{-s\alpha - t\beta} x_{\alpha+1,\beta} = \omega^s y_{s,t},$$

$$y_{s,t} I_3 = \sum_{\alpha,\beta} \omega^{s\alpha - t\beta} x_{\alpha,\beta+1} = \omega^t y_{s,t},$$

$$y_{s,t} P = \sum_{\alpha,\beta} \omega^{-s\alpha - t\beta} \eta\omega^{\beta} x_{\alpha,\beta} = \eta y_{s,t-1},$$

$$y_{s,t} Q = \sum_{\alpha,\beta} \omega^{-s\alpha - t\beta} \omega^{-\alpha} x_{\alpha,\beta} = y_{s+1,t}.$$

Here the maximum Abelian self-conjugate sub-group $\{P^p, I_2, I_3\}$ is completely reduced. The representation is equivalent to the preceding, since it is obtained by a transformation of the variables.

Finally, take as variables

$$z_{\alpha,t} = \sum_{\beta} \omega^{-t\beta} x_{\alpha,\beta}.$$

Then

$$z_{\alpha,t} I_1 = \sum_{\beta} \omega^{-t\beta} x_{\alpha,\beta-\alpha} = \omega^{-t\alpha} z_{\alpha,t},$$

$$z_{\alpha,t} I_2 = \sum_{\beta} \omega^{-t\beta} x_{\alpha+1,\beta} = z_{\alpha+1,t},$$

$$z_{\alpha,t} I_3 = \sum_{\beta} \omega^{-t\beta} x_{\alpha,\beta+1} = \omega^t z_{\alpha,t},$$

$$z_{\alpha,t} P = \sum_{\beta} \omega^{-t\beta} \eta\omega^{\beta} x_{\alpha,\beta} = \eta z_{\alpha,t-1},$$

$$z_{\alpha,t} Q = \sum_{\beta} \omega^{-t\beta} \omega^{-\alpha} x_{\alpha,\beta} = \omega^{-\alpha} z_{\alpha,t}.$$

This representation is again equivalent to the original one, and in it the maximum Abelian self-conjugate sub-group $\{P^p, Q, I_1, I_2\}$ is completely reduced.

9.* From the preceding result a process may be deduced for constructing the irreducible groups of linear substitutions which are simply isomorphic with any group of prime power order whose central is cyclical.

Suppose that G is any abstract group of prime power order whose central is the cyclical group generated by C. Let A be *any* maximum Abelian self-conjugate sub-group of G.

If the order of C is p^c, A may contain operations C', of order $p^{c'}$ $(c' > c)$, such that

$$C'^{p^{c'-c}} = C.$$

Among these operations let C' be chosen so that c' is as great as possible. Then a set of independent generators of A,

$$C',\ C_1,\ C_2,\ \ldots$$

may be chosen of which C' is one. The sub-group $\{C_1, C_2, \ldots\}$ of A, whose order is $p^{a-c'}$, if p^a is that of A, contains no self-conjugate operation

* §§ 9, 10 have been modified since the paper was communicated to the Society.

of G; and any sub-group of A whose order exceeds $p^{a-c'}$ necessarily contains self-conjugate operations, other than E, of G.

Suppose now that B is an arbitrarily chosen sub-group of A of order $p^{a-c'}$, subject to the condition of containing no self-conjugate operation of G except E. Then A admits an irreducible representation (necessarily in a single symbol) in which every operation of B corresponds to

$$x' = x,$$

while C' corresponds to $$x' = \omega x,$$

ω being an assigned primitive $p^{c'}$-th root of unity.

Now, among the representations of A occurring within the irreducible representations of G, every irreducible representation of A must arise. Hence the representation of A which has been just constructed must occur in some of the irreducible representations of G. Let Γ be one of these. In this representation C multiplies one of the symbols by $\omega^{p^{c'-c}}$, *i.e.*, by a primitive p^c-th root of unity. It follows that, in Γ, no power of C except E corresponds to the identical substitution, so that Γ and G are simply isomorphic.

Let Γ be expressed as a group of monomial substitutions, for which A is completely reduced, on the symbols

$$x_0,\ x_1,\ \ldots,\ x_{p^a-1};$$

and suppose that the assigned representation of A is that which is given by x_0.

Denote by Γ_0 the sub-group of Γ which changes x_0 into a multiple of itself; and by Γ_0' that sub-group which leaves x_0 unchanged. Then Γ_0/Γ_0' is a cyclical group whose order is equal to the index of the highest root of unity that occurs in the coefficients of the monomial substitutions. Since Γ_0' leaves x_0 unchanged it contains no self-conjugate substitution of Γ, except E; but any sub-group of Γ_0, containing Γ_0' and of greater order, necessarily contains self-conjugate substitutions of Γ.

Since the substitutions of A do not permute the symbols, A is the greatest self-conjugate sub-group of Γ contained in Γ_0. Also, since B is the greatest sub-group of A which leaves x_0 unchanged, B is the greatest common sub-group of A and Γ_0'. Hence, both A and Γ_0' being contained self-conjugately in Γ_0, B is also contained self-conjugately in Γ_0.

Suppose now that T is any substitution of Γ_0. Then

$$T^{-1}BT = B,$$

$$T^{-1}\{C', B\}T = \{C', B\};$$

and therefore $$T^{-1}C'T = C'^k b,$$

where b is some substitution of B.

Now C' changes x_0 into ωx_0, b changes x_0 into itself, and T changes x_0 into a multiple of itself; so that $T^{-1}C'T$ and $C'^k b$ change x_0 in ωx_0 and $\omega^k x_0$ respectively. Hence $k = 1$, and every substitution of Γ_0 transforms C' into C' multiplied by some substitution of B.

Let Σ_i be a substitution of Γ which changes x_0 into a multiple of x_i, and suppose that

$$\Sigma_i^{-1} B \Sigma_i = B,$$
$$\Sigma_i^{-1} C' \Sigma_i = C'b',$$

where b' is some substitution of B. Then every substitution of B leaves x_i unchanged, and C' changes x_i into ωx_i; in other words, the representations of A in x_0 and x_i are equivalent. It has been shewn in § 7 that this is not the case: and therefore the substitutions of Γ_0 are the only ones which at once transform B into itself and give the identical isomorphism of A/B, or, in other words, which satisfy the relations

$$T^{-1} B T = B,$$
$$T^{-1} C' T = C'b,$$

where b belongs to B.

Moreover it has been seen above that Γ_0' is a maximum self-conjugate sub-group of Γ_0, subject to the condition of containing no self-conjugate substitution of Γ, and that Γ_0/Γ_0' is cyclical.

10. Return now to the abstract group G with a cyclical central. When the maximum Abelian self-conjugate sub-group A and its sub-group B, of as large an order as possible subject to the condition of containing no power of C', are assigned, the operations of G satisfying the conditions

$$T^{-1} B T = B,$$
$$T^{-1} C' T = C'b,$$

are determinate. Suppose that they constitute a sub-group H, and that K is any maximum self-conjugate sub-group of H subject to the condition of containing no self-conjugate operation of G. Among the sub-groups K there is necessarily one (or more) such that H/K is cyclical and of order p^d equal to or greater than p^c, the order of C'. In respect of K let H fall into the sets

$$K,\ DK,\ D^2K,\ \ldots,\ D^{p^d-1}K;$$

and in respect of H let G fall into the sets

$$H,\ HS_1,\ HS_2,\ \ldots,\ HS_{p^n-1}.$$

Then the p^n symbols

$$(E + \omega^{-1}D + \omega^{-2}D^2 + \ldots + \omega^{-p^d+1}D^{p^d-1})\,KS_i \quad (i = 0, 1, \ldots, p^n - 1),$$

where ω is a primitive p^d-th root of unity, undergo, on post-multiplication

by the operations of G, a transitive group of monomial substitutions. This group is simply isomorphic with G, since the substitution corresponding to C multiplies each symbol by the same p^c-th root of unity. The representations of A in the p^n symbols are all distinct, and therefore* the group of monomial substitutions is irreducible.

When A and B are assigned, subject to the assumed conditions H is unique, but K is not necessarily unique. To ensure obtaining *all* the irreducible groups simply isomorphic with G, it would be necessary to take for B a representative of each set of conjugate sub-groups of G, contained in A, which satisfy the necessary conditions; for K each self-conjugate sub-group of H satisfying the necessary conditions; and for ω each primitive p^d-th root of unity.

11. [*Added on May 20th*, 1912.]—The process of the preceding paragraph will now be illustrated by applying it to a particular case. It would be tedious, and indeed would serve no useful purpose, to construct all the distinct irreducible representations (simply isomorphic with it) of the group chosen. My object is rather to shew how to construct a representation of the group which contains an assigned representation of a maximum Abelian self-conjugate sub-group, and to shew that the degree of the irreducible representation will depend on the chosen representation of the Abelian sub-group.

If p is an odd prime, the relations

$$P^p = Q^p = R^p = E, \quad P^{-1}Q^{-1}PQ = R, \quad P^{-1}R^{-1}PR = Q^{-1}R^{-1}QR = E$$

define a non-Abelian group of order p^3 whose operations are all of order p. Let

$$\{P_1, Q_1\}, \ \{P_2, Q_2\}, \ \ldots, \ \{P_p, Q_p\}$$

be p such groups, no two of which have a common operation, except E, while every operation of any one of these groups is permutable with every operation of any other. The direct product of these groups is a group of order p^{3p}. Further suppose S such that

$$S^p = E,$$

$$S^{-1}P_1S = P_2, \quad S^{-1}P_2S = P_3, \quad \ldots, \quad S^{-1}P_pS = P_1,$$

$$S^{-1}Q_1S = Q_2, \quad S^{-1}Q_2S = Q_3, \quad \ldots, \quad S^{-1}Q_pS = Q_1.$$

These relations clearly involve

$$S^{-1}R_1S = R_2, \quad S^{-1}R_2S = R_3, \quad \ldots, \quad S^{-1}R_pS = R_1.$$

The group $\{S, P_1, Q_1\}$ or G is then a group of order p^{3p+1}. It has a cyclical central, for $R_1R_2 \ldots R_p$ and its power are the only self-conjugate operations of the group; and every operation of the group is represented just once by the form

$$S^\kappa \prod_i P_i^{\alpha_i} Q_i^{\beta_i} R_i^{\gamma_i}, \tag{i}$$

where the indices take all values from 0 to $p-1$. It is clear that G has a number of maximum Abelian self-conjugate sub-groups. Denote by A the sub-group

$$\{P_1, P_2, \ldots, P_p, R_1, R_2, \ldots, R_p\}.$$

The defining relations shew that this is self-conjugate and Abelian of type $(1, 1, \ldots, 1)$. Moreover it is easy to verify that no operation of the form (i) which does not belong to A is permutable with every operation of A. Hence A is a maximum Abelian self-conjugate sub-group. In this case the c' of § 9 is 1, and any sub-group of order p^{2p-1} of A, which does not contain

* See note at the end of the paper.

$R_1 R_2 \dots R_p$, may be taken for B. The following sub-groups, satisfying these conditions, will be considered, viz.,

$$\{P_1, P_2, \dots, P_p, R_2, R_3, R_4, \dots, R_p\},$$
$$\{P_1, P_2, \dots, P_p, R_1^{-1}R_2, R_3, R_4, \dots, R_p\},$$
$$\{P_1, P_2, \dots, P_p, R_1^{-1}R_2, R_2^{-1}R_3, R_4, \dots, R_p\},$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$\{P_1, P_2, \dots, P_p, R_1^{-1}R_2, R_2^{-1}R_3, \dots, R_{p-2}^{-1}R_{p-1}, R_p\}.$$

If the first of these groups is taken for B, it is clearly contained self-conjugately in

$$\{Q_2, Q_3, \dots, Q_p, B\},$$

and this group contains no self-conjugate operation of G. No operation of the form (i) in which α is different from zero transforms B into itself; and if an operation of the form (i), in which $\alpha = 0$, $\beta_1 \neq 0$, is combined with B, the resulting group contains self-conjugate operations of G. Hence $\{Q_2, Q_3, \dots, Q_p, B\}$ is a group of as great order as possible which contains B self-conjugately, and does not contain any self-conjugate operation of G. Call this group K and denote $R_1 R_2 \dots R_p$ by C. Then the p^2 symbols

$$(E + \omega^{-1}C + \dots + \omega^{-p+1}C^{p-1}) K Q_1^{\alpha} S^{\beta} \text{ or } x_{\alpha,\beta}$$

undergo on post-multiplication by the operations of G an irreducible group of monomial substitutions simply isomorphic with G. It is generated by

$$\begin{aligned} x'_{\alpha,\beta} S &= x_{\alpha,\beta+1}, \\ x'_{\alpha,\beta} Q_1 &= x_{\alpha+1,\beta} \qquad (\beta = 0), \\ &= x_{\alpha,\beta} \qquad (\beta \neq 0), \\ x'_{\alpha,\beta} P_1 &= \omega^{\alpha+1} x_{\alpha,\beta} \quad (\beta = 0), \\ &= x_{\alpha,\beta} \qquad (\beta \neq 0). \end{aligned}$$

If the second of the groups is taken for B, it may be shewn by similar considerations that

$$\{Q_3, Q_4, \dots, Q_p, B\}$$

is the greatest group that contains it self-conjugately without at the same time containing a self-conjugate operation of G. In this case the p^3 symbols

$$(E + \omega^{-1}C + \dots + \omega^{-p+1}C^{p-1}) K Q_1^{\alpha} Q_2^{\beta} S^{\gamma},$$

where K now denotes the group $\{Q_3, Q_4, \dots, Q_p, B\}$, undergo on post-multiplication an irreducible group of monomial substitutions simply isomorphic with G. The discussion of the other forms of B is on similar lines. For the last form it is found that any group containing $\{Q_p, B\}$, and of greater order, within which B is self-conjugate, necessarily contains self-conjugate operations of G. Hence in this case $\{Q_p, B\}$ may be taken for K, and the p^p symbols

$$(E + \omega^{-1}C + \dots + \omega^{-p+1}C^{p-1}) K Q_1^{\alpha_1} Q_2^{\alpha_2} \dots Q_{p-1}^{\alpha_{p-1}} S^{\alpha_p}$$

undergo on post-multiplication an irreducible group of linear substitutions simply isomorphic with G. To the various sub-groups of A which have been chosen for B, there correspond in this case irreducible representations* of G of degrees $p^2, p^3, \dots, p^p$, which are all simply isomorphic with G.

12. If G is any group of prime power order with a non-cyclical central,

* This example is in direct contradiction with a theorem given by Prof. Fite (*Bulletin Amer. Math. Soc.*, Dec., 1911), to the effect that an irreducible group of order p^m and degree p^2 cannot be simply isomorphic with an irreducible group of any other degree. I have not been able to follow the argument by which Prof. Fite supports his theorem.

no irreducible group of linear substitutions can be simply isomorphic with G. In any irreducible representation Γ, of G, the operations of some self-conjugate sub-group H of G must correspond to the identical substitution, and Γ is simply isomorphic with G/H, which must therefore be a group with a cyclical central. All those irreducible representations of G which are simply isomorphic with G/H can be constructed by the method of the preceding paragraph; so that the determination of the irreducible representations of G is completed when those self-conjugate sub-groups H are determined such that G/H is a group with a cyclical central.

Let P be any operation of G, and H' any self-conjugate sub-group of G which does not contain P. If P' is the operation of G/H' to which P corresponds, and if $\{P'\}$ is not the sole self-conjugate sub-group of order p of G/H', then G/H' has a self-conjugate sub-group that does not contain P'; and G has a self-conjugate sub-group containing H, and of greater order, which does not contain P. It follows that if H is a self-conjugate sub-group of G of maximum order, subject to the condition of not containing P, then the central of G/H is cyclical. Moreover, every self-conjugate sub-group H of G, such that G/H has a cyclical central, may be thus determined on taking for P in turn a representative of each of the conjugate sets of G.

II. *The Group of p-Isomorphisms of an Abelian Group of Type* (1, 1, ..., 1).

13. The group of p-isomorphisms of an Abelian group of type (1, 1, 1, ..., 1) has the same relative importance in connection with groups of p-isomorphisms in general, as the Abelian group of type (1, 1, 1, ..., 1) has in connection with Abelian groups.

It is obvious that the generators $P_i (i = 1, 2, \ldots, n)$ of an Abelian group of order p^n and type (1, 1, 1, ..., 1) may be so chosen that the series of groups G_i of § 1 are

$$\{P_i, P_{i+1}, \ldots, P_n\} \quad (i = 2, 3, \ldots, n).$$

The isomorphism which leaves $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$ unchanged and transforms P_i into $P_i P_j$ $(j > i)$, will be denoted by $I_{i,j}$. It changes each of the above series of sub-groups into itself.

It may be easily verified, by carrying out the isomorphisms, that the relations

$$\begin{aligned} I_{m,n}^{-1} I_{i,j} I_{m,n} &= I_{i,j}, \qquad (n \neq i,\ m \neq j) \\ I_{m,i}^{-1} I_{i,j} I_{m,i} &= I_{i,j} I_{m,j}^{-1}, \\ I_{j,n}^{-1} I_{i,j} I_{j,n} &= I_{i,j} I_{i,n}, \\ I_{i,j}^{p} &= E, \end{aligned} \tag{i}$$

hold for all values of i, j, m, n. (It is to be noticed that, from the definition of $I_{i,j}$, $j-i$ is necessarily a positive number.)

A simple induction suffices to shew that every operation of the group K_n of p-isomorphisms is expressible in the form

$$I_{1,2}^{\alpha} I_{1,3}^{\beta} \dots I_{1,n}^{\gamma} \dots I_{i,i+1}^{\delta} \dots I_{i,n}^{\epsilon} \dots I_{n-1,n}^{\theta};$$

and that no two such operations, with distinct indices, give the same isomorphism. The order of the group of p-isomorphisms is therefore $p^{\frac{1}{2}n(n-1)}$; and this is, in fact, the greatest power of p which divides the order of the group of isomorphisms of the Abelian group.

The relations satisfied by $I_{i,j}$ enable the product of any two operations of the group to be calculated.

The $n-1$ operations

$$I_{1,2},\ I_{2,3},\ I_{3,4},\ \dots,\ I_{n-2,n-1},\ I_{n-1,n}$$

generate the group.

The relations (i) shew that so long as $j-i$ is less than $n-1$, $I_{i,j}$ is not a self-conjugate operation. From this it follows at once that $I_{1,n}$ and its powers are the only self-conjugate operations.

Since $$I_{i,j}^{-1} I_{1,i} I_{i,j} = I_{1,i} I_{1,j},$$

and $$I_{j,k}^{-1} I_{1,i} I_{j,k} = I_{1,i},$$

the sub-group $\{I_{1,2}, I_{1,3}, \dots, I_{1,n}\}$ is self-conjugate. It is an Abelian group of order p^{n-1} and type (1, 1, ..., 1). Every operation of the group gives an isomorphism of this self-conjugate sub-group, and its own operations are the only ones which give the identical isomorphism.

Now $\{I_{2,3}, I_{2,4}, \dots, I_{n-1,n}\}$ is a sub-group of order $p^{\frac{1}{2}(n-1)(n-2)}$ simply isomorphic with the group of p-isomorphisms of an Abelian group of order p^{n-1} and type (1, 1, ..., 1) and having no operation, except E, in common with $\{I_{1,2}, I_{1,3}, \dots, I_{1,n}\}$.

Hence the group of p-isomorphisms of an Abelian group of order p^n and type (1, 1, ..., 1) is simply isomorphic with the p-holomorph of an Abelian group of order p^{n-1} and type (1, 1, ..., 1).

14. From the defining relations (i), when written in the form

$$I_{m,n}^{-1} I_{i,j} I_{m,n} I_{i,j}^{-1} = E, \qquad (n \neq i,\ m \neq j),$$

$$I_{m,i}^{-1} I_{i,j} I_{m,i} I_{i,j}^{-1} = I_{m,j}^{-1},$$

$$I_{j,n}^{-1} I_{i,j} I_{j,n} I_{i,j}^{-1} = I_{i,n},$$

it follows that the operations $I_{i,j}$, for which

$$j-i \geqslant k,$$

combine among themselves to constitute a self-conjugate sub-group D_k. There are $\frac{1}{2}(n-k+1)(n-k)$ such I's, and therefore the order of this sub-group is $p^{\frac{1}{2}(n-k+1)(n-k)}$. In particular, when k is 2, the group D_2 is the derived group of K_n, when k is 4 the group D_4 is the derived group of D_2, and so on. Any one of these sub-groups is clearly Abelian or non-Abelian according as $2k$ is or is not greater than $n-1$. Hence, if

$$2^m \leqslant n-1 < 2^{m+1},$$

D_{2^m} is Abelian, and the derived series of K_n is

$$D_2,\ D_{2^2},\ \ldots,\ D_{2^m},\ E.$$

Denoting by $[a/b]$ the greatest positive integer in a/b, D_k is Abelian if

$$k = [\tfrac{1}{2}(n-1)]+1.$$

Hence, for all values of n, K_n has an Abelian self-conjugate sub-group of order $p^{\frac{1}{2}(n-[\frac{1}{2}(n-1)])(n-[\frac{1}{2}(n-1)]-1)}$, and type (1, 1, 1, ..., 1).

15. Any operation of K_n, when represented as an actual isomorphism of the group $\{P_1, P_2, \ldots, P_n\}$ is expressible in the form

$$\begin{pmatrix} P_1, & P_2, & \ldots, & P_{n-1}, & P_n \\ P_1 P_2^{\alpha} P_3^{\beta} \ldots P_n^{\gamma}, & P_2 P_3^{\alpha'} \ldots P_n^{\beta'}, & \ldots, & P_{n-1} P_n^{\alpha''}, & P_n \end{pmatrix}.$$

By taking as new generators for the Abelian group

$$Q_1 = P_1,$$
$$Q_2 = P_2^{\alpha} P_3^{\beta} \ldots P_n^{\gamma},$$
$$Q_i = P_i, \qquad (i > 2),$$

this becomes $\begin{pmatrix} Q_1, & Q_2, & \ldots, & Q_{n-1}, & Q_n \\ Q_1 Q_2 & Q_2 Q_3^{\alpha'} \ldots Q_n^{\beta'}, & \ldots, & Q_{n-1} Q_n^{\alpha''}, & Q_n \end{pmatrix}$.

Repeating this process, any operation of K_n can be represented as an actual isomorphism in the form

$$\begin{pmatrix} Q_1, & Q_2, & \ldots, & Q_{r-1}, & Q_r, & Q_{r+1}, & \ldots, & Q_{r+s-1}, & Q_{r+s}, & \ldots \\ Q_1 Q_2, & Q_2 Q_3, & \ldots, & Q_{r-1} Q_r, & Q_r, & Q_{r+1} Q_{r+2}, & \ldots, & Q_{r+s-1} Q_{r+s}, & Q_{r+s}, & \ldots \end{pmatrix},$$

where r, s, ... is any partition of n. The order of this isomorphism is clearly the least common multiple of the orders of the isomorphisms

$$\begin{pmatrix} Q_1, & \ldots, & Q_{r-1}, & Q_r \\ Q_1 Q_2, & \ldots, & Q_{r-1} Q_r, & Q_r \end{pmatrix}, \quad \begin{pmatrix} Q_{r+1}, & \ldots, & Q_{r+s-1}, & Q_{r+s} \\ Q_{r+1} Q_{r+2}, & \ldots, & Q_{r+s-1} Q_{r+s}, & Q_{r+s} \end{pmatrix}, \quad \ldots.$$

Now a simple calculation shews that the m-th power of the isomorphism

$$\begin{pmatrix} Q_1, & Q_2, & \dots, & Q_{n-1}, & Q_n \\ Q_1Q_2, & Q_2Q_3, & \dots, & Q_{n-1}Q_n, & Q_n \end{pmatrix}$$

replaces Q_i by $Q_iQ_{i+1}^m Q_{i+2}^{\frac{m(m-1)}{1.2}} Q_{i+3}^{\frac{m(m-1)(m-2)}{1.2.3}} \dots Q_{i+m}$,

where, if any suffixes are greater than n, the corresponding terms are to be omitted from the product. Hence the p-th power of the isomorphism is identity if $n < p+1$; while if $n \geqslant p+1$, the p-th power is

$$\begin{pmatrix} Q_1, & Q_2, & \dots, & Q_{n-p}, & Q_{n-p+1}, & \dots, & Q_n \\ Q_1Q_{p+1}, & Q_2Q_{p+2}, & \dots, & Q_{n-p}Q_n, & Q_{n-p+1}, & \dots, & Q_n \end{pmatrix}.$$

If $$kp+1 \leqslant n < (k+1)p+1,$$

this is the product of

$$\begin{pmatrix} Q_1, & Q_{p+1}, & \dots, & Q_{(k-1)p+1}, & Q_{kp+1} \\ Q_1Q_{p+1}, & Q_{p+1}Q_{2p+1}, & \dots, & Q_{(k-1)p+1}Q_{kp+1}, & Q_{kp+1} \end{pmatrix},$$

and similar isomorphisms with the same or a smaller number of terms. If $k < p$, the p-th power of the last isomorphism is identity: if $k \geqslant p$, its p-th power may be similarly constructed. Hence, if

$$p^{m-1}+1 \leqslant n < p^m+1,$$

the order of $$\begin{pmatrix} Q_1, & Q_2, & \dots, & Q_{n-1}, & Q_n \\ Q_1Q_2, & Q_2Q_3, & \dots, & Q_{n-1}Q_n, & Q_n \end{pmatrix}$$

is p^m.

It follows that, if p^{m-1} is the highest power of p which does not exceed $n-1$, then K_n contains operations of order p^m and no operations of higher order.

16. Since the central of K_n is cyclical, K_n is simply isomorphic with an irreducible group of linear substitutions.* The order of K_n being a power of a prime, its irreducible representations may be exhibited as groups of monomial substitutions. The simplest concrete representation of K_n is probably an irreducible group of monomial substitutions, and such a representation may be constructed as follows.†

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* *Theory of Groups*, 2nd edition. Note F.

† *L.c.*, p. 324.

Denote the sub-group $\{I_{2,3}, I_{2,4}, \ldots, I_{n-1,n}\}$ of K_n by H, and write $x_{0,0,\ldots,0}$ for

$$(E+\omega^{-1}I_{1,n}+\omega^{-2}I_{1,n}^2+\ldots+\omega^{-p+1}I_{1,n}^{p-1})H,$$

where ω is an assigned primitive p-th root of unity.

By post-multiplication there arise from $x_{0,0,\ldots,0}$ just p^{n-2} distinct symbols; and since the two sub-groups $\{I_{1,n}, H\}$ and $\{I_{1,2}, I_{1,3}, \ldots, I_{1,n-1}\}$ have no operation in common, except E, these may be taken to be $x_{a_1, a_2, \ldots, a_{n-2}}$, where

$$x_{0,0,\ldots,0}\, I_{1,2}^{a_1} I_{1,3}^{a_2} \ldots I_{1,n-1}^{a_{n-2}} = x_{a_1, a_2, \ldots, a_{n-2}}.$$

Now, if $i < n-1$,

$$I_{1,2}^{a_1} I_{1,3}^{a_2} \ldots I_{1,n-1}^{a_{n-2}} I_{i,i+1} = I_{1,2}^{a_1} \ldots I_{1,i}^{a_{i-1}} I_{i,i+1} I_{1,i+1}^{a_i} \ldots I_{1,n-1}^{a_{n-2}}$$

$$= I_{i,i+1} I_{1,2}^{a_1} \ldots I_{1,i}^{a_{i-1}} I_{1,i+1}^{a_i+a_{i-1}} \ldots I_{1,n-1}^{a_{n-2}}.$$

Hence, if $i < n-1$,

$$x_{a_1, a_2, \ldots, a_{n-2}} I_{i,i+1} = x_{a_1, a_2, \ldots, a_{i-1}, a_i+a_{i-1}, a_{i+1}, \ldots, a_{n-2}}.$$

Similarly it is found that

$$x_{a_1, a_2, \ldots, a_{n-2}} I_{n-1,n} = \omega^{a_{n-2}} x_{a_1, a_2, \ldots, a_{n-2}}.$$

The generating operations of the group are thus represented as monomial substitutions on the p^{n-2} symbols.

The Abelian self-conjugate sub-group which replaces every symbol by a multiple of itself is $\{I_{1,n}, I_{2,n}, \ldots, I_{n-1,n}\}$. Now

$$I_{1,2}^{a_1} I_{1,3}^{a_2} \ldots I_{1,n-1}^{a_{n-2}} I_{i,n} = I_{i,n} I_{1,n}^{a_{i-1}} I_{1,2}^{a_1} \ldots I_{1,n-1}^{a_{n-2}},$$

where, if $i = 1$, a_{i-1} is to be replaced by 0.

Hence $$x_{a_1, a_2, \ldots, a_{n-2}} I_{i,n} = \omega^{a_{i-1}} x_{a_1, a_2, \ldots, a_{n-2}} \quad (i > 1),$$

and $$x_{a_1, a_2, \ldots, a_{n-2}} I_{1,n} = \omega x_{a_1, a_2, \ldots, a_{n-2}}.$$

From this it follows that the p^{n-2} representations of $\{I_{1,n}, I_{2,n}, \ldots, I_{n-1,n}\}$ in the p^{n-2} symbols are all *distinct*. It will be shewn below* that, when this condition is satisfied in a transitive group of monomial substitutions, the group is irreducible.

* The proof of this statement is given in a note at the end of the paper.

Hence K_n is simply isomorphic with the irreducible group of monomial substitutions on p^{n-2} symbols $x_{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}}$ generated by

$$I_{i, i+1} \quad (i = 1, 2, \ldots, n-1);$$

where
$$x_{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}} I_{i, i+1} = x_{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i + \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n-2}} \quad (i < n-1),$$

$$x_{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}} I_{n-1, n} = \omega^{\alpha_{n-2}} x_{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}}.$$

The suffixes are to be reduced, mod p.

III. *On the Derived Group.*

17. So far as I know, no investigation has hitherto been made as regards the limitations, if any, on the nature of a group which can be the derived group of a group of prime power order. Two general theorems are here proved.

THEOREM.—A non-Abelian group whose central is cyclical cannot be the derived group of a group of prime power order.

Suppose that a group G of prime power order has for its derived group G_1, a non-Abelian group with a cyclical central, order p^c, generated by C.

Since $G_1/\{C\}$ has $1+kp$ self-conjugate sub-groups of order p, G_1 must have at least one sub-group of order p^{c+1} which is self-conjugate in G and contains C. This sub-group is necessarily Abelian and of type either $(p+1)$ or $(p, 1)$.

In the latter case let it be $\{C, S\}$, where

$$S^p = E, \qquad SC = CS.$$

In G_1, the operations

$$S, \; SC^{p^{c-1}}, \; SC^{2p^{c-1}}, \; \ldots, \; SC^{(p-1)p^{c-1}},$$

form a conjugate set. No operation of G can transform S into a power of itself, and therefore these operations also form a conjugate set for G. The operations of G which are permutable with S therefore form a self-conjugate sub-group H of index p. Such a sub-group necessarily contains the derived group of G. On the other hand, G_1 by supposition contains operations which are not permutable with S. Hence the second case leads to a contradiction.

Taking now the first case, let the sub-group of order p^{c+1} be $\{C'\}$, where $C'^p = C$. Every operation of G transforms C' into a power of itself. The operations of G which are permutable with C' therefore con-

stitute a self-conjugate sub-group H', and G/H' is cyclical. Hence H' must contain G_1; and, since by supposition G_1 has operations which are not permutable with C', there is again a contradiction.

The theorem is therefore proved.

18. From this result a further limitation on the nature of a group, which can be the derived group of a prime power group, may be deduced. Let G be a group of prime power order, G_1 the derived group of G, G_2 the derived group of G_1, G_3 that of G_2, and so on. If G_2 is not E there must be a self-conjugate sub-group K of G contained in G_2 and containing G_3, such that the order of G_2/K is p. Denote G/K by G' and let

$$G',\ G_1',\ G_2',\ \ldots$$

be the derived series of G', and

$$G,\ H_1,\ H_2,\ \ldots$$

the corresponding sub-groups of G. Then * H_1 is the smallest sub-group of G containing K such that G/H_1 is Abelian. Hence, since G_1 contains K, H_1 is G_1. Similarly H_2 is G_2; and, since K is of index p in G_2, the derived series of G' is

$$G',\ G_1',\ G_2',\ E,$$

where the order of G_2' is p. Now, if the order of G_1'/G_2' were p^2, G_1' would be a non-Abelian group of order p^3. Such a group, having a cyclical central, could not be the derived group of G'. Hence the order of G_1'/G_2', and therefore also of G_1/G_2, must be greater than p^2.

THEOREM.—A non-Abelian group, the index of whose derived group is p^2, cannot be the derived group of a prime power group.

19. It is an immediate consequence of either of the preceding theorems that a group of order $p^n\,(n < 6)$ is necessarily either Abelian or metabelian. If a group of order p^6 is neither Abelian nor metabelian, the orders of its first and second derived groups must be p^4 and p. That such groups exist is shewn by the following example. Let A and B denote the two substitutions

$$\left.\begin{aligned} x'_{\alpha,\beta} &= x_{\alpha,\beta+1} \\ x'_{\alpha,\beta} &= \omega^{\frac{1}{2}\alpha(\alpha-1)}\, x_{\alpha+\beta,\beta} \end{aligned}\right\} \quad (\alpha,\ \beta = 0,\ 1,\ \ldots,\ p-1),$$

where the suffixes are reduced (mod p), and ω is a primitive p-th root of unity, p being an odd prime. Then $A^{-1}B^{-1}AB$ or C is

$$x'_{\alpha,\beta} = x_{\alpha+1,\beta}.$$

The derived group of $\{A, B\}$ is generated by C and its conjugates. It is a self-conjugate sub-

* Compare, *Theory of Groups*, 2nd edition, p. 56.

group of index p^2, since both A and B are of order p. The self-conjugate sub-group H generated by the commutators of C is of index p^3. Now $A^{-1}C^{-1}AC$ is E, and $B^{-1}CBC^{-1}$, or P, is

$$x'_{\alpha,\beta} = \omega^{\alpha-\beta} x_{\alpha,\beta}.$$

H is generated by P and its conjugates. Moreover $A^{-1}PA$ and $B^{-1}PB$ are respectively

$$x'_{\alpha,\beta} = \omega^{\alpha-\beta+1} x_{\alpha,\beta},$$

and

$$x'_{\alpha,\beta} = \omega^{\alpha-2\beta} x_{\alpha,\beta}.$$

Hence if the two substitutions

$$x'_{\alpha,\beta} = \omega^{\beta} x_{\alpha,\beta},$$

and

$$x'_{\alpha,\beta} = \omega x_{\alpha,\beta},$$

be denoted by Q and R, then

$$A^{-1}PA = PR, \quad A^{-1}QA = QR^{-1},$$

$$B^{-1}PB = PQ^{-1}, \quad B^{-1}QB = Q,$$

while

$$A^{-1}RA = B^{-1}RB = R.$$

Hence H is the Abelian group $\{P, Q, R\}$ of order p^3, and $\{A, B\}$ is of order p^6. The derived group of $\{A, B\}$, viz., $\{C, P, Q, R\}$ is of order p^4; and since

$$C^{-1}PC = PR, \quad C^{-1}QC = Q, \quad C^{-1}RC = R,$$

its derived group, viz., $\{R\}$, is of order p.

20. It is a direct consequence of the latter of the two preceding theorems that the order (p^n) of a prime power group G whose i-th derived group is distinct from E cannot be less than p^{3i}. When i is 2 the foregoing example shews that there are groups of order p^6 whose second derived groups are distinct from E; but it seems probable that for greater values of i the actual lower limit of the order exceeds p^{3i}.

If Γ is an irreducible group of linear substitutions of prime power order in p^n variables, its derived group Γ_1 is necessarily reducible, and must transform the variables in irreducible sets of p^{n-1} or fewer variables. So, in each set of variables irreducibly transformed by Γ_1, its derived group Γ_2 is necessarily reducible and must transform the variables in irreducible sets of p^{n-2} or fewer. Hence the n-th derived group of Γ, if distinct from E, is necessarily Abelian; and in any case the $n+1$-th derived group in E. Hence if the greatest of the degrees of the irreducible representations of a prime power group G is p^n, then in each irreducible representation the substitution corresponding to each of the operations of the $n+1$-th derived group of G is the identical substitution. The $n+1$-th derived group of G is therefore E, and the derived series of G can consist at most of $n+1$ terms.

Note to pp. 234, 240.

Let G be a group of monomial substitutions and A the Abelian sub-group of G which replaces each symbol by a multiple of itself. Then G/A is simply isomorphic with the group of permutations which arises from G when each coefficient in each of its substitutions is replaced by unity. Denote by n the degree of G, *i.e.*, the number of symbols on which it operates. The Abelian group A is completely reduced, and to each symbol operated on by G corresponds an irreducible representation of A.

Suppose G is such that the n irreducible representations of A thus arising are all distinct from each other.

Let S be any substitution of G which replaces just t symbols by multiples of themselves, and, so far as these symbols are concerned, let S be

$$x_1' = \omega_1 x_1, \quad x_2' = \omega_2 x_2, \quad \ldots, \quad x_t' = \omega_t x_t.$$

Let P be any substitution of A, and, so far as the same t symbols are concerned, let P be

$$x_1' = \alpha_1 x_1, \quad x_2' = \alpha_2 x_2, \quad \ldots, \quad x_t' = \alpha_t x_t.$$

Then in G the characteristic χ_{SP} of SP is

$$\alpha_1\omega_1 + \alpha_2\omega_2 + \ldots + \alpha_t\omega_t.$$

Hence the value of $\Sigma\chi\bar{\chi}$ for all symbols of the set SP, where for P each substitution of A is taken in turn, is

$$\sum_{i,j} \omega_i \bar{\omega}_j \sum_P \alpha_i \bar{\alpha}_j,$$

where $\sum_P$ extends to all the substitutions of A. Since the n representations of A in the n symbols are all distinct,

$$\sum_P \alpha_i \bar{\alpha}_j = 0 \quad (j \neq i),$$

$$\sum_P \alpha_i \bar{\alpha}_i = n_A,$$

where n_A is the order of A ; and therefore

$$\sum_P \chi_{SP} \bar{\chi}_{SP} = t n_A.$$

If m_t is the number of permutations of G/A, which leave just t symbols unchanged, it follows that

$$\sum_S \chi_s \bar{\chi}_s = \sum_t m_t t n_A,$$

where the summation on the left extends now to all the substitutions of G.

Now, if the permutation group G/A permutes the symbols in τ transitive sets,*

$$\sum_t m_t t = \tau n_{G/A},$$

where $n_{G/A}$ is the order of G/A. Hence

$$\sum_s \chi_s \bar{\chi}_s = \tau n_{G/A} n_A = \tau n_G,$$

where n_G is the order of G.

It follows that if G/A is transitive, so that $\tau = 1$, then G is irreducible. Hence :—

THEOREM.—A transitive group of monomial substitutions of degree n, in which the n irreducible representations of the Abelian sub-group which does not displace the symbols are all distinct, is irreducible.

It may be noticed that, though the condition that G/A be transitive is obviously necessary in order that G may be irreducible, the further condition that the n representations of A be distinct, though sufficient, is not necessary. In fact, there are irreducible groups of monomial substitutions in which A consists of the identical substitution only.

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* *Theory of Groups*, 2nd edition, p. 190.

ON THE CASES OF EXCEPTION IN JACOBI'S THEOREM CONCERNING DOUBLE RESIDUES

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IN a former paper* I gave a proof of Jacobi's double residue theorem, and shewed how the chief general propositions of the theory of point groups in a plane could be deduced from it, all the points being taken as isolated. In the present paper the same methods are applied to the case when the points coalesce in any way.

Some theorems are first proved concerning matrices whose constituents are polynomials in one letter. These theorems are known when the polynomials are of the first degree; they are here applied to the discussion of two ternary quantics and their eliminant, and they lead to Jacobi's theorem both in its original form and also in the modified form which is the main object of the paper.

With a view to applications the form of the result is examined when the numerator polynomial breaks up into factors, and this discussion leads to the modified theorems of Cayley and Bacharach, Riemann and Roch, the latter being so extended as to include the former.

Jacobi's theorem may be proved by evaluating over a suitable closed region the double integral

$$\iint \frac{\omega}{\phi\psi}\, dt\, du,$$

or the single integral

$$\int \frac{\omega}{\phi \dfrac{\partial\psi}{\partial u}}\, dt,$$

the notation being that of § 4 of this paper. See Baker's *Multiply-Periodic Functions*, p. 178, or *Camb. Phil. Proc.*, Vol. XIV, pp. 387–390. The treatment in the present paper is purely algebraic.

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* *Camb. Phil. Proc.*, Vol. XV, pp 472–481.

Properties of a Matrix whose Constituents are Polynomials.

1. Let x, y, a, b denote rows of quantities, there being n in each row. Let M denote a matrix of n rows and columns, whose coefficients are polynomials in a variable t, and N the associated matrix formed from M by the interchange of rows and columns. Let D be the value of the determinant of M and N, which is itself a polynomial in t and which will be supposed not to vanish identically.

Then if we solve the equations $Mx = a$, or $Ny = b$, where a, b are rows of polynomials in t, we have the values of x, y as rows of polynomials divided by the common denominator D. When D has been resolved into factors the values of x, y can be resolved into partial fractions, and these partial fractions will now be investigated.

2. If D has the factor $t-t'$, then when $t = t'$, the equations $Mx = a$, $Ny = b$ cannot be solved for all values of a, b, but $Mx = 0$, $Ny = 0$ can be satisfied by rows x, y which do not wholly vanish. That is, a row of polynomials ξ can be found, not all containing the factor $t-t'$, but such that $t-t'$ is a factor of the row $M\xi$. There may be several rows such as ξ; let ξ_1 be one of them such that $M\xi_1$ contains the highest possible power of $t-t'$, say $(t-t')^{r_1}$, so that

$$M\xi_1 = (t-t')^{r_1} a_1,$$

where a_1 does not vanish when $t = t'$.

Let ξ_2 be a row, not a multiple of ξ_1, such that $M\xi_2$ contains $(t-t')^{r_2}$, the highest power of $t-t'$ that is possible when multiples of ξ_1 are excluded, and so on until all the linearly independent solutions of $Mx = 0$ when $t = t'$ are exhausted.

It is convenient to complete the tale of n rows ξ by means of rows linearly independent of each other, and of the original rows ξ, so that for these Mx does not vanish when $t = t'$; that is, there are n indices $r_1, r_2, \ldots$ in descending order of magnitude, some at the end being zero. If, then, D is multiplied by the determinant of the n rows ξ, which does not vanish with $t-t'$, the product may be written as the determinant of the n rows $M\xi$, and its rows will contain the factor $t-t'$ to the powers $r_1, r_2, \ldots$. Hence D contains $t-t'$ to the power $r_1+r_2+\ldots$.

D does not contain any higher power, for if it did we could by a combination of the rows ξ replace at least one of them by another for which $M\xi$ would contain a higher power of $t-t'$, and this is excluded by the method of construction.

Also $t-t'$ is contained in all the first minors of D to the power $r_2+r_3+\ldots$, in the second minors to the power $r_3+r_4+\ldots$, and so on.

Similarly, the equations $Ny = 0$ are satisfied when $t = t'$ by rows η to which the same considerations apply, and by noting the degree of the factor $t-t'$ in D and its minors we are led to the conclusion that the series of indices $r_1, r_2, \ldots$ is the same for η as for ξ.

3. Since the determinant of the rows η does not vanish when $t = t'$, the set of equations $Mx = a$ is equivalent to the set

$$\eta Mx = \eta a,$$

which may also be written $$x N\eta = \eta a.$$

Here the left sides contain the factor $t-t'$ to the powers $r_1, r_2, \ldots$, and thus it is necessary that the same should be true of the right sides if x is to be limited when $t \to t'$. No further condition is necessary, since the determinant of the coefficients on the left does not vanish when the factors in question have been taken out.

Hence, if $\eta_1 a, \eta_2 a, \ldots$ contain $t-t'$ to the powers $r_1, r_2, \ldots$, the fractions with powers of $t-t'$ in the denominator are absent from the expressions for x, and, in general, the numerators of these fractions are linear in the coefficients of the powers

$$0, 1, \ldots, r_1-1 \text{ of } t-t' \text{ in } \eta_1 a,$$

$$0, 1, \ldots, r_2-1 \text{ of } t-t' \text{ in } \eta_2 a,$$

and so on.

The number of these coefficients, that is, of the conditions that x should be limited when $t \to t'$, is $r_1+r_2+\ldots$, the degree of the factor $t-t'$ in D, and all these conditions are independent, for (1) in any particular expression of the type ηa the coefficient of any power of $t-t'$ involves coefficients from a that have not occurred with lower powers, and these are multiplied by the absolute terms in η, while (2) the absolute terms in the rows $\eta_1, \eta_2, \ldots$ are linearly independent: the absolute terms here referred to are those of the expansions in powers of $t-t'$, and the polynomials a are supposed quite general and of as high a degree as may be necessary.

Application to the Theory of the Eliminant.

4. Let ϕ, ψ be two polynomials in t, u of degrees m, n in u, which need not be irreducible, and let x denote collectively the $m+n$ coefficients

of powers of u in two polynomials λ, μ in t, u of degrees $n-1$, $m-1$ in u, and a those in a polynomial ω in t, u of degree $m+n-1$ in u.

The identity $$\lambda\phi+\mu\psi \equiv \omega \qquad (1)$$

gives a system of $m+n$ equations for x of the type $Mx = a$ which has just been discussed.

The determinant D is the u-eliminant of ϕ, ψ in the form given by Sylvester's dialytic method, but with rows and columns interchanged.

The conclusion from the matrix theory in the present case is that if D contains a factor $(t-t')^q$, and not $(t-t')^{q+1}$, then the number of conditions arising therefrom to be satisfied by the coefficients of ω is exactly q if the identity is to be a possible one, and that otherwise the expressions for x derived from the equations

$$Mx = a$$

contain q rows of partial fractions, each having a power of $t-t'$ as denominator, and each independent of a save for a coefficient whose vanishing is one of the conditions referred to.

5. When $t = t'$, D vanishes and thus x can be determined so that

$$\lambda\phi+\mu\psi = 0.$$

Since λ, μ are of lower degrees than ψ, ϕ, this implies that ϕ, ψ have a common factor, say $u-u'$. If the identity (1) is to be possible, it is clear that ω must have the same factor when $t = t'$, but without making this assumption we may find the corresponding row of partial fractions as follows when the factor $t-t'$ is simple.

By separating these partial fractions from the rest we divide λ, μ, each into two parts, say

$$\frac{\alpha}{t-t'}+\lambda_1, \qquad \frac{\beta}{t-t'}+\mu_1,$$

where $t-t'$ does not occur as a denominator in λ_1, μ_1, and α, β are polynomials in u, of degrees $n-1$, $m-1$ respectively, with constant coefficients.

We have then

$$\omega(t-t') = \{\alpha+\lambda_1(t-t')\}\,\phi+\{\beta+\mu_1(t-t')\}\,\psi,$$

and when $t = t'$, $$\alpha\phi(t', u)+\beta\psi(t', u) = 0.$$

Thus* $\alpha = C\psi(t', u)/(u-u')$ and $\beta = -C\phi(t', u)/(u-u')$,

where C is a constant, and the expression for ω contains the term

$$\frac{C}{(t-t')(u-u')}[\phi(t, u)\psi(t', u)-\psi(t, u)\,\phi(t', u)],$$

or $$C\left[\frac{\phi(t, u)-\phi(t', u)}{t-t'}\,\frac{\psi(t', u)-\psi(t', u')}{u-u'} - \frac{\psi(t, u)-\psi(t', u)}{t-t'}\,\frac{\phi(t', u)-\phi(t', u')}{u-u'}\right].$$

The expression in brackets is only fractional in appearance: when the divisions are carried out and we put $t = t'$, $u = u'$, it becomes $CJ(t', u')$, where J is the Jacobian of ϕ, ψ. Thus

$$\omega(t', u') = CJ(t', u'),^{\dagger}$$

for $\lambda_1\phi+\mu_1\psi = 0$, when $t = t'$, $u = u'$, since ϕ, ψ vanish then.

If, then, all the factors of D are simple, we have

$$\omega(t, u) = \Sigma\frac{\omega(t', u')}{J(t', u')} \times \frac{\phi(t, u)\,\psi(t', u)-\psi(t, u)\,\phi(t', u)}{(t-t')(u-u')}+\eta\phi+\theta\psi, \quad (2)$$

where η, θ are polynomials in t and u.

6. The condition to be satisfied by ω in order that the fractions with $t-t'$ as denominator may be absent is

$$\omega(t', u') = 0,$$

that is, the minors of a column in D are proportional to

$$1,\ u',\ u'^2,\ \ldots,\ u'^{m+n-1},$$

when $t = t'$, in accordance with Sylvester's theory.

7. Suppose now that m, n, $m+n-1$ are the degrees of ϕ, ψ, ω in t *and* u, and that the axes and line at infinity have no special position with respect to the curves $\phi = 0$, $\psi = 0$, so that in particular no common point lies at infinity and no two common points are on the same line $t = \text{const.}$

* The common factor in $\phi(t', u)$, $\psi(t', u)$ can only be a simple one, such as $u-u'$, for otherwise α, β would contain two arbitrary coefficients, all the first minors of D would vanish when $t = t'$, and $t-t'$ would be a repeated factor in D.

† This method can be carried out when $t-t'$ is not repeated as a factor in D, and it follows that in that case $J(t', u')$ cannot vanish.

Then the terms of fractional form in the expression for ω that has been found (2) are really polynomials in t, u of degree $m+n-2$ only, so that the terms of degree $m+n-1$ in ω are identical with those in $\eta\phi+\theta\psi$. In particular, if ω is of the degree $m+n-2$ only, these terms must be absent in $\eta\phi+\theta\psi$, and therefore η, θ must be of degrees $n-2$, $m-2$, since the assemblages of highest terms in ϕ, ψ have no common factor.

Now compare the terms of degree $m+n-2$ on the two sides of (2). In all the fractions on the right these are the same, being independent of t', u', except for the factors $\dfrac{\omega(t', u')}{J(t', u')}$, whose sum therefore appears as the coefficient of a polynomial which for the moment we will call U. Distinguishing the terms of highest degree by a bar, we have then

$$\bar{\omega} = U\Sigma \frac{\omega(t', u')}{J(t', u')} + \bar{\eta}\bar{\phi} + \bar{\theta}\bar{\psi}.$$

There are $m+n-1$ independent coefficients in $\bar{\omega}$, and hence there must be the same number on the right, that is, the $n-1$ coefficients in $\bar{\eta}$, the $m-1$ in $\bar{\theta}$ and $\Sigma \dfrac{\omega(t', u')}{J(t', u')}$ must be linearly independent and must all vanish if all the coefficients in $\bar{\omega}$ do so. That is, if ω is of the degree $m+n-3$ only,

$$\Sigma \frac{\omega(t', u')}{J(t', u')} = 0,$$

which is *Jacobi's theorem* in its ordinary form.

8. Jacobi's theorem leads to a rule, which corresponds to Newton's method in the theory of equations, and which might be useful in practice, for finding the sum of the values of such a product as $t^\alpha u^\beta$ at the different intersections. To find $\Sigma\omega(t', u')$ express $\omega(t, u)J(t, u)$ in the form

$$\lambda\phi+\mu\psi+cJ+\chi,$$

where λ, μ, χ are polynomials, χ of degree $m+n-3$ at most, and c is a constant. This can be done by successive reduction of the highest terms, and then

$$\Sigma\omega(t', u') = mnc.$$

9. In discussing the extension of Jacobi's theorem to cases when the intersections are not all isolated, it will be convenient to speak of ω as reconcilable with ϕ, ψ, ... at (t', u') when ω can be expressed in the form $\lambda\phi+\mu\psi+\ldots$, where λ, μ are not necessarily finite polynomials, but may be infinite series of ascending powers and products of $t-t'$, $u-u'$, say *regular* series. The identity $\omega = \lambda\phi+\mu\psi+\ldots$ is to be considered as merely

formal, no question of convergency being material. A fundamental result, now to be proved, is that *the number of linearly independent polynomials not reconcilable with ϕ, ψ at a point (t', u'), where ϕ, ψ vanish, is exactly the degree to which the factor $t-t'$ enters in D.*

If ϕ, ψ do not both vanish at (t', u'), suppose ϕ not to; then, in the identity

$$\omega = \lambda\phi + \mu\psi,$$

μ may be taken arbitrarily and λ found by long division, term by term: thus any polynomial is reconcilable with ϕ, ψ at (t', u') in this case.

10. LEMMA I.—If D vanishes identically, then ϕ, ψ must have a common polynomial factor.

For now polynomials λ, μ in t, u exist of degrees in u lower than those of ψ, ϕ, such that

$$\lambda\phi + \mu\psi = 0$$

identically. Suppose any common factors of λ, μ to have been taken out. Then by the regular H.C.F. process applied to λ, μ as polynomials in u, other polynomials η, θ in t, u are found such that

$$\eta\lambda + \theta\mu = \kappa,$$

a polynomial in t only, and thus

$$\kappa\phi = \eta\lambda\phi + \theta\mu\phi = \mu(\theta\phi - \eta\psi),$$
$$\kappa\psi = -\lambda(\theta\phi - \eta\psi).$$

Since λ, μ have no common factor they cannot both vanish identically for any particular value of t, and therefore all the factors of κ must divide $\theta\phi - \eta\psi$, and since μ is of lower degree in u than ϕ,

$$\phi = \mu\chi, \qquad \psi = -\lambda\chi,$$

where χ is a polynomial in t, u, as was to be proved.* In the present discussion the case of ϕ, ψ having a common factor is excluded, so that D cannot vanish identically.

LEMMA II.—If there is a formal identity

$$\eta\phi + \xi\psi = 0,$$

where η, ξ are regular series in $t-t'$, $u-u'$, then these series must be expressible in the forms $\theta\psi$, $-\theta\phi$, where θ is a regular series in $t-t'$, $u-u'$.

* The proof is on the same lines as that of the similar theorem for polynomials in u with constant coefficients, which has been used in § 5.

For there is an identity

$$D = \rho\phi + \sigma\psi,$$

where ρ, σ are polynomials, and thus

$$D\eta = (\sigma\eta - \rho\xi)\psi, \qquad D\xi = -(\sigma\eta - \rho\xi)\phi.$$

Any power of $t-t'$ which is a factor in D must occur in each term of the series $\sigma\eta - \rho\xi$, since $t-t'$ cannot be a common factor of ϕ, ψ. The division by the other factor of D can be carried out in an ascending series.

Thus $\eta = \theta\psi$, $\xi = -\theta\phi$, where $\theta = (\sigma\eta - \rho\xi)/D$, which is a regular series in $t-t'$, $u-u'$.

11. Since any fraction whose denominator is a power of $t-t''$ can be expanded in positive powers of $t-t'$, when $t' \neq t''$, it follows that if the $t-t'$ fractions are absent from the expressions for λ, μ deduced from (1), then ω is reconcilable with ϕ, ψ at (t', u'). What we have to prove is the converse statement.

Suppose then that ω, a polynomial in t, u of degree $m+n-1$ at most, is reconcilable with ϕ, ψ at (t', u'), that is,

$$\omega = \alpha\phi + \beta\psi,$$

where α, β are regular series in $t-t'$, $u-u'$.

We have $D\omega = \rho\phi + \sigma\psi$, where ρ, σ are polynomials in t, u of degrees $n-1$, $m-1$ in u.

Thus $$\phi(\rho - D\alpha) + \psi(\sigma - D\beta) = 0,$$

and, by Lemma II, $$\rho = D\alpha + \theta\psi, \quad \sigma = D\beta - \theta\phi,$$

where θ is a regular series in $t-t'$, $u-u'$.

Here put $t = t'$, then $\rho(t', u) = \theta(t', u)\psi(t', u)$ for D vanishes, and therefore $\rho(t', u)$ contains the factor $u-u'$ to at least as high a degree as $\psi(t', u)$.

Also, when $t = t'$, $\rho\phi + \sigma\psi = 0$, and therefore $\rho(t', u)$ contains all the factors of $\psi(t', u)$ that are not factors of $\phi(t', u)$, that is, all the factors of $\psi(t', u)$ but $u-u'$, since the axes of t, u have no special position. Hence $\rho(t', u)$ contains all the factors of $\psi(t', u)$ and must vanish identically, since it is only of degree $n-1$. So also $\theta(t', u)$ and $\sigma(t', u) = 0$ identically.

The same argument applies to the terms in ρ, σ, θ of lowest degree in

$t-t'$, so long as that degree is less than q, the power of $t-t'$ which is a factor of D.

Now $\lambda = \rho/D$, $\mu = \sigma/D$, and ρ, σ both contain as high a power of $t-t'$ as D, so that the $t-t'$ series of partial fractions is absent in λ and μ when ω is reconcilable with ϕ, ψ at (t', u') and there is no other intersection for which $t = t'$. It has been supposed that the degree of ω is not greater than $m+n-1$; any polynomial of higher degree can have its degree reduced by subtracting an expression of the form $\eta\phi+\xi\psi$, where η, ξ are polynomials.

It is therefore proved that q, the degree to which the factor $t-t'$ occurs in D, is the same as the number of linearly independent polynomials which are not reconcilable with ϕ, ψ at (t', u').

Geometrically q is the number of intersections for which (t', u') counts.

If ω is reconcilable with ϕ, ψ at *every* intersection, then

$$\omega = \lambda\phi+\mu\psi,$$

where λ, μ are polynomials, all the partial fractions disappearing.

12. An important part is now played by Kronecker's interpolation function $\Delta(t, u, \tau, \upsilon)$ which may be defined as

$$\begin{vmatrix} \phi(t, u) & \phi(\tau, u) & \phi(\tau, \upsilon) \\ \psi(t, u) & \psi(\tau, u) & \psi(\tau, \upsilon) \\ 1 & 1 & 1 \end{vmatrix} \div (t-\tau)(u-\upsilon).*$$

Subtracting in turn the third and first columns from the second in the determinant, we see that

$$(t-t')\,\Delta(t, u, t', u') \quad \text{and} \quad (u-u')\,\Delta(t, u, t', u')$$

are everywhere reconcilable with ϕ, ψ, if

$$\phi(t', u') = \psi(t', u') = 0,$$

but an important fact now to be proved is that $\Delta(t, u, t', u')$ is not reconcilable with ϕ, ψ at (t', u').

For suppose that $\Delta(t, u, t', u') = \lambda\phi+\mu\psi,$

where λ, μ are regular series in $t-t'$, $u-u'$. Then

$$(t-t')(u-u')(\lambda\phi+\mu\psi) = \phi(t, u)\,\psi(t', u) - \phi(t', u)\,\psi(t, u),$$

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* For expressions of this nature compare A. L. Dixon, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, pp. 51-69, 476-492; Vol. 8, pp. 341 352.

and, by Lemma II,

$$\psi(t', u) = \quad (t-t')(u-u')\lambda+\theta\psi(t, u),$$

$$\phi(t', u) = -(t-t')(u-u')\mu+\theta\phi(t, u),$$

where θ is a regular series in $t-t'$, $u-u'$.

Putting $u = u'$ we have $\theta(t, u') = 0$, since $u-u'$ is not a factor of both $\phi(t, u)$ and $\psi(t, u)$. Thus there is no absolute term in the series θ.

Putting $t = t'$, we have $\theta(t', u) = 1$, so that θ must have an absolute term 1, which is absurd.

Hence $\Delta(t, u, t', u')$ is not reconcilable with ϕ, ψ at (t', u').

At any other point of intersection (t'', u''), $\Delta(t, u, t', u')$ is reconcilable with ϕ, ψ for $(t-t')\Delta$ and $(u-u')\Delta$ are so, and the reciprocal of $t-t'$ or $u-u'$ is a regular series in $t-t''$ or $u-u''$.

13. Now let U be such an operation that $U\omega$ denotes collectively the linear combinations of the coefficients in a polynomial ω which must vanish if ω is to be reconcilable with ϕ, ψ at (t', u'), the variables in ω, ϕ, ψ being t, u, and let Υ be the similar operation when the variables are τ, v, Υ_1 any particular linear combination of the members of the row Υ with constant coefficients.

We have

$$(t-\tau)\Delta(t, u, \tau, v) = \begin{vmatrix} \phi(t, u), & \dfrac{\phi(\tau, u)-\phi(\tau, v)}{u-v}, & \phi(\tau, v) \\ \psi(t, u), & \dfrac{\psi(\tau, u)-\psi(\tau, v)}{u-v}, & \psi(\tau, v) \\ 1, & 0, & 1 \end{vmatrix},$$

where the second column is fractional only in appearance, so that each term in the expanded determinant is destroyed by U or by Υ.

Thus $$U\Upsilon_1\{(t-\tau)\Delta(t, u, \tau, v)\} = 0,$$

or $$U\{(t-t')\Upsilon_1\Delta\} = U\Upsilon_1\{(\tau-t')\Delta\}.$$

From this it may be proved that $\Upsilon_1\Delta$ cannot be reconcilable with ϕ, ψ at (t', u'). If it were, then $(t-t')\Upsilon_1\Delta$ would be so, that is, if

$$U\Upsilon_1\Delta = 0,$$

then $$U\{(t-t')\Upsilon_1\Delta\} = 0,$$

and $$U\Upsilon_1\{(\tau-t')\Delta\} = 0.$$

Thus $\Upsilon_1\{(\tau-t')\Delta\}$ is reconcilable and similarly so is $\Upsilon_1\{(\upsilon-u')\Delta\}$, and, in general, $\Upsilon_1\{(\tau-t')^\alpha(\upsilon-u')^\beta\Delta\}$, where α, β are any positive integers.

Now, for at least one pair of indices α, β,

$$\Upsilon_1\{(\tau-t')^\alpha(\upsilon-u')^\beta\Delta\}$$

is a mere multiple of the absolute term in Δ as expanded in a series regular in $\tau-t'$, $\upsilon-u'$, and therefore this absolute term, namely $\Delta(t, u, t', u')$, is reconcilable with ϕ, ψ at (t', u'), which has been proved impossible.

Hence $\Upsilon_1\Delta(t, u, \tau, \upsilon)$ is not reconcilable with ϕ, ψ at (t', u'), whatever linear combination Υ_1 may be of the row Υ, but *the row*

$$\Upsilon\Delta(t, u, \tau, \upsilon)$$

furnishes the complete set of q polynomials in t, u that are not reconcilable with ϕ, ψ at (t', u').

A like argument shews that at (t'', u''), any other intersection, these functions are all reconcilable with ϕ, ψ.

14. Now any polynomial ω may be made reconcilable at (t', u') with ϕ and ψ by subtracting from it constant multiples of such a complete set as $\Upsilon\Delta(t, u, \tau, \upsilon)$ has been proved to be: let the row of multipliers be C, so that

$$\omega-C\Upsilon\Delta(t, u, \tau, \upsilon)$$

is reconcilable with ϕ, ψ at (t', u').

The row $\Upsilon\Delta(t, u, \tau, \upsilon)$ includes $\Delta(t, u, t', u')$ itself, a polynomial of degree $m+n-2$ in t, u: all the $q-1$ other members of the row may be taken as of lower degree than this, since any term in $\Delta(t, u, \tau, \upsilon)$ which actually contains τ or υ must be of degree $m+n-3$ at most in t, u.

By carrying out the same process at each point of intersection we arrive at a polynomial

$$\omega-\Sigma C\Upsilon\Delta(t, u, \tau, \upsilon),$$

in which Σ refers to the different intersections, and for this polynomial all the partial fractions in λ, μ (§ 4) are wanting, that is,

$$\omega = \Sigma C\Upsilon\Delta(t, u, \tau, \upsilon)+\eta\phi+\xi\phi,$$

where η, ξ are polynomials.

The reasoning by which Jacobi's theorem was proved (§ 7) still applies, since the terms of degree $m+n-2$ in $\Delta(t, u, t', u')$ are still independent of (t', u'). Thus, if ω is of the degree $m+n-3$ only, the sum of the co-

efficients of such functions as $\Delta(t, u, t', u')$, say of the *last** coefficients in ω, at the different intersections, is zero: this is *Jacobi's theorem in its generalized form*, and it clearly reduces to the ordinary form when all the intersections are simple.† Some care is needed in forming the last coefficient; an illustration is given below in Example III, § 25.

It follows from the theorem that, *if ω satisfies the conditions of reconcilability at all the intersections with the exception of the last at one intersection, this last coefficient must vanish also, and $\omega = \lambda\phi + \mu\psi$, where λ, μ are polynomials.*

The use of the phrase "last coefficient" accords with the theory of priority to be developed in § 15. It may be worth while to recapitulate the method of its formation at a point (t', u').

Form the expression $\Delta(t, u, \tau, v) - \Delta(t, u, t', u')$, and write down the conditions for this, as a function of τ, v, to be reconcilable with $\phi(\tau, v)$ and $\psi(\tau, v)$ at (t', u'). These conditions are that $q-1$ polynomials in t, u should vanish, say the polynomials

$$\omega_1, \omega_2, \ldots, \omega_{q-1}.$$

Any polynomial ω can be expressed in the form

$$a_1\omega_1 + a_2\omega_2 + \ldots + a_{q-1}\omega_{q-1} + a_q\omega_q + \varpi,$$

where ϖ is reconcilable with ϕ, ψ at (t', u') and

$$\omega_q = \Delta(t, u, t', u').$$

Then a_q is the "last coefficient" of ω at (t', u').

Priority among the Conditions of Reconcilability.

15. Among the conditions that ω should be reconcilable with ϕ, ψ at (t', u') a certain order is to be observed. The conditions include those for the reconcilability of $(t-t')^\alpha (u-u')^\beta \omega$, where α, β are any positive integers, and indeed of $\omega\chi$ where χ is any polynomial. When two condi-

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* From the above discussion of $\Upsilon\Delta$ (§ 13) it appears that $\Delta(t, u, \tau, v)$ can be divided into three parts—

(1) $\Delta(t, u, t', u')$ with coefficient unity.

(2) Other members of the row $\Upsilon\Delta(t, u, \tau, v)$, multiplied by powers or products of $\tau - t'$ and $v - u'$, which are not reconcilable at (t', u') with $\phi(\tau, v)$, $\psi(\tau, v)$.

(3) A part reconcilable at (t', u') with $\phi(\tau, v)$ and $\psi(\tau, v)$, and involving t, u also.

Hence in $\Delta(t, u, \tau, v)$ the "last" coefficient at (t', u') with respect to t, u is a function of τ, v differing from 1 by an expression reconcilable at (t', u') with ϕ and ψ.

† In the other extreme case when $\phi = t^m$, $\psi = u^n$, so that

$$t' = u' = 0 \quad \text{and} \quad \Delta(t, u, t', u') = t^{m-1}u^{n-1},$$

the theorem states that when ω is of degree $m+n-3$ it cannot contain a term in $t^{m-1}u^{n-1}$, which is evident.

tions (1), (2) are so related that ω must satisfy (1) as a condition necessary that (2) may be satisfied by $\omega\chi$, where χ is an arbitrary polynomial, then (1) may be called *earlier* than (2), and (2) *later* than (1). The vanishing of ω itself, that is, of the absolute term in the expansion at (t', u'), is the earliest condition to be satisfied at that point, for α, β may always be so chosen that this is the only coefficient affected by the condition (2) as applied to $(t-t')^\alpha (u-u')^\beta \omega$.

The whole system of conditions at (t', u') may be called a complete one in the sense that if satisfied by ω it is also satisfied by $\omega\chi$, where χ is any polynomial. It includes what may be called minor complete systems which have the property of completeness in the same sense but form only part of the whole system. Any minor complete system must contain all conditions which are earlier than any one of its members, and it does not cease to be complete if it is reduced by the cancelling of a condition when it contains no later condition than the one cancelled. Thus any minor complete system may be taken to belong to a sequence of complete systems, in which each contains the system before it and an additional condition, the first of the sequence consisting of one condition and the last of q.

16. Again, if $(t-t')\omega$ and $(u-u')\omega$ are both reconcilable at (t', u'), then ω can be made reconcilable there by subtracting a constant multiple of $\Delta(t, u, t', u')$.

For let $E(t)$ be D divided by the highest power of $t-t'$ that is a factor in D. Then, since the $t-t'$ fractions are absent from the expressions of § 4 for $(t-t')\omega$ and $(u-u')\omega$, we have

$$E(t)(t-t')\omega = \alpha\phi+\beta\psi,$$

$$E(t)(u-u')\omega = \gamma\phi+\delta\psi,$$

where α, β, γ, δ are polynomials.

Thus $$\phi[\alpha(u-u')-\gamma(t-t')] = \psi[\delta(t-t')-\beta(u-u')]$$

and

$$\alpha(u-u')-\gamma(t-t') = \rho\psi,$$

$$\delta(t-t')-\beta(u-u') = \rho\phi,$$

where ρ is a polynomial.

Hence

$$\alpha(u-u')-\rho\{\psi(t', u)-\psi(t', u')\} = \gamma(t-t')+\rho\{\psi(t, u)-\psi(t', u)\}$$

$$= \sigma(t-t')(u-u'), \text{ say},$$

where σ is a polynomial; for the first member of the equality has the factor $u-u'$, and the second has $t-t'$.

Similarly,

$$\beta(u-u')+\rho\{\phi(t', u)-\phi(t', u')\} = \delta(t-t')-\rho\{\phi(t, u)-\phi(t', u)\}$$
$$= \theta(t-t')(u-u'),$$

where θ is also a polynomial.

Hence

$$(t-t')(u-u')(\sigma\phi+\theta\psi)$$
$$= (t-t')(\gamma\phi+\delta\psi)+\rho\phi(t,u)\{\psi(t,u)-\psi(t',u)\}-\rho\psi(t,u)\{\phi(t,u)-\phi(t',u)\}$$
$$= E(t)(t-t')(u-u')\omega-\rho(t-t')(u-u')\Delta(t, u, t', u'),$$

or
$$E(t)\omega = \rho\Delta(t, u, t', u')+\sigma\phi+\theta\psi,$$

which proves the theorem since ρ, σ, θ are polynomials, and $E(t')$ is not zero, for the reciprocal of $E(t)$ is a regular series in $t-t'$, and the only irreconcilable term on the right is that which contains the absolute term of ρ.

17. Let $\omega_1, \omega_2, \ldots, \omega_q$ be a complete set of the polynomials not reconcilable with ϕ, ψ at (t', u'), so that any polynomial ω whatever can be made reconcilable by subtracting such an expression as

$$a_1\omega_1+a_2\omega_2+\ldots+a_q\omega_q,$$

where $a_1, a_2, \ldots, a_q$ are constants. The whole system of conditions for ω to be reconcilable is

$$a_1 = a_2 = \ldots = a_q = 0,$$

and we shall suppose $\omega_1, \omega_2, \ldots, \omega_q$, such that the minor system

$$a_1 = a_2 = \ldots = a_p = 0$$

is complete for all values of p less than q.

18. It follows that ω_q must be substantially identical with $\Delta(t, u, t', u')$. For the conditions $a_1 = a_2 = \ldots = a_{q-1} = 0$ form a minor complete system restricting ω to the form $a_q\omega_q$, and therefore $\omega_q(t-t')$ and $\omega_q(u-u')$ satisfy the same system, and constants a, b exist, such that $\omega_q(t-t')+a\omega_q$ and $\omega_q(u-u')+b\omega_q$ are reconcilable. If a, b are not both zero, it follows that ω_q is itself reconcilable, which is contrary to hypothesis. Hence $\omega_q(t-t')$ and $\omega_q(u-u')$ are both reconcilable, that

is, ω_q only differs by reconcilable expressions from a constant multiple of $\Delta(t, u, t', u')$ according to the theorem just now proved (§ 16).

There is then no loss of generality in putting

$$\omega_q = \Delta(t, u, t', u').$$

By subtracting suitable multiples of ω_q from $\omega_1, \omega_2, \ldots, \omega_{q-1}$, we secure that the "last" coefficient in each of these is zero, and then the "last" coefficient in ω will be a_q: this step does not affect $a_1, a_2, \ldots, a_{q-1}$.

19. Putting $$a_1\omega_1+a_2\omega_2+\ldots+a_q\omega_q = \omega$$

and $$b_1\omega_1+b_2\omega_2+\ldots+b_q\omega_q = \chi,$$

we have $\omega\chi$ made reconcilable by subtracting a new expression of the same form, say

$$c_1\omega_1+c_2\omega_2+\ldots+c_q\omega_q,$$

and $c_1, c_2, \ldots, c_q$ are all linear in the a's and also in the b's. Let the "last" coefficient in $\omega\chi$, that is

$$\begin{aligned} c_q &= a_1B_1+a_2B_2+\ldots+a_qB_q \\ &= b_1A_1+b_2A_2+\ldots+b_qA_q, \end{aligned}$$

where $A_1, \ldots, A_q$ are linear and homogeneous in $a_1, \ldots, a_q$ with constant coefficients, and $B_1, \ldots, B_q$ are the same functions of $b_1, \ldots, b_q$.

The determinant of the coefficients in these functions, that is,

$$\frac{\partial(A_1, \ldots, A_q)}{\partial(a_1, \ldots, a_q)}$$

does not vanish.

For to any irreconcilable polynomial ω there corresponds an index p, such that $\omega\{a(t-t')+b(u-u')\}^p$ is reconcilable for all values of the constants a, b, while $\omega\{a(t-t')+b(u-u')\}^{p-1}$ is not. That is, for certain indices α, β, $\omega(t-t')^\alpha(u-u')^\beta$ is not reconcilable while $\omega(t-t')^{\alpha+1}(u-u')^\beta$ and $\omega(t-t')^\alpha(u-u')^{\beta+1}$ are so. Hence, if $b_1, b_2, \ldots, b_q$ are so chosen that $\chi-(t-t')^\alpha(u-u')^\beta$ is reconcilable, c_q does not vanish: that is, $A_1, \ldots, A_q$ cannot all vanish unless $a_1, \ldots, a_q$ do so, which is the thing to be proved.

Another way of stating this result is to say that the conditions for ω to be reconcilable with ϕ, ψ at (t', u') are the conditions that c_q, the "last" coefficient in $\omega\chi$, may vanish for an arbitrary polynomial χ.

20. Now let ω be restricted, but only by the minor complete system

$$a_1 = a_2 = \ldots = a_p = 0,$$

and consider the conditions to be satisfied by χ if $\omega\chi$ is to be reconcilable. Since

$$c_q = a_{p+1}B_{p+1} + \ldots + a_qB_q,$$

it is necessary that $B_{p+1} = B_{p+2} = \ldots = B_q = 0.$

The other conditions are all of the form $c_q' = 0$, where c_q' is the coefficient of ω_q in $(t-t')^\alpha (u-u')^\beta \omega\chi$, and may therefore be written

$$a_1'B_1 + a_2'B_2 + \ldots + a_q'B_q = 0,$$

where a_1', a_2', ..., a_q' are for $(t-t')^\alpha (u-u')^\beta \omega$ what a_1, a_2, ..., a_q are for ω. Now, by hypothesis, a_1', a_2', ..., a_p' vanish, because a_1, a_2, ..., a_p vanish, and thus the conditions

$$B_{p+1} = B_{p+2} = \ldots = B_q = 0$$

are sufficient as well as necessary. These conditions form a minor complete system, for clearly if $\omega\chi$ is reconcilable, so is $\omega.\chi(t-t')^\alpha (u-u')^\beta$. The two systems

$$a_1 = a_2 = \ldots = a_p = 0$$

and

$$A_{p+1} = A_{p+2} = \ldots = A_q = 0,$$

will be called *associated* systems.

21. The extended residue theorem as applied to $\omega\chi$ is that the sum of all such coefficients as c_q with respect to the different intersections is zero. It has now been found that c_q is the sum of q terms, each of which is a product of the form aB, the first factor being a linear function of the coefficients of ω, and the second a linear function of the coefficients of χ; when $q > 1$, the reduction of c_q to this form is not always unique.

We may, by a convention, speak of the curves $\phi = 0$, $\psi = 0$ as having q coincident or consecutive intersections at (t', u'), and we may distribute these, assigning p to ω and $q-p$ to χ, the meaning being that ω is to satisfy a minor complete system of p conditions, and χ the associated complete system of $q-p$ conditions. The total number of intersections is the degree of the eliminant, mn, and this is also the total number of terms of form aB in the residue theorem.

Thus, if χ passes through all the points assigned to it, and ω through all but one of those assigned to it, all these terms vanish but one: hence that term must also vanish, and ω must pass through the excepted point, unless χ passes through an additional point in the conventional sense.

Thus we have the modified *theorem of Cayley and Bacharach :—If ω is a polynomial of degree r, less than $m+n-2$, restricted by "complete" systems of conditions at the different points where ϕ, ψ vanish, the whole*

number of conditions being $mn-\frac{1}{2}(m+n-r-1)(m+n-r-2)$, *then* ω *is of the form* $\lambda\phi+\mu\psi$, *where* λ, μ *are polynomials, unless there is a polynomial* χ, *of degree* $m+n-r-3$, *which satisfies all the*

$$\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

conditions which are included in the "associated" complete systems. (Compare *Camb. Phil. Proc.*, Vol. x·, p. 474.)

22. *Modified Theorem of Riemann and Roch :—Of the* mn *intersections of* ϕ, ψ *in the conventional sense take a number* h, *that is, take minor, or other, complete systems* (A) *of conditions at their common points, amounting to* h *conditions in all. Let* B *denote the systems associated with* A, *amounting to* $mn-h$ *in all.*

Let k *be the number of arbitrary coefficients in an* r*-ic satisfying* B, *where* $r < m+n-2$, *and* l *the number of arbitrary coefficients in an* $(m+n-r-3)$*-ic satisfying* A. *Then shall*

$$k = \tfrac{1}{2}(r+1)(r+2)-mn+h+l.$$

For, if ω, χ are of degrees r, $m+n-r-3$, and we denote by $A\chi$, $B\omega$ the rows of linear functions in their coefficients that have to vanish on account of the conditions specified, Jacobi's theorem applied to $\omega\chi$ gives a result that may be written

$$A\chi . B'\omega + A'\chi . B\omega = 0.$$

Suppose then that $B\omega = 0$, and this equation includes k relations involving $A\chi$, but of these some are illusory, namely, those in which $B'\omega = 0$. The number of these, if $r > m-3$, and also $> n-3$, is

$$\tfrac{1}{2}(r-m+1)(r-m+2)+\tfrac{1}{2}(r-n+1)(r-n+2),$$

for ω may be $\sigma\psi$ or $\rho\phi$, where σ, ρ are of degrees $r-n$, $r-m$.

Hence the conditions $A\chi = 0$ are in number only

$$h-k+\tfrac{1}{2}(r-m+1)(r-m+2)+\tfrac{1}{2}(r-n+1)(r-n+2),$$

and $l \geqslant \tfrac{1}{2}(m+n-r-1)(m+n-r-2)-h+k-\tfrac{1}{2}(r-m+1)(r-m+2)$
$$-\tfrac{1}{2}(r-n+1)(r-n+2),$$

or $$l \geqslant mn-\tfrac{1}{2}(r+1)(r+2)+k-h. \qquad (3)$$

Again, suppose $A\chi = 0$, and it follows that

$$A'\chi B\omega = 0,$$

so that the expressions $B\omega$ are connected by l linear relations of which none can be illusory, since χ is of lower degree than either ϕ or ψ, and cannot therefore satisfy all the relations $A\chi = A'\chi = 0$.

Thus k, the number of arbitrary coefficients in ω when $B\omega = 0$, is

$$\geqslant \tfrac{1}{2}(r+1)(r+2) - mn + h + l. \qquad (4)$$

By comparing (3) and (4) we have the result stated, that is, the modified and extended theorem of Riemann and Roch, which includes, substantially, that of Cayley and Bacharach.

23. The following is a statement of the *theorem* which appears to be the *converse of Jacobi's*.

Take a number of pairs of polynomials in t, u, say ϕ_1 and ψ_1 both vanishing at (t_1, u_1), ϕ_2 and ψ_2 both vanishing at (t_2, u_2), and so on. Let B denote the conditions that a polynomial may be reconcilable with ϕ_1 and ψ_1 at (t_1, u_1) with ϕ_2 and ψ_2 at (t_2, u_2), and so on, and let these conditions be altogether mn in number. Any given polynomial will have a "last" coefficient at each of the points (t_1, u_1), (t_2, u_2), ... with respect to the pair of polynomials thus associated with that point. If the "last" coefficients at the different points in all polynomials of degree $m+n-3$ satisfy a certain homogeneous linear relation with constant coefficients, none of which is zero, then the number of m-ics and $(n-3)$-ics which satisfy B is together at least

$$\tfrac{1}{2}(m-n+1)(m-n+2)+1,$$

and the number of n-ics and $(m-3)$-ics which satisfy B is at least

$$\tfrac{1}{2}(n-m+1)(n-m+2)+1,$$

For the proof see *Camb. Phil. Proc.*, Vol. xv, pp. 479, 480.

The Jacobian.

24. When (t, u) and (τ, v) coincide, $\Delta(t, u, \tau, v)$ becomes the Jacobian $\partial(\phi, \psi)/\partial(t, u)$. An interesting property of this Jacobian is that the "last" coefficient in it at (t', u') is q: this may be proved as follows.

Since

$$(t-\tau)\,\Delta(t, u, \tau, v) = \lambda\{\phi(t, u) - \phi(\tau, v)\} + \mu\{\psi(t, u) - \psi(\tau, v)\},$$

where λ, μ are polynomials in t, u, τ, v, it follows that the "last" coefficient with respect to t, u in $(t-t')^{\alpha-1}(u-u')^{\beta}(t-\tau)\,\Delta(t, u, \tau, v)$ is the same as in $-(t-t')^{\alpha-1}(u-u')^{\beta}\{\lambda\phi(\tau, v) + \mu\psi(\tau, v)\}$, and therefore, with with respect to τ, v, is reconcilable with ϕ, ψ.

Thus, as a function of τ, v, the last coefficient with respect to t, u in

$$(t-t')^{\alpha}(u-u')^{\beta}\,\Delta(t, u, \tau, v)$$

differs only by a reconcilable polynomial from that in

$$(\tau-t')(t-t')^{\alpha-1}(u-u')^{\beta}\Delta(t, u, \tau, v),$$

or similarly, that in

$$(v-u')(t-t')^{\alpha}(u-u')^{\beta-1}\Delta(t, u, \tau, v),$$

or, by successive steps, that in

$$(\tau-t')^{\alpha}(v-u')^{\beta}\Delta(t, u, \tau, v),$$

which is $(\tau-t')^{\alpha}(v-u')^{\beta}$ (§ 14, note).

Since any polynomial ω is the sum of such terms as $A(t-t')^{\alpha}(u-u')^{\beta}$, it follows that the last coefficient at (t', u') with respect to t, u in $\omega(t, u)\Delta(t, u, \tau, v)$ differs from $\omega(\tau, v)$ by a polynomial reconcilable with ϕ, ψ at (t', u').

Thus, if we write

$$\Delta(t, u, \tau, v) \equiv \omega_1\bar{\Omega}_1+\omega_2\bar{\Omega}_2+\ldots+\omega_q\bar{\Omega}_q \pmod{\phi, \psi},$$

where $\omega_1, \omega_2, \ldots, \omega_q$ are the polynomials of § 18 in t, u, and the bar indicates that τ, v are to be substituted for t, u in polynomials $\Omega_1, \ldots, \Omega_q$, we have

$$\bar{\omega} \equiv \text{last coefficient in } (a_1\omega_1+\ldots+a_q\omega_q)(\omega_1\bar{\Omega}_1+\ldots+\omega_q\bar{\Omega}_q) \pmod{\bar{\phi}, \bar{\psi}}$$
$$\equiv A_1\bar{\Omega}_1+A_2\bar{\Omega}_2+\ldots+A_q\bar{\Omega}_q \text{ (§ 19)}.$$

Thus the substitutions Ω for ω and A for a are contragredient.

Now the last coefficient in $\omega\chi$ is

$$a_1B_1+\ldots+a_qB_q,$$

and has the value 1, if $a_p = B_p = 1,$

$$a_r = B_r = 0 \quad (r \neq p),$$

that is, if $\omega = \omega_p, \quad \chi = \Omega_p.$

Any other coefficient in $\omega\chi$ is of the form

$$a_1'B_1+\ldots+a_q'B_q,$$

where $a_1', \ldots, a_p'$ must vanish, since $a_1, \ldots, a_{p-1}$ vanish. Thus

$$\omega_p\Omega_p-\Delta(t, u, t', u')$$

is reconcilable with ϕ, ψ at (t', u').

Now, at (t', u'), $\Delta(t, u, t, u) \equiv \omega_1\Omega_1+\ldots+\omega_q\Omega_q \pmod{\phi, \psi}$

$$\equiv q\Delta(t, u, t', u'),$$

that is, the "last" coefficient in the Jacobian at any common point is q, the number of intersections for which that point counts.

The sum of all the numbers such as q is mn, and in accordance with this the sum of the highest terms in the Jacobian, that is, those of degree $m+n-2$, differs from mn times the sum of the highest terms in $\Delta(t, u, \tau, v)$ by a polynomial which can be expressed in the form

$$\lambda\bar{\phi}+\mu\bar{\psi},$$

where $\bar{\phi}$ is the sum of the highest terms in ϕ,

and $\bar{\psi}$,, ,, ψ,

and λ, μ are polynomials.

Illustrations.

25. A few special illustrations of the general theory may be given.

I. Suppose the curves $\phi = 0$, $\psi = 0$ to have contact, of order α only, at the origin, but no other special feature, that is, suppose ϕ, ψ to be reconcilable at the origin with

$$u-p_1 t-p_2 t^2-\ldots-p_\alpha t^\alpha,\ t^{\alpha+1},$$

where $p_1, p_2, \ldots, p_\alpha$ are constants.

Then it is easy to see that there are $\alpha+1$ polynomials not reconcilable at the origin with ϕ, ψ, and that these may be taken to be

$$1,\ t,\ t^2,\ \ldots,\ t^\alpha,$$

or, more generally, to be expressions beginning with these terms when arranged in ascending powers.

If
$$\omega = a_0+a_1 t+a_2 t^2+\ldots+a_\alpha t^\alpha,$$
$$\chi = b_0+b_1 t+ \ldots\ldots\ldots +b_\alpha t^\alpha,$$

then the coefficient of t^α in $\omega\chi$ is

$$a_0 b_\alpha+a_1 b_{\alpha-1}+\ldots+a_\alpha b_0,$$

and the "last" coefficient is a constant multiple of this, at least when all the earlier coefficients vanish.

The statement that ω passes through β consecutive points at the origin on the curve ϕ or ψ means that

$$a_0,\ a_1,\ \ldots,\ a_{\beta-1},$$

all vanish, and the "associated" system of conditions for χ is that $b_0, b_1, \ldots, b_{\alpha-\beta}$ all vanish, or that χ pass through $\alpha-\beta+1$ consecutive points of ϕ or ψ at the origin.

II. Let the curves have a common node at the origin, and let the axes be so chosen that

$$\phi \equiv p_1 t^2 + q_1 u^2 + \ldots,$$

$$\psi \equiv p_2 t^2 + q_2 u^2 + \ldots,$$

the term in tu being absent, and

$$p_1 q_2 - p_2 q_1 \neq 0.$$

Then four irreconcilable polynomials at the origin are 1, t, u, tu, and so are

$$1 + at + bu + ctu, \quad a't + b'u + c'tu, \quad a''t + b''u + c''tu, \quad tu,$$

unless $$a'b'' = a''b'.$$

Either of these sets may be taken for ω_1, ω_2, ω_3, ω_4 in § 17, but the further condition of § 18 is not generally satisfied by them.

Now, if

$$\omega = a_1 + a_2 t + a_3 u + a_4 tu,$$

$$\chi = b_1 + b_2 t + b_3 u + b_4 tu,$$

the coefficient of tu in $\omega\chi$ is $a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1$, and this is proportional to the last coefficient if the three earlier ones vanish.

If, then, $\omega\chi$ is to be reconcilable with ϕ and ψ at the origin,

(1) when ω must pass through the origin, but is not otherwise restricted, χ must have a double point at the origin;

(2) if ω touches $u = \mu t$ at the origin, χ must touch $u = -\mu t$ at the origin;

(3) if a double point at the origin is assigned to ω, then χ must pass through the origin.

Hence many theorems may be deduced such as the following:—If two given quartics have a common node at the origin, no quartic can pass through the origin, touching a given line there, and also pass through eleven of the other twelve common points without having a node at the origin and passing through the twelfth other common point, unless the given tangent at the origin, and the line from the origin to the twelfth point, are conjugate in the involution to which belong the tangents at the origin to the given quartics.

III. Take $$\phi = t^2 + au^3, \quad \psi = u^2 + bt^3,$$

$$\omega = c + et + fu + gt^2 + htu + ku^2 + lt^3 + mt^2 u + ntu^2 + pu^3,$$

and let us verify Jacobi's theorem.

We have

$$\Delta(t, u, \tau, v) = (t+\tau)(u+v) - ab(t^2+t\tau+\tau^2)(u^2+uv+v^2),$$

and in order to find the "last" coefficient in ω at (0, 0) we must pick out the coefficients of $1, \tau, v, \tau v$ in $\Delta(t, u, \tau, v)$; they are

$$tu - abt^2u^2, \quad u - abtu^2, \quad t - abt^2u, \quad 1 - abtu,$$

and on account of the second term in the fourth of these, the last coefficient in ω is not h but $h+abc$: the other second terms are reconcilable with ϕ, ψ at the origin.

The unreconcilable part of ω may, in fact, be written

$$c(1-abtu)+e.t+f.u+(h+abc)tu.$$

The intersections other than (0, 0) are given by

$$t^5 = -a^{-2}b^{-3}, \quad tu = a^{-1}b^{-1},$$

and

$$J = 4tu - 9abt^2u^2,$$

which has the value $-5a^{-1}b^{-1}$ at each of the five points.

The sum of the values of ω at the five points is

$$5c+5ha^{-1}b^{-1},$$

and the sum of the values of ω/J is therefore $-h-abc$, that is, equal and opposite to the last coefficient at the origin, as it should be.

This example illustrates the method of finding the last coefficient, and shews that it is not safe to neglect all but the lowest terms in ϕ, ψ for the purpose.

IV. For the ordinary theory of adjoint polynomials of a curve whose only singularities are ordinary multiple points, take ϕ to have a multiple point of order i at the origin, and ψ to have one of order $i-1$ at the origin, but not to touch any branch of ϕ there. Let $\phi_0, \psi_0, \ldots$ denote the terms of lowest degree in $\phi, \psi, \ldots$.

Suppose ω to be restricted by the condition of having a multiple point of order $i-1$ at the origin: to find the associated system of conditions, suppose $\omega\chi$ to be reconcilable at the origin with ϕ and ψ, and χ_0 to be of degree j. Since ω_0 may be $t^{i-1}, t^{i-2}u, \ldots$, we must have

$$t^{i-1}\chi_0 = \lambda\phi_0+\mu\psi_0,$$

$$t^{i-2}u\chi_0 = \rho\phi_0+\sigma\psi_0, \ldots,$$

where $\lambda, \rho, \mu, \sigma$ are homogeneous polynomials of degrees $j-1, j-1, j, j$. Thus

$$(u\lambda - t\rho)\phi_0 + (u\mu - t\sigma)\psi_0 = 0,$$

and $u\lambda = t\rho$, $u\mu = t\sigma$, if $j < i-1$. In this way it follows that the coefficients of ϕ_0, such as λ, ρ, in the expressions for

$$t^{i-1}\chi_0,\ t^{i-2}u\chi_0,\ t^{i-3}u^2\chi_0,\ \ldots,\ u^{i-1}\chi_0,$$

form a geometrical progression, and are, in fact, proportional to

$$t^{i-1},\ t^{i-2}u,\ t^{i-3}u^2,\ \ldots,\ u^{i-1},$$

and the same is true of μ, σ, ... the coefficients of ψ_0. This is impossible if $j < i-1$, and hence χ_0 must be of degree $i-1$, and χ must also have a multiple point of order $i-1$ at the origin, the associated system of conditions being the same as the original.

Thus, if ϕ has only ordinary multiple points, and ψ is adjoint to ϕ, that is, has an $(i-1)$-ple point wherever ϕ has an i-ple point, and if k is the number of arbitrary coefficients in a polynomial ω of degree r ($< m+n-2$ but $> m-3$ or $n-3$), which is also adjoint to ϕ and passes through all but h of the other intersections of ϕ and ψ, and l is the number of arbitrary coefficients in a polynomial χ of degree $m+n-r-3$, which is also adjoint to ϕ and passes through the h excepted intersections, then

$$k = \tfrac{1}{2}(r+1)(r+2) - mn + h + l.$$

ON DIFFERENTIAL OPERATORS WHICH GENERATE ALL SEMINVARIANTS, AND ALL TERNARY COVARIANT SOURCES

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1. It is known—see, for instance, my *Algebra of Quantics*, §§ 193, 194—that, if

$$\Omega \equiv a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \ldots,$$

and

$$\vartheta \equiv a_1 \frac{\partial}{\partial a_0} + a_2 \frac{\partial}{\partial a_1} + a_3 \frac{\partial}{\partial a_2} + \ldots,$$

then any gradient G, *i.e.*, any rational integral homogeneous isobaric function, in the unending series of letters $a_0, a_1, a_2, \ldots$ is an ΩF, where F is another gradient. In fact, if i is the degree of G,

$$G = \Omega \left\{ \frac{1}{i} \vartheta - \frac{1}{i^2} \frac{\vartheta^2\Omega}{1.2} + \frac{1}{i^3} \frac{\vartheta^3\Omega^2}{1.2.3} - \ldots \right\} G. \tag{1}$$

The F thus obtained as an $\Omega^{-1}G$ may well involve letters more advanced in the sequence $a_0, a_1, a_2, \ldots$ than G itself does.

An F such that $G = \Omega F$, with a given G, is not as a rule unique; and another operator for the derivation of one from G will first be constructed which possesses two advantages, (1) that of involving less complex differential operations, and (2) that of securing an F which does not involve to a higher degree than the first any letter more advanced than those present in G.

Again, it is known (*loc. cit.*) that, with any G,

$$\left\{ 1 - \frac{1}{i} \vartheta\Omega + \frac{1}{i^2} \frac{\vartheta^2\Omega^2}{1.2} - \frac{1}{i^3} \frac{\vartheta^3\Omega^3}{1.2.3} + \ldots \right\} G \tag{2}$$

is annihilated by Ω, so that (if not zero) it is a seminvariant of the binary quantic

$$(a_0, a_1, a_2, \ldots, a_{p+r})(x, y)^{p+r},$$

where a_p is the most advanced letter in the derived gradient, and r is any one of 0, 1, 2, 3, ..., and also one of the infinite quantic

$$a_0+a_1z+a_2\frac{z^2}{1.2}+a_3\frac{z^3}{1.2.3}+\ldots.$$

Moreover all seminvariants are thus obtained.

Another effective operator for deriving all seminvariants from gradients will be exhibited, having the two advantages mentioned earlier.

After this we shall see that an operator can be constructed for the generation of all *ternary* covariant sources. Hitherto I believe such operators have been looked for in vain.

A. *The Seminvariant Problem.*

2. Taking any number n, denote by ψ_n the operator

$$\psi_n \equiv \frac{\partial}{\partial a_n}-\frac{1}{n}\Omega\frac{\partial}{\partial a_{n-1}}+\frac{1}{n(n-1)}\Omega^2\frac{\partial}{\partial a_{n-2}}-\ldots+(-1)^n\frac{1}{n!}\Omega^n\frac{\partial}{\partial a_0}. \quad (3)$$

Operation with ψ_n on a gradient G diminishes weight by n. Consequently if w is the weight of G,

$$\psi_{w+1}G = 0,$$

seeing that there are no gradients of weight -1. A particular gradient G may be annihilated by a lower ψ_n than ψ_{w+1}. In such a case we may replace w in what immediately follows by $n-1$. From the form of ψ_n, we have

$$\psi_{w+1}G+\frac{1}{w+1}\Omega\psi_wG = \frac{\partial}{\partial a_{w+1}}G, \text{ i.e., } \frac{1}{w+1}\Omega\psi_wG = 0,$$

$$\psi_wG + \frac{1}{w}\Omega\psi_{w-1}G = \frac{\partial}{\partial a_w}G,$$

$$\psi_{w-1}G+\frac{1}{w-1}\Omega\psi_{w-2}G = \frac{\partial}{\partial a_{w-1}}G,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\psi_1G + \Omega\psi_0G = \frac{\partial}{\partial a_1}G,$$

$$\psi_0G = \frac{\partial}{\partial a_0}G.$$

Hence, multiplying by a_{w+1}, a_w, a_{w-1}, ..., a_1, a_0 respectively, and adding,

$$\Omega\left\{\frac{1}{w+1}a_{w+1}\psi_w+\frac{1}{w}a_w\psi_{w-1}+\ldots+a_1\psi_0\right\}=iG; \tag{4}$$

and this is a relation $G=\Omega F$ such as desired. Letters more advanced than those occurring in G present themselves to the first degree, if at all, in F, because $\psi_w G$, $\psi_{w-1}G$, ... do not involve the more advanced letters.

3. The deduction of a generator of seminvariants is immediate. Take any gradient F of type w, i, so that ΩF, if not zero, is of type $w-1$, i, and contains no more advanced letter than F does. In the last result put ΩF for G, and $w-1$ for w. It at once follows that

$$\left\{1-\frac{1}{i}\left(\frac{1}{w}a_w\psi_{w-1}+\frac{1}{w-1}a_{w-1}\psi_{w-2}+\ldots+a_1\psi_0\right)\Omega\right\}F \tag{5}$$

is annihilated by Ω, so that it is a seminvariant or zero.

A much simpler form can be given to the generator here obtained as follows. Using the fact that, for every n,

$$\Omega\frac{\partial}{\partial a_n}-\frac{\partial}{\partial a_n}\Omega=-(n+1)\frac{\partial}{\partial a_{n+1}},$$

we see that

$$\begin{aligned}\psi_n\Omega=&\left\{(n+1)\frac{\partial}{\partial a_{n+1}}+\Omega\frac{\partial}{\partial a_n}\right\}-\frac{1}{n}\Omega\left\{n\frac{\partial}{\partial a_n}+\Omega\frac{\partial}{\partial a_{n-1}}\right\}\\&+\frac{1}{n(n-1)}\Omega^2\left\{(n-1)\frac{\partial}{\partial a_{n-1}}+\Omega\frac{\partial}{\partial a_{n-2}}\right\}\\&-\ldots+(-1)^n\frac{1}{n!}\Omega^n\left\{\frac{\partial}{\partial a_1}+\Omega\frac{\partial}{\partial a_0}\right\}\\=&(n+1)\frac{\partial}{\partial a_{n+1}}+(-1)^n\frac{1}{n!}\Omega^{n+1}\frac{\partial}{\partial a_0}.\end{aligned}$$

Hence, by another use of Euler's theorem as to homogeneous functions,

$$\begin{aligned}&\left(\frac{1}{w}a_w\psi_{w-1}+\frac{1}{w-1}a_{w-1}\psi_{w-2}+\ldots+a_1\psi_0\right)\Omega\\&\quad=i-a_0\frac{\partial}{\partial a_0}+a_1\Omega\frac{\partial}{\partial a_0}-\frac{a_2}{1.2}\Omega^2\frac{\partial}{\partial a_0}+\ldots+(-1)^{w-1}\frac{1}{w!}a_w\Omega^w\frac{\partial}{\partial a_0}.\end{aligned}$$

Thus the seminvariant, or zero, derived from F, may be written

$$\left\{a_0-a_1\Omega+\frac{a_2}{1.2}\Omega^2-\ldots\right\}\frac{\partial}{\partial a_0}F, \tag{6}$$

where the operator may be regarded either as infinitely continued, or as

ending with the term $(-1)^w \frac{1}{w!} a_w \Omega^w \frac{\partial}{\partial a_0}$, or earlier if F is known to be annihilated by a lower power of Ω than Ω^{w+1}.

The generator, continued to infinity, produces all seminvariants from gradients. For the first form in which it was obtained shows that it produces any F satisfying $\Omega F = 0$ from itself. The second form, a form in which the annihilation by Ω is instantaneously verified, gives the same information, seeing that when $\Omega F = 0$ it simply produces from F

$$\frac{1}{i}\left(a_0 \frac{\partial}{\partial a_0} + a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} + \ldots\right) F.$$

It will be remembered that the generator in (2), which produces the most general seminvariant of a type w, i from the most general gradient of that type, yields a complete system of seminvariants from a complete system of power-ending products, one from one. The generator now obtained in (5) or (6) equally produces the most general seminvariant from the most general gradient, but it does not exhibit the one-to-one correspondence of seminvariants and power enders, for it clearly annihilates those power enders which do not involve a_0. From a system of power enders we can derive an equally numerous system of products all involving a_0 by replacing in each the final power a_n^r by the corresponding $a_0 a_n^{r-2} a_{2n}$, and it is a matter for investigation whether operation as in (6) on the products thus obtained, *i.e.*, the products in which the final letter has even weight and is unrepeated, the other letters of weights greater than half this do not occur, and the first letter is present, derives from them non-vanishing linearly independent results. I have not succeeded in establishing this.

Observe that a seminvariant derived from a gradient as in (5) or (6) does not involve, to a higher degree than the first, any letter beyond those present in the gradient.

Notice also that, if F is general of degree i, $\frac{\partial}{\partial a_0} F$ is general of degree $i-1$. Thus, if G is the general gradient of type w, $i-1$, then

$$\left(a_0 - a_1\Omega + \frac{a_2}{1.2}\Omega^2 - \ldots\right) G$$

is a sum of arbitrary multiples of all seminvariants of type w, i.

In this last result is contained the known theorem that any seminvariant of weight w is the w-th transvectant (lineo-linear invariant) of the w-ic $(a_0, a_1, a_2, \ldots, a_w)(x, y)^w$, and a w-ic which is a rational integral function of a quantic or quantics chosen from among that w-ic and its successive x-derivatives.

B. *The Problem of Ternary Sources.*

4. Let us call those rational integral homogeneous doubly isobaric functions of the coefficients in a ternary quantic which have the four annihilators Ω_{yx}, Ω_{xy}, Ω_{xz}, Ω_{yz} (*Algebra of Quantics*, Chap. XVI) ternary covariant sources, because every such function determines a covariant in which it is the coefficient of the highest power of z.

A covariant source in the coefficients of a p-ic is not only the coefficient of z^{ϖ}, the highest power of z, in a covariant of the p-ic, but also the coefficient of the highest power of z in a covariant of any $(p+r)$-ic which has, but for the altered appropriate multinomial coefficient multipliers, the same coefficients as the p-ic in its terms up to the p-th order in x and y.

As in the case of binary quantics there is convenience in allowing the idea that the order $p+r$ may be increased even to infinity.

Let us give double suffixes to coefficients, and write a ternary p-ic

$$\sum_{r=0,\,s=0}^{r+s=p} \frac{p!}{r!\,s!\,(p-r-s)!}\, C_{rs}\, z^{p-r-s} x^r y^s ;$$

but let us consider an infinitely extended double system of coefficients

$$\begin{array}{cccc} C_{00} & & & \\ C_{10}, & C_{01} & & \\ C_{20}, & C_{11}, & C_{02} & \\ C_{30}, & C_{21}, & C_{12}, & C_{03} \\ \dots & \dots & \dots, & \end{array}$$

where the first p rows contain the coefficients present in a p-ic, for any p. Let us also adapt the four operators to this infinitely extended system, writing

$$\Omega_{yx} \equiv \sum_{r=1}^{r\to\infty} \left\{ C_{r0} \frac{\partial}{\partial C_{r-1\,1}} + 2C_{r-1\,1} \frac{\partial}{\partial C_{r-2\,2}} + \dots + rC_{1\,r-1} \frac{\partial}{\partial C_{0\,r}} \right\},$$

$$\Omega_{xy} \equiv \sum_{r=1}^{r\to\infty} \left\{ rC_{r-1\,1} \frac{\partial}{\partial C_{r\,0}} + (r-1)\, C_{r-2,\,2} \frac{\partial}{\partial C_{r-1\,1}} + \dots + C_{0\,r} \frac{\partial}{\partial C_{1\,r-1}} \right\},$$

$$\Omega_{xz} \equiv \sum_{r=0,\,s=0}^{r+s\to\infty} \left\{ (r+1)\, C_{rs} \frac{\partial}{\partial C_{r+1\,s}} \right\},$$

$$\Omega_{yz} \equiv \sum_{r=0,\,s=0}^{r+s\to\infty} \left\{ (s+1)\, C_{rs} \frac{\partial}{\partial C_{r\,s+1}} \right\},$$

so that the Ω_{yx}, Ω_{xy}, Ω_{xz}, Ω_{yz} appropriate to a p-ic are the parts of these containing no mention of any C_{rs} with $r+s > p$.

Any homogeneous rational integral function of the coefficients with the same equal sums of first and second suffixes throughout, which is annihilated by the extended Ω_{yx}, Ω_{xy}, Ω_{xz}, Ω_{yz}, will be a covariant source for a p-ic, and every $(p+r)$-ic, if it involves no coefficient for which $r+s > p$.

5. We can now prove it possible to find by direct operation all covariant sources in the infinite double system of coefficients, though not, without solving systems of linear equations, to dissociate from the rest those which appertain to a p-ic.

A rational integral homogeneous function of degree i which is annihilated by Ω_{yx} and Ω_{xy}, being an invariant of the binary quantics

$$C_{00}(x, y)^0,\ (C_{10}, C_{01})(x, y)^1,\ (C_{20}, C_{11}, C_{02})(x, y)^2,\ \ldots,$$

consists of terms in each of which the sums of first and of second suffixes are equal, and the same for all terms. Also every such function is a

$$\phi F,$$

where
$$\phi \equiv 1 - \frac{\Omega_{xy}\Omega_{yx}}{1.2} + \frac{\Omega_{xy}^2\Omega_{yx}^2}{1.2^2.3} - \frac{\Omega_{xy}^3\Omega_{yx}^3}{1.2^2.3^2.4} + \ldots,$$

and F consists of terms of the degree and equal sums of suffixes in question. Conversely, every non-vanishing ϕF is annihilated by Ω_{yx} and Ω_{xy}.

If now we take for F a function, of the degree and equal sums of suffixes, which is annihilated by Ω_{xz} and Ω_{yz}, the derived ϕF will itself be annihilated by Ω_{xz} and Ω_{yz} (as well as by Ω_{yx} and Ω_{xy}). To prove this we will show that, if Ω_{xz} and Ω_{yz} annihilate F, they also annihilate $\Omega_{xy}\Omega_{yx}F$, $\Omega_{xy}^2\Omega_{yx}^2F$, ... We have

$$\Omega_{xz}\Omega_{yz} - \Omega_{yx}\Omega_{xz} = \Omega_{yz},$$

and
$$\Omega_{yz}\Omega_{yx} - \Omega_{yx}\Omega_{yz} = 0,$$

so that, if Ω_{xz} and Ω_{yz} annihilate F, they also annihilate $\Omega_{yx}F$. Similarly they annihilate $\Omega_{xy}F$. By a succession of uses of these facts, it follows that they annihilate every $\Omega_{xy}^m\Omega_{yx}^n\Omega_{xy}^p \ldots F$.

Moreover, conversely, every function with all four annihilators is a ϕF with F limited as above, being the result of operating on itself with ϕ.

It follows that, if we can write down the sum of arbitrary multiples of all functions of degree i and the given sums of suffixes which are annihilated by Ω_{xz} and Ω_{yz}, thus obtaining the most general F of its type

which has those annihilators, and if we derive from it the corresponding ϕF, this will be a sum of arbitrary multiples of all the functions of the type which possess all four annihilators.

6. Now, if G is any finite rational integral function of coefficients chosen from the double system,

$$\left\{ C_{00} - (C_{10}\Omega_{xz} + C_{01}\Omega_{yz}) + \frac{1}{1.2}(C_{20}\Omega_{xz}^2 + 2C_{11}\Omega_{xz}\Omega_{yz} + C_{02}\Omega_{yz}^2) + \dots \right\} G,$$

which call

$$\psi G,$$

is annihilated by Ω_{xz} and Ω_{yz}. The verification is immediate, when we remember that

$$\Omega_{xz}\Omega_{yz} = \Omega_{yz}\Omega_{xz},$$

and that, for some finite and every greater value of r, Ω_{xz}^r and Ω_{yz}^r must annihilate any G of finite degree in given C's.

Conversely, every F annihilated by Ω_{xz} and by Ω_{yz} is a ψG. It involves C_{00} (*Quantics*, § 119); and we will prove the statement made by showing that for such an F,

$$iF = \psi\left(\frac{\partial}{\partial C_{00}} F\right).$$

From

$$\Omega_{xz}\frac{\partial}{\partial C_{r-1\,s}} - \frac{\partial}{\partial C_{r-1\,s}}\Omega_{xz} = -r\frac{\partial}{\partial C_{rs}},$$

and

$$\Omega_{yz}\frac{\partial}{\partial C_{r\,s-1}} - \frac{\partial}{\partial C_{r\,s-1}}\Omega_{yz} = -s\frac{\partial}{\partial C_{rs}},$$

it follows that, when the operation is on something annihilated by both Ω_{xz} and Ω_{yz},

$$\Omega_{xz}\frac{\partial}{\partial C_{r-1\,s}} = -r\frac{\partial}{\partial C_{rs}}, \quad \text{and} \quad \Omega_{yz}\frac{\partial}{\partial C_{r\,s-1}} = -s\frac{\partial}{\partial C_{rs}},$$

for every r and s, and that consequently, by $r+s$ successive applications,

$$\frac{1}{r!\,s!}\Omega_{xz}^r\Omega_{yz}^s\frac{\partial}{\partial C_{00}} = (-1)^{r+s}\frac{\partial}{\partial C_{rs}},$$

so that

$$\psi\frac{\partial}{\partial C_{00}}F = \left\{ C_{00}\frac{\partial}{\partial C_{00}} + C_{10}\frac{\partial}{\partial C_{10}} + C_{01}\frac{\partial}{\partial C_{01}} + C_{20}\frac{\partial}{\partial C_{20}} + C_{11}\frac{\partial}{\partial C_{11}} + C_{02}\frac{\partial}{\partial C_{02}} + \dots \right\} F = iF,$$

as stated.

Accordingly, observing that operation with ψ adds one to degree, while it does not alter either sum of suffixes, we have proved that $\phi\psi$ is an operator which derives from the most general G, of degree $i-1$ and equal sums of suffixes q throughout, the most general covariant source of degree i and equal sums of suffixes q.

It is the same thing to say that if G is the most general gradient of type i, q, q in the double system of coefficients, then

$$\phi\psi \frac{\partial}{\partial C_{00}} G$$

is the most general covariant source of the same type.

It is not the case that all covariant sources of the type which appertain to a p-ic are thus obtained from the most general G which includes only coefficients in the p-ic, except in the very limited class of cases when $2q \not> p$; for, in general, the results of operation on this G will involve, to the first degree only, more advanced coefficients than those present in the p-ic.

To obtain all covariant sources for a p-ic we might, after obtaining the most general $\psi \frac{\partial}{\partial C_{00}} G$ for a given q and i, equate to zero the coefficients in it of those products which involve letters absent from the p-ic, thus determining some of the arbitraries as linear functions of the rest, and leaving the most general $\psi \frac{\partial}{\partial C_{00}} G$ which contains only coefficients in the p-ic. Operation with ϕ on this limited $\psi \frac{\partial}{\partial C_{00}} G$ would give what is required.

THE SCATTERING OF LIGHT BY A LARGE CONDUCTING SPHERE

(SECOND PAPER.)

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IN the first paper on this subject it was shewn that the disturbance radiated from a large perfectly conducting sphere on which plane waves are incident, when calculated from the harmonic series to which the electromagnetic theory leads, is in accord, to a first approximation, with the result of a calculation by the ordinary method of optics.

This conclusion was shewn to be valid when the point, at which the effect is determined, does not lie close to the axis, defined as the diameter of the sphere drawn parallel to the direction of propagation of the incident waves.

In this paper, points near the axis are examined, and they are found to present no exception to the statement.

As it is necessary to make frequent reference to formulæ in the first paper, such formulæ will be denoted by a square bracket [ ]. The notation is the same as before.

Corresponding to a system of plane waves travelling along the direction x, and incident on the positive side of the sphere, containing an electric polarisation (f_0, g_0, h_0) and magnetic force $(\alpha_0, \beta_0, \gamma_0)$ given by

$$\left.\begin{aligned}(f_0, g_0, h_0) &= (0, 0, e^{ikx}) \\ (\alpha_0, \beta_0, \gamma_0) &= -4\pi V(0, e^{ikx}, 0)\end{aligned}\right\}, \tag{1}$$

it was shewn in [16] that the polarisation in the scattered waves is

$$(f, g, h) = -\frac{(x, y, z)}{k^2 r^3} e^{-\iota kr} \Sigma_1^\infty 2m \cos(\phi_n + \chi_n) \sin\theta \sin\phi \frac{dP_n}{d\mu} e^{\iota\xi}$$

$$+ ie^{-\iota kr} \Sigma_1^\infty \frac{2m}{kn,\ n+1} \cos(\phi_n + \chi_n) \frac{\partial}{\partial(x, y, z)} \sin\theta \sin\phi \frac{dP_n}{d\mu} e^{\iota\xi}$$

$$- e^{-\iota kr} \Sigma_1^\infty \frac{2m}{krn,\ n+1} \sin\phi_n \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y},\ \ldots,\ \ldots\right)$$

$$\times \sin\theta \cos\phi \frac{dP_n}{d\mu} e^{\iota\xi - \iota\chi_n}$$

$$= S_1 + S_2 + S_3 \text{ (say)}, \tag{2}$$

where $m = n + \frac{1}{2}$, $\xi = \phi_n + \chi_n + n\pi$.

The differentiations may be taken outside the summations on the understanding that they act only on θ and ϕ, and not on r. The angles θ and ϕ are given by

$$x = r\cos\theta, \quad y = r\sin\theta\cos\phi, \quad z = r\sin\theta\sin\phi. \tag{3}$$

In order to include points near the axis, we may employ, instead of the asymptotic expansion of $P_n(\mu)$, the exact Mehler-Dirichlet formula,

$$P_n(\mu) = \frac{2}{\pi} \int_0^\theta \frac{\cos m\omega\, d\omega}{\sqrt{2}\sqrt{(\cos\omega - \cos\theta)}}, \tag{4}$$

which therefore takes the place of

$$(2/m\pi \sin\theta)^{\frac{1}{2}} \cos(m\theta - \tfrac{1}{4}\pi).$$

The value of v for a series of oscillatory type

$$\Sigma\, ue^{\iota zv}, \tag{5}$$

($z = ka$), thus contains ω in place of θ.

From the differential equation satisfied by the zonal harmonic,

$$\sin^2\theta \frac{dP_n}{d\mu} = -n.n+1 \int_1^\mu P_n\, d\mu,$$

and by a recurrence formula,

$$2n+1 \int_1^\mu P_n\, d\mu = \int_1^\mu \left(\frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu}\right) d\mu = P_{n+1} - P_{n-1}.$$

Hence

$$\frac{2n+1}{n.n+1} \sin^2\theta \frac{dP_n}{d\mu} = P_{n-1} - P_{n+1} = \frac{2}{\pi} \int_0^\theta \frac{\cos(m-1)\omega - \cos(m+1)\omega}{\sqrt{2}\sqrt{(\cos\omega - \cos\theta)}} d\omega$$

$$= \frac{4}{\pi} \int_0^\theta \frac{\sin\omega \sin m\omega\, d\omega}{\sqrt{2}\sqrt{(\cos\omega - \cos\theta)}}. \tag{6}$$

Let U denote an operation which, acting on F, has the effect

$$U.F \equiv \frac{4e^{-ikr}}{\pi}\left(-\frac{\alpha}{k^2r^3}+\frac{\iota}{kn,\,n+1}\frac{\partial}{\partial\alpha}\right)\frac{n.n+1\sin\phi}{\sin\theta}\int_0^\theta \frac{\sin\omega.Fd\omega}{\sqrt{2}\sqrt{(\cos\omega-\cos\theta)}}, \tag{7}$$

where α denotes (x, y, z) as in the first paper. Then, by (2),

$$S_1+S_2 = \Sigma_{n=1}^{\infty} U\cos(\phi_n+\chi_n)\sin m\omega.e^{\iota\phi_n+\iota\chi_n+\iota n\pi}, \tag{8}$$

where no factor of the operation U is oscillatory. We shewed before that χ_n does not oscillate in the neighbourhood of possible zero points, which are determined from [22]. In fact, it is of order z^{-1} there, where $z = ka$. A comparison with [22–27] shews that such points, when they occur, are given by $t = \sin\frac{1}{2}\omega$, where t denotes m/z. For the oscillatory portions of the exponents only differ from their old values by a substitution of ω for θ.

Moreover, these exponents being

$$\left.\begin{aligned} z(v_1, v_2) &= 2\phi_n+m\pi\pm m\omega \\ z(v_3, v_4) &= m\pi\pm m\omega \end{aligned}\right\}, \tag{9}$$

it follows, as in [27], that only zv_2 leads to a point of this nature, when ω is not zero or π. If ω is zero, zv_1 also leads to such a point $t = 0$. We shall return to this case later. It arises when $m\theta$ is very small, or $z\theta$ small. If $t = (n+\frac{1}{2})/z$,

$$\phi_n = z\{(1-t^2)^{\frac{1}{2}}+t\sin^{-1}t-\tfrac{1}{2}\pi t\}+\tfrac{1}{4}\pi, \tag{10}$$

gives the value of ϕ_n near the point $t = \sin\frac{1}{2}\omega$. It would cease to do so if ω were nearly π. This case does not arise in the present paper. The complete value of S_1+S_2 is

$$S_1+S_2 = -\tfrac{1}{4}\Sigma_1^{\infty} U\{e^{\iota v_1+2\iota\chi_n}-e^{\iota v_2+2\iota\chi_n}+e^{\iota v_3}-e^{\iota v_4}\}, \tag{11}$$

and rejecting the three series without a zero point, we may write, approximately,

$$S_1+S_2 = \tfrac{1}{4}\Sigma_1^{\infty} U.e^{\iota v_2+2\iota\chi_n}. \tag{12}$$

If the suffix refers to the zero point henceforth, v_2 being now v in (12), then, by [26], the leading term in the sum is

$$S_1+S_2 = \tfrac{1}{2}U_0\left(\frac{\pi z}{2v_0''}\right)^{\frac{1}{2}} e^{\iota zv_0+2\iota\chi_0+\frac{1}{4}\iota\pi}, \tag{13}$$

the accent denoting $\partial/\partial t$ or $z\partial/\partial m$, and v_0'' being positive.

In this formula, with $\chi_0 = 0$,

$$zv = zv_2 = 2\phi_n + m\pi - m\omega$$
$$= 2z\{(1-t^2)^{\frac{1}{2}} + t\sin^{-1}t\} - z\omega t + \tfrac{1}{2}\pi,$$

so that, when $t = \sin\frac{1}{2}\omega$,

$$zv_0 = 2z\cos\tfrac{1}{2}\omega + \tfrac{1}{2}\pi, \qquad v_0' = 0,$$
$$v_0'' = 2(1-t^2)^{-\frac{1}{2}} = 2\sec\tfrac{1}{2}\omega,$$

and is positive, so that

$$S_1 + S_2 = \tfrac{1}{4}iU_0(\pi z\cos\tfrac{1}{2}\omega)^{\frac{1}{2}}\, e^{2iz\cos\frac{1}{2}\omega + \frac{1}{4}i\pi}. \tag{14}$$

Since $n.n+1 = m^2 - \frac{1}{4}$, the value of this product at the zero point $m = z\sin\frac{1}{2}\omega$ is effectively $z^2\sin^2\frac{1}{2}\omega$, so that

$$S_1 + S_2 = -\frac{ia z^2}{k^2 r^3}\left(\frac{z}{\pi}\right)^{\frac{1}{2}} \frac{\sin\phi}{\sin\theta}\, e^{-ikr + \frac{1}{4}i\pi} \int_0^\theta \frac{d\omega \sin\omega \sin^2\frac{1}{2}\omega(\cos\frac{1}{2}\omega)^{\frac{1}{2}}}{\sqrt{2}\sqrt{(\cos\omega - \cos\theta)}}\, e^{2iz\cos\frac{1}{2}\omega}$$
$$- \frac{1}{k}\left(\frac{z}{\pi}\right)^{\frac{1}{2}} e^{-ikr + \frac{1}{4}i\pi} \frac{\partial}{\partial a}\frac{\sin\phi}{\sin\theta} \int_0^\theta \frac{d\omega \sin\omega(\cos\frac{1}{2}\omega)^{\frac{1}{2}}}{\sqrt{2}\sqrt{(\cos\omega - \cos\theta)}}\, e^{2iz\cos\frac{1}{2}\omega}. \tag{15}$$

The main term of each integral arises from the upper limit if θ is not too small. Writing $\cos\omega = \cos\theta + \lambda^2$,

$$I_1 \equiv \int_0^\theta \frac{d\omega \sin\omega \sin^2\frac{1}{2}\omega(\cos\frac{1}{2}\omega)^{\frac{1}{2}}}{\sqrt{2}\sqrt{(\cos\omega - \cos\theta)}}\, e^{2iz\cos\frac{1}{2}\omega}$$
$$= \sqrt{2}\sin^2\tfrac{1}{2}\theta(\cos\tfrac{1}{2}\theta)^{\frac{1}{2}} \int_0^{2^{\frac{1}{2}}\sin\frac{1}{2}\theta} d\lambda\, e^{iz\sqrt{[2(1+\cos\theta+\lambda^2)]}},$$

with an error of relative order z^{-1} by the usual theory of such integrals. To the same order,

$$I_1 = (2\cos\tfrac{1}{2}\theta)^{\frac{1}{2}}\sin^2\tfrac{1}{2}\theta\, e^{2iz\cos\frac{1}{2}\theta} \int_0^{2^{\frac{1}{2}}\sin\frac{1}{2}\theta} d\lambda\, e^{\frac{1}{2}iz\lambda^2\sec\frac{1}{2}\theta}.$$

The error involved in supposing that the integral extends to infinity is $(z\theta)^{-1}$ relatively to the final result. To this order,

$$I_1 = \tfrac{1}{2}\left(\frac{\pi}{z}\right)^{\frac{1}{2}} \sin\theta\sin\tfrac{1}{2}\theta\, e^{2iz\cos\frac{1}{2}\theta + \frac{1}{4}i\pi}, \tag{16}$$

on reduction.

The first term of $S_1 + S_2$ thus becomes, in (15),

$$\frac{a a^2}{2r^3}\sin\phi\sin\tfrac{1}{2}\theta\, e^{-ikr + 2ika\cos\frac{1}{2}\theta}. \tag{17}$$

To the same order $(z\theta)^{-1}$, the second term becomes, in a similar manner,

$$-\frac{\iota}{k}\frac{\partial}{\partial a}\frac{\sin\phi\cos\frac{1}{2}\theta}{\sin\theta}e^{-2\iota ka\cos\frac{1}{2}\theta}. \tag{18}$$

By addition, we obtain the formula [31], which may be developed as before. The sum S_1+S_2 has therefore been correctly estimated, subject to a relative error $(z\theta)^{-1}$, the maximum error in the present investigation. A similar detailed treatment may be given to S_3, with the same result. Its reproduction here does not seem necessary.

We may conclude, therefore, that the extent to which the formulæ of the first paper have been shewn to be valid near the positive side of the axis is determined by the relative smallness of $(z\theta)^{-1}$ compared with unity. Thus for the numerical case of visible light and a sphere of an inch diameter, the error would not exceed one per cent. if $(z\theta)^{-1}=10^{-2}$, or as $z=1{\cdot}4\,10^6$, if θ were of order 10^{-4}, so that θ^2 is itself negligible to the same order.

We shall see that the error is actually of relative order z^{-1}.

With this conclusion, we may adopt a simpler method of investigating points so close to the axis, in order to discover whether the nature of the effect changes at all. It cannot be expected to do so.

When $z\theta$ becomes small, another series neglected above, because it had no zero point, becomes equally important. This is the series whose exponent contains

$$2\phi_n+m\pi+m\omega, \tag{19}$$

where ω is never greater than θ, for if $\theta=0$, this series also has a zero point $t=0$.

Therefore we cannot obtain a correct result by proceeding to the limit, when $\theta=0$, of the Fresnel integral in the last investigation, and instead of doing this, and subsequently adding the other series, it is simpler to find the effect on the axis as follows.

We may ignore the series S_1 at once, for it is proportional to $1/r^2$. This was ultimately ignored before.

If a denotes the Cartesian coordinates (x, y, z) in turn,

$$\partial/\partial a\,(\sin\theta\sin\phi\,dP_n/d\mu)$$

may be calculated by differentiating $\sin\theta\sin\phi$ only, if θ is small, and it is readily shewn that

$$\left.\begin{aligned}\partial\theta/\partial a &= (-\sin\theta,\ \cos\theta\cos\phi,\ \cos\theta\sin\phi)/r\\ \partial\phi/\partial a &= (0,\ -\sin\phi,\ \cos\phi)/r\sin\theta\end{aligned}\right\}. \tag{20}$$

Thus $$\partial/\partial\alpha(\sin\theta\sin\phi) = \sin\theta\cos\phi\,\partial\phi/\partial\alpha + \cos\theta\sin\phi\,\partial\theta/\partial\alpha$$
$$= (0,\ 0,\ 1)/r,$$

without approximation on the axis. Therefore, when $\theta = 0$, since $dP_n/d\mu$ may be replaced, to order θ^2, by $\frac{1}{2}n.n+1$, we find

$$\partial/\partial\alpha(\sin\theta\sin\phi\,dP_n/d\mu) = n(n+1)(0,\ 0,\ 1)/2r, \tag{21}$$

and in a similar manner,

$$(y\,\partial/\partial z - z\,\partial/\partial y,\ \ldots,\ \ldots)(\sin\theta\cos\phi\,dP_n/d\mu) = n(n+1)(0,\ 0,\ 1)/2\,; \tag{22}$$

and therefore, with (2),

$$(f,\ g,\ h) = \frac{e^{-ikr}}{kr}(0,\ 0,\ 1)\,\Sigma_1^\infty\, m\cos(\phi_n+\chi_n)\,e^{\iota\phi_n+\iota\chi_n+im\pi}$$
$$+\frac{e^{-\iota kr}}{kr}(0,\ 0,\ 1)\,\Sigma_1^\infty\, im\sin\phi_n\,e^{i\phi_n+im\pi}$$
$$= \frac{e^{-\iota kr}}{2kr}(0,\ 0,\ 1)\,\Sigma_1^\infty\, m(1+e^{2\iota\chi_n})\,e^{2i\phi_n+\iota m\pi}$$

or $$f = 0,\quad g = 0,\quad h = \frac{e^{-\iota kr}}{2kr}\Sigma_1^\infty\, m(1+e^{2\iota\chi_n})\,e^{2\iota\phi_n+\iota m\pi}. \tag{23}$$

This formula for points on the axis is exact to the order $1/r$.

The direction of vibration in the scattered light on the axis is therefore the same as in the incident light. This would be expected from the symmetry.

The series (23) may be shewn to be absolutely convergent, for let us consider the terms near infinity. In such terms, where m greatly exceeds z, we may write*

$$\tan\phi_n = e^{2t_n},\quad t_n = -\tfrac{1}{2}\log 2 + z(\sinh\beta - \beta\cosh\beta),\quad \beta = m/z, \tag{24}$$

where t_n is large and negative. Moreover,

$$\partial R_n/\partial z = -e^{-2t_n},$$
$$\tan\chi_n = -\tfrac{1}{2}\,\partial R_n/\partial z = \tfrac{1}{2}e^{2t_n},$$

and tends to infinity. Thus χ_n tends to $\frac{1}{2}\pi$, and $1+e^{2\iota\chi_n}$ to zero. Effectively,

$$\tan\chi_n = \tfrac{1}{4}e^{2z\beta\cosh\beta} = \tan\left(\frac{\pi}{2} - u\right),$$

* *Phil. Mag.*, April, 1910, p. 525.

where $$u = 4e^{-2z\beta \cosh \beta},$$

so that ultimately $$1+e^{2\iota\chi_n} = 8\iota e^{-2z\beta \cosh \beta}.$$

Thus the modulus series of (23) is ultimately of the form

$$\frac{4}{kr} \Sigma m e^{-m \cosh m/z}, \tag{25}$$

and is absolutely convergent.

Writing $$a_m = \frac{m}{2kr} (1+e^{2\iota\chi_n}) e^{2\iota\phi_n + \iota m\pi - \iota kr}, \tag{26}$$

then $$h = \Sigma_1^\infty a_m. \tag{27}$$

Consider now the series

$$h' = \Sigma_1^\infty a_m e^{-m^2\xi/z}, \tag{28}$$

where the index of the exponential contains a new positive parameter ξ. Then since the series h is convergent, the generalised Abel-theorem* shews that as ξ tends to zero, h' tends to the limit h.

Now the zero point in the series h or h' is at $t = 0$, in whose neighbourhood χ_n may be neglected, and $1+e^{2\iota\chi_n} = 2$. With $m = zt = kat$, then near $t = 0$,

$$\begin{aligned} 2\phi_n + m\pi &= 2z \{(1-t^2)^{\frac{1}{2}} + t \sin^{-1} t\} + \tfrac{1}{2}\pi \\ &= 2z + \tfrac{1}{2}\pi + zt^2, \end{aligned}$$

and replacing the summation h' by the first approximation to its equivalent integral,†

$$h' = \frac{\iota a}{r} e^{2\iota z - \iota kr} \int_0^\infty zt e^{-(\xi-\iota) zt^2} dt \quad (\xi > 0). \tag{29}$$

The value of the integral is $\frac{1}{2}(\xi-\iota)$, so that

$$h' = \frac{\iota a}{2r(\xi-\iota)} e^{2\iota z - \iota kr}, \tag{30}$$

whose limit when $\xi = 0$ becomes

$$h = -\frac{a}{2r} e^{-\iota kr + 2\iota ka}, \tag{31}$$

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* *Vide, e.g.*, Bromwich, *Theory of Infinite Series*, p. 113.

† *Messenger of Mathematics*, Oct., 1907. The corresponding integral for h is of a divergent type, and its consideration presents some difficulties. The use of the Abel-theorem which provides the simplest means of avoiding them, was suggested to me by Dr. Bromwich.

which is therefore the value of h. This is identical with the limiting value of [40] when θ is small. Accordingly, [40] is correct for points actually on the axis.

This investigation throws an interesting light on the relative importance of various parts of a harmonic series in problems of this nature. Near the axis, another part of the series becomes important, and is exactly balanced by the change in the former part, so that the result is unaltered. The type of integral in (29) is also different, but follows naturally by the general method of the analysis.

We note that the series without zero points have disappeared together on the axis, where the zero points of series which possess them are at $t = 0$.

ON THE CURVES WHICH LIE ON A CUBIC SURFACE

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THE group of the twenty-seven lines of a cubic surface was obtained from the theory of the trisection of the hyperelliptic functions in the form of a linear group in six variables by Herr Burkhardt (*Math. Ann.*, Vol. XLI, 1893, pp. 320–326). Recently Prof. Burnside has shewn that such a representation can be obtained by regarding as variables the symbols used to represent the lines, and operating upon them in a certain conventional way (*Quar. Jour. of Math.*, 1909, pp. 246–250; *Theory of Groups*, 2nd edition, 1911, p. 485). It is noticeable, however, that when we apply to the cubic surface a simple idea, of which the generalisation is one of the most brilliant results which have lately arisen from the co-operation of French and Italian geometers, the equations in question present themselves quite naturally; and with them those for the linear group of seven variables just discovered by Prof. Burnside (*Proc. London Math. Soc.*, Vol. 10, 1911, pp. 302–305). Mentioning the fact to Prof. Burnside, he suggested to me the propriety of pointing out that this is so. The method employed, though so extremely simple a particular case, furnishes a ready method of formulating the relations among the curves of a cubic surface, and gives interesting examples of the arithmetical theory of quadratic forms. I take the opportunity of mentioning also, in a note at the conclusion, some results in regard to certain quadric surfaces associated with the cubic surface.

1. The central idea may be illustrated by referring first to the case of a quadric surface. It is known that any algebraic curve lying on the surface, intersecting, say, one system of generators each in λ points, and the other system each in μ points ($\lambda > \mu$), may be obtained as the residual intersection of the quadric with a surface (of order λ) intersecting it in $\lambda - \mu$ lines coincident with a single generator. If, then, the fact that two curves A, B of the surface are coresidual be expressed by writing $A \equiv B$, we have for any curve C of the surface an equivalence

$$C + (\lambda - \mu)\, v \equiv \lambda (u + v),$$

where u, v denote generators of opposite systems; that is,

$$C \equiv \lambda u + \mu v.$$

This fact, however, once established for the cases of curves of the first few orders, may be obtained by induction, by shewing that every curve of the quadric surface may be regarded as residual of a curve of lower order; and this follows if we shew that a surface of order $m-1$ can be drawn through every curve of order m of the quadric surface. In fact, a surface of order r ($\geqslant 2$) contains effectively, for intersection with the quadric,

$$\tfrac{1}{6}(r+1)(r+2)(r+3)-1-\tfrac{1}{6}(r-1)\,r(r+1),$$

or $(r+1)^2-1$ constants; when $r = m-1$, this is m^2-1, which is equal to or greater than $rm+1$, the difference being $m-2$. This proves the result.

The symbol $\lambda u+\mu v$ represents a system of coresidual curves; any curve of this system intersects any curve of a system $\lambda' u+\mu' v$ in

$$\lambda\lambda'[u, u]+(\lambda\mu'+\lambda'\mu)[u, v]+\mu\mu'[v, v]$$

points, where $[u, v]$, $= 1$, is the number of intersections of the lines u, v, and so on, that is in $\lambda\mu'+\lambda'\mu$ points. In particular, two curves of the same system $\lambda u+\mu v$ intersect in

$$n = 2\lambda\mu$$

points; and all curves of this system have an order $m = \lambda+\mu$. If π be the genus of the curves of the system, we have

$$\pi = 1+\tfrac{1}{2}(n-2m) = (\lambda-1)(\mu-1) = \tfrac{1}{4}m^2-m+1-(\lambda-\tfrac{1}{2}m)^2,$$

and when m is given this is greatest when $\lambda = \frac{1}{2}m$ or $\lambda = \frac{1}{2}(m+1)$, giving respectively $\pi = \frac{1}{4}(m-2)^2$, $\pi = \frac{1}{4}(m-1)(m-3)$. The corresponding expressions are $\frac{1}{2}m(u+v)$ and $\frac{1}{2}(m+1)(u+v)-v$, the curves in the former case being the complete intersection with surfaces of order $\frac{1}{2}m$, and in the latter case the residual of a single generator by surfaces of order $\frac{1}{2}(m+1)$. The cubic curves of the surface are represented by expressions of one of the types $2u+v$, $u+2v$; the quartic curves of genus unity by the expression $2u+2v$, the quartic curves of genus zero by expressions $3u+v$ or $u+3v$, the former, $3(u+v)-2v$, representing intersection with a cubic surface described through two generators v; a cubic $2u+v$ intersects a quartic $2u+2v$ in $4+2 = 6$ points, and intersects a quartic $3u+v$ in $2+3 = 5$ points, and so on. The quartic curve of genus zero with a double point, obtained by another quadric surface touching the original surface, would need to be represented by an expression of the form $2u+2v-w$, of which the constituent w would furnish a contribution

-2 to the number n; but we only consider the various curves that arise in their most general form.

2. Passing to the case of a general cubic surface, and using the ordinary Schläfli notation for the lines, so that representative double sixes are

$$\left.\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix}\right\}, \tag{I}$$

$$\left.\begin{matrix} a_1 & a_2 & a_3 & c_{56} & c_{64} & c_{45} \\ c_{23} & c_{31} & c_{12} & b_4 & b_5 & b_6 \end{matrix}\right\}, \tag{II}$$

$$\left.\begin{matrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{matrix}\right\}, \tag{III}$$

since the composite curve $c_{1i}+a_1+b_i$ is the complete intersection of the cubic surface with a plane, it is coresidual with $c_{1i}+a_i+b_1$; thus we write

$$b_i \equiv a_i+b_1-a_1.$$

Similarly, $$c_{ij}+a_i+b_j \equiv c_{12}+a_1+b_2,$$

gives $$c_{ij}+a_i+a_j \equiv c_{12}+a_1+a_2;$$

we write $$u \equiv c_{12}+a_1+a_2,$$

and hence $$c_{ij} \equiv u-a_i-a_j.$$

Then, as c_{12}, c_{34}, c_{56} are coplanar,

$$c_{12}+a_2+b_1 \equiv c_{12}+c_{34}+c_{56} \equiv c_{12}+2u-a_3-a_4-a_5-a_6,$$

or $$b_i-a_i \equiv 2u-(a_1+a_2+a_3+a_4+a_5+a_6);$$

we put for abbreviation

$$s = a_1+a_2+a_3+a_4+a_5+a_6,$$

and so have $$b_i \equiv 2u-s+a_i.$$

Hence all the twenty-seven lines of the surface are expressible in terms of the seven elements

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \quad a_6, \quad u,$$

of which the last, in virtue of $u \equiv c_{12}+a_1+a_2$, may be replaced by the line c_{12}. This is in accordance with the known fact that all the lines may be rationally determined when seven of them, properly taken, are given.

It is known that there exist seventy-two systems of coresidual twisted

cubic curves upon the surface, each system being associated with a row of one of the double sixes; that associated with the row $a_1, a_2, a_3, a_4, a_5, a_6$ consists of coresidual curves each of which, not meeting any of these six lines, meets each of $b_1, b_2, b_3, b_4, b_5, b_6$ twice, and each of the remaining fifteen lines c_{ij} once; this system of cubics is the residual intersection of the cubic surface with quadrics having common a twisted cubic curve which may be taken to consist of the three lines c_{12}, b_1, b_2, and one cubic of the system is $c_{12}+a_1+a_2$; the symbol u above thus stands for the system of cubic curves in question. In fact, starting with the hypothesis of a twisted cubic curve upon the surface, and a line a_1 of the surface not intersecting this, as is permissible, since cubic surfaces can be drawn through these two, it is easy to show that thereby a double six such as (I) is determined, with the intersections stated; or the fact is obvious from the known representation of the cubic surface upon a plane. The cubic curves associated with the row $b_1, b_2, \ldots, b_6$, are coresidual with $c_{12}+b_1+b_2$ and thus represented by the symbol $5u-2s$.

3. It is now to be shewn that every general algebraic curve upon the cubic surface belongs to a system represented by a symbol

$$\lambda u+\lambda_1 a_1+\ldots+\lambda_6 a_6,$$

wherein $\lambda, \lambda_1, \ldots, \lambda_6$ are integers. Some examples may first be given—

(a) Any plane section is $\equiv c_{12}+a_2+b_1$, $\equiv 3u-s$.

(b) The conics residual of a_1 are $\equiv b_2+c_{12} \equiv 3u-s-a_1$. Those residual of b_1 are $\equiv a_2+c_{12} \equiv u-a_1$.

(c) The twisted cubics associated with the row $a_1, \ldots, a_6$ are $\equiv c_{12}+a_1+a_2$, that is u, and those associated with the row $b_1, \ldots, b_6$ are $\equiv c_{12}+b_1+b_2 \equiv 5u-2s$, as already remarked.

(d) The twisted cubics associated with the row $a_1, a_2, a_3, c_{56}, c_{64}, c_{45}$ of the double six (II) are $\equiv b_3+a_1+a_2 \equiv 2u-(a_4+a_5+a_6)$. Those associated with the other row of the same double six are $\equiv b_3+c_{23}+c_{31} \equiv 4u-s-(a_1+a_2+a_3)$.

(e) The twisted cubics associated with the row $a_1, b_1, c_{23}, c_{24}, c_{25}, c_{26}$ of the double six (III) are $\equiv c_{12}+a_1+b_1 \equiv 3u-s+a_1-a_2$. Those associated with the other row of the same double six are

$$\equiv c_{12}+a_2+b_2, \equiv 3u-s+a_2-a_1.$$

(f) The complete intersection of the cubic surface with a surface of order $k \equiv k(3u-s)$. In particular any twisted cubic of the system

associated with one row of a double six lies, with any cubic of the system associated with the other row, upon a quadric surface, the sum of the appropriate symbols for these systems of cubics being in each case $2(3u-s)$.

If we suppose the theorem proved by trial for all curves of the first few orders, the general theorem will follow by induction, provided we shew that every curve of the cubic surface is the residual of a curve, or set of curves, each of lower order than the original. This we proceed to do.

A surface of order r contains effectively, so far as its intersection with the cubic surface is concerned,

$$\tfrac{1}{6}(r+1)(r+2)(r+3)-1-\tfrac{1}{6}(r-2)(r-1)r = \tfrac{3}{2}r(r+1)$$

constants; if this is greater than rm, the number of the intersections with a curve of order m, then a surface of order r can certainly be drawn through the curve of order m. Now, when $m = 3k+1$, the value

$$r = \tfrac{2}{3}(m-1),$$

gives
$$\tfrac{3}{2}r(r+1)-rm-1 = \tfrac{1}{2}r-1,$$

which is positive; when $m = 3k+2$, the value $r = \frac{1}{3}(2m-1)$ gives

$$\tfrac{3}{2}r(r+1)-rm-1 = r-1,$$

which is also positive; in the former case the residual curve is of order $m-2$, and in the latter case of order $m-1$. When $m = 3k$, we have, with $r = \frac{2}{3}m$,

$$\tfrac{3}{2}r(r+1)-rm-1 = m-1,$$

and the residual curve is of order m; but the surface of order r may also be put through a straight line of the cubic surface, requiring $r+1$ conditions at most, when the line does not intersect the original curve of order m, provided

$$m-1-(\tfrac{2}{3}m+1), \text{ or } \tfrac{1}{3}m-2,$$

is positive or zero, or $m \geqslant 6$; then the residual curve will consist of the line and a curve of order $m-1$. The theorem is then proved, the case of all cubics of the surface being supposed examined independently.

4. The number of intersections of two systems of curves each represented by an expression of the form $\lambda u+\lambda_1 a_1+\ldots+\lambda_6 a_6$ can evidently be found when we know the number of intersections of every pair of the

seven fundamental systems. It is known that two of the coresidual cubics u meet in one point; that these cubics do not meet any of $a_1, \ldots, a_6$; and that two of these lines do not intersect. But we are to consider that any one of these lines meets itself in -1 points; this is a general result, into the proof of which we do not enter, the line transforming to a point in the plane representation of the cubic surface; in other words, if P, Q denote two systems of each coresidual curves, the number of intersections of two curves, one from each of the systems coresidual respectively with a_i+P and a_i+Q, is

$$[P, Q]+[a_i, P]+[a_i, Q]-1,$$

as may be verified in manifold examples. Thus two curves, one from each of the systems $\lambda u+\lambda_1 a_1+\ldots+\lambda_6 a_6$, $\mu u+\mu_1 a_1+\ldots+\mu_6 a_6$, meet in $\lambda\mu-\lambda_1\mu_1-\ldots-\lambda_6\mu_6$ points, and two curves of the former system meet in

$$n = \lambda^2-\lambda_1^2-\ldots-\lambda_6^2$$

points; the order of the curves of this system is

$$m = 3\lambda+\lambda_1+\ldots+\lambda_6.$$

If π be the genus of the curves of this system, it is at once seen that $2\pi-2+m$ is equal to the grade n. The number $2\pi-2-n$, which is of such great importance in the general theory of systems of curves upon surfaces, does not seem to bear any name; perhaps we may call it the *canonical number* of a curve; leaving aside the consideration of multiple and exceptional curves, it is for a curve on a surface of order N equal to $m(N-4)$, in particular $-3m$, $-2m$, $-m$ for a curve respectively on a plane, or quadric, or cubic surface. The canonical number for a system of coresidual curves arising by adding two systems of coresidual curves is the sum of their canonical numbers, a fact so frequently arising as to be worthy to be regarded as a principle. A surface which possesses a coresidual system of curves, or even a single curve of genus greater than zero, whose canonical number is negative, can be birationally transformed to a ruled, or rational, surface. In our case, the canonical number for cubic curves u is $-2-1=-3$, and for any of $a_1, \ldots, a_6$, it is $-2+1=-1$; thus for a system $\lambda u+\lambda_1 a_1+\ldots+\lambda_6 a_6$ it is

$$-3\lambda-\lambda_1-\ldots-\lambda_6,$$

or the negative of the order, as already remarked. Thus the genus of a curve of order m and grade n upon the cubic surface is $1+\frac{1}{2}(n-m)$. Further, the cubic surface being representable upon a plane, any curve of grade n and genus π belongs to a system of coresidual curves of freedom $r = n-\pi+1 = \frac{1}{2}(n+m)$.

It is shewn by Herr Noether (*Berlin. Abhandl.*, 1882) that the greatest genus that can arise for curves of a given order m upon a given surface, arises for curves whose residual is the whole or part of a plane section of the surface. It is interesting to see how easily this result for the case of a cubic surface arises from the ideas here used. Such curves, on the cubic surface, when $m = 3k+\epsilon$, $\epsilon = 0$ or ± 1, are those expressed by a symbol

$$\tfrac{1}{3}(m-\epsilon)(3u-s)+\epsilon l,$$

where l is a line of the surface; they have for grade, the greatest arising for given m, the value $\frac{1}{3}m^2$ or $\frac{1}{3}(m^2-4)$, and for genus the value

$$\tfrac{1}{6}(m^2-3m+6) \quad \text{or} \quad \tfrac{1}{6}(m-1)(m-2).$$

To prove this result we notice that from

$$n = \lambda^2-\Sigma\lambda_i^2, \qquad m = 3\lambda+\Sigma\lambda_i,$$

we have $$n = \tfrac{1}{3}m^2-\tfrac{1}{18}(\Sigma\lambda_i+2m)^2-\Sigma(\tfrac{1}{6}\Sigma\lambda_i-\lambda_i)^2,$$

so that $n \leqslant \frac{1}{3}m^2$; when $m = 3k$, this value is attained by taking

$$\lambda_1 = \ldots = \lambda_6 = -k;$$

when m is not a multiple of 3, n cannot be as great as $\frac{1}{3}(m^2-1)$, the greatest integer in $\frac{1}{3}m^2$, because

$$\tfrac{1}{3}(m^2-1)-m = 1+\tfrac{1}{3}(m-4)(m+1)$$

is then odd, whereas $n-m$, equal to $\lambda^2-\lambda-2\lambda-\Sigma(\lambda_i^2+\lambda_i)$ is even. Thus n is $\leqslant \frac{1}{3}(m^2-4)$; and we have seen that this value is attainable. The result stated is thus proved; it has an application to a certain quadratic form as is seen incidentally in the next section.

5. To obtain directly all irreducible curves of order m on the cubic surface, since $\pi = 1+\frac{1}{2}(n-m) \geqslant 0$, or $n \geqslant m-2$, we should require to solve in integers the conditions

$$3\lambda+\Sigma\lambda_i = m, \qquad \lambda^2-\Sigma\lambda_i^2 \geqslant m-2,$$

and hence $$\tfrac{1}{3}m^2-\tfrac{1}{18}(\Sigma\lambda_i+2m)^2-\Sigma(\lambda_i-\tfrac{1}{6}\Sigma\lambda_i)^2 \geqslant m-2.$$

Let* $$\Sigma\lambda_i+2m, \quad \text{or} \quad 3(m-\lambda),$$

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* Prof. Burnside suggests that if we put

$$\mu_i = \lambda_i + \tfrac{1}{3}m,$$

the conditions $$3\lambda + \Sigma\lambda = m, \quad \lambda^2-\Sigma\lambda_i^2 \geqslant m-2,$$

require solutions of the form $\mu_i = \frac{1}{3}m +$ positive integer (leading to positive integer values of λ)

be denoted by $-3y$, and putting $m = 3k+\epsilon$, with $\epsilon = 0$, or ± 1, let $x_i = \lambda_i + k$; then the condition for y, x_1, ..., x_5 is that

$$\sum_{i=1}^{5} (x_i + \tfrac{1}{2}y + \tfrac{1}{3}\epsilon)^2 + \left(\sum_1^5 x_i + \tfrac{5}{2}y + \tfrac{5}{3}\epsilon\right)^2 + \tfrac{1}{2}y^2 + \tfrac{2}{3}\epsilon^2$$

should be equal to or less than

$$3k^2 + (2\epsilon - 3)k + \epsilon^2 - \epsilon + 2,$$

and we are to remember, for the determination of λ, λ_1, ..., λ_5 and λ_6, that

$$\lambda = m + y, \qquad \lambda_i = x_i - k, \qquad \sum_{i=1}^{6} (\lambda_i + \tfrac{1}{2}y + \tfrac{1}{3}m) = 0.$$

Conversely, every one of the necessarily finite number of solutions of this inequality gives rise to an appropriate symbol. The first of the two quantities, being equal to

$$\sum_1^5 x_i^2 + \left(\sum_1^5 x_i\right)^2 + 2(3y + 2\epsilon) \sum_1^5 x_i + 8y^2 + 10\epsilon y + 4\epsilon^2,$$

is always an even integer, at least equal to $2\epsilon^2$. It is this least value which corresponds to the curves of given order, of greatest genus.

For instance, to obtain the straight lines of the surface in this way, we should require to solve in integers

$$\sum_{i=1}^{5} (x_i + \tfrac{1}{2}y + \tfrac{1}{3})^2 + \left(\sum_1^5 x_i + \tfrac{5}{2}y + \tfrac{5}{3}\right)^2 + \tfrac{1}{2}y^2 + \tfrac{2}{3} = 2,$$

which is the same as

$$\sum_1^5 x_i^2 + \left(\sum_1^5 x_i + 3y + 2\right)^2 = (y+1)^2 + 1.$$

Thus, as $\frac{1}{2}y^2 < 2$, this the only possible values of y are 0 or 1 or -1, and corresponding thereto we easily find respectively the fifteen lines c_{ij}, the six

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of the inequality

$$9\Sigma\mu_i^2 - (\Sigma\mu_i)^2 \leqslant 3(m^2 - 3m + 6),$$

of which the left side, divided by 9, is

$$\frac{1}{9.8}(8\mu_1 - \mu_2 - \ldots - \mu_6)^2 + \frac{1}{8.7}(7\mu_2 - \mu_3 - \ldots - \mu_6)^2 + \frac{1}{7.6}(6\mu_3 - \mu_4 - \ldots - \mu_6)^2$$
$$+ \frac{1}{6.5}(5\mu_4 - \mu_5 - \mu_6)^2 + \frac{1}{5.4}(4\mu_5 - \mu_6)^2 + \frac{3}{4}\mu_6^2.$$

Taking successively

$$\mu_2 = \mu_3 = \ldots = \mu_6 = 0,$$
$$\mu_3 = \ldots = \mu_6 = 0, \ \ldots,$$

the actual calculation can be made straightforwardly to the end.

lines b_i and the six lines a_i, thus exhausting the twenty-seven possible solutions of the indeterminate equation.

Or, again, to find the cubic curves upon the surface we are to solve

$$\sum_1^5 (x_i+\tfrac{1}{2}y)^2+\left(\sum_1^5 x_i+\tfrac{5}{2}y\right)^2+\tfrac{1}{2}y^2 \leqslant 2,$$

or the left side is to be zero or 2. Of these the former value gives the plane cubics $3u-s$ for which $y=0$, $x_i=0$. The latter value requires $y=0$ or ± 1 or ± 2; the values $y=0$, $y=1$, $y=2$ lead respectively to the equations

$$\Sigma x_i^2+(\Sigma x_i)^2 = 2,$$

$$\Sigma x_i^2+(\Sigma x_i+3)^2 = 3,$$

$$\Sigma (x_i+1)^2+(\Sigma x_i+5)^2 = 0,$$

and to any solution (y, x_i) of the original condition there is another $(-y, -x_i)$. We hence easily find the seventy-two systems of coresidual cubics previously set down (§ 3).

We do not pursue this method, but extract from Herr Noether's memoir (*Berlin. Abhandl.*, 1882) the following list of curves on the cubic surface, up to those of the seventh order. Reference may also be given to Clebsch, "Die Geometrie auf der Flächen dritter Ordnung," *Crelle*, Bd. LXV, 1866. We note also for curves of genus $\pi > 1$ the adjoint system obtained by subtraction of the symbol $3u-s$. A curve of order m and genus π is denoted by R_m^π; the freedom of the system is

$$r, = m+\pi-1.$$

Quartics.

R_4^1, $3u-s+l$, where l is an arbitrary line of the surface; residual of conic $3u-s-l$.

R_4^0, $6u-2s-l-m$, where l, m are two non-intersecting lines.

Quintics.

R_5^2, $6u-2s-l$. The line l meets the quintic in three points. Adjoint system, conics $3u-s-l$.

R_5^1, $3(3u-s)-R_4^0$; for instance, $7u-3s+a_1+a_2$. The quartic residual R_4^0, $= 2u-a_1-a_2$, meets the quintic in $14-2-2$, $= 10$, points.

R_5^0, $3(3u-s)-R_3^0-l$; for instance, $3(3u-s)-u-a_1$, $= 8u-3s-a_1$. The formula shews that the line a_1, meets the quintic in four points, and the cubic u meets the quintic in eight points.

Sextics.

R_6^4, $6u-2s$, by arbitrary quadric surface. Adjoint system, plane sections $3u-s$.

R_6^3, $3(3u-s)-R_3^0$; for example, $8u-3s$. Adjoint system, cubic curves residual of R_3^0, $5u-2s$.

R_6^2, $3(3u-s), -R_1^0-R_2^0$; for example,

$$3(3u-s)-a_1-(a_1+b_2), = 7u-2s-2a_1-a_2.$$

The line, a_1, meets the sextic in four points, the conic meets it in four points. Adjoint system, $4u-s-2a_1-a_2$, consists of the fixed component line $u-a_1-a_2$, or c_{12}, together with conics, $3u-s-a_1$, lying in planes through the line a_1.

R_6^1, $3(3u-s)-R_1^0-R_1^0-R_1^0$; for example, $9u-3s-a_1-a_2-a_3$. Each of a_1, a_2, a_3 meets the sextic in four points, each of a_4, a_5, a_6 meets the sextic in three points.

R_6^0, $4(3u-s)-R_5^0-R_1^0$, where R_1^0 intersects the R_5^0; for example, $4(3u-s)-(8u-3s-a_1)-b_1, = 2u$; each of b_1, b_2, ..., b_6 meets the sextic in four points. If we take $c_{12}, = u-a_1-a_2$, for R_1^0, we obtain $3u-s+2a_1+a_2$.

R_6^0, residual of $3u-s+2a_1+a_2$ just obtained, namely,

$$9u-3s-2a_1-a_2;$$

this has, beside the four point chord a_2, the five point chord a_1. A cubic surface, other than the fundamental one, can be described through four points of this chord a_1 and fourteen points of the sextic, and this, meeting the sextic in $5+14 = 6.3+1$ points, will contain it entirely.

Septimics.

In previous cases the maximum genus agrees with that in space. For septimics the maximum genus in space is 6, on the cubic surface only 5. The following are examples of the various cases enumerated by Herr Noether :—

R_7^5, $3(3u-s)-R_2^0$, $8u-3s+a_1$.

R_7^4, $3(3u-s)-R_1^0-R_1^0$, $9u-3s-a_1-a_2$.

R_7^3, $3(3u-s)-2R_1^0$, $9u-3s-2a_1$.

$4(3u-s)-R_5^0$, $4u-s+a_1$.

$$R_7^2, \quad 4(3u-s)-R_4^0-R_1^0, \quad 5u-2s+2a_1+2a_2.$$

$$R_7^1, \quad 4(3u-s)-R_3^0-R_1^0-R_1^0, \quad 11u-4s-a_1-a_2.$$

$$R_7^1, \quad 4(3u-s)-2R_2^0-R_1, \quad 9u-4s+3a_1+a_2.$$

$$R_7^0, \quad 4(3u-s)-R_3^0-2R_1, \quad 11u-4s-2a_1.$$

6. It is natural to enquire as to the possible birational transformations of the cubic surface into itself. For instance, such a transformation is given by joining a variable point P of the surface to a fixed point O of the surface, and seeking the residual point P' of the surface on the joining line; this transformation, however, replaces the *point* O by the *curve* in which the tangent plane at O cuts the surface, and replaces any arbitrary curve of the surface by a curve having a multiple point at O. Without entering into the general question, we consider the abstract process* of replacing u by the curve u' given by

$$u' = pu+p_1a_1+\ldots+p_6a_6,$$

---

* Prof. Burnside suggests the following alternative to the text, which I have nevertheless retained because the process there followed, though disproportionately heavy in this case, is of use in other work.

We consider the abstract process of replacing $u, a_1, \ldots$ by $u', a_1', \ldots$, where

$$u' = pu+p_1a_1+\ldots,$$
$$a_i' = q_iu+q_{i1}a_1+\ldots \quad (i = 1, 2, \ldots, 6).$$

So replacing $\lambda u+\lambda_1a_1+\ldots$ by $\lambda'u+\lambda_1'a_1+\ldots$, where

$$\left.\begin{aligned}\lambda' &= \lambda p+\lambda_1q_1+\ldots,\\ \lambda_i' &= \lambda p_i+\lambda_1q_{1i}+\ldots \quad (i = 1, 2, \ldots, 6)\end{aligned}\right\}. \qquad \text{(i)}$$

Equations (i) define a linear substitution on the λ's. If this is to have the required geometrical significance, (i) to any integral values of the λ's there must correspond integral values of the λ''s, and conversely; (ii) $3\lambda+\lambda_1+\lambda_2+\ldots$ and $\lambda^2-\lambda_1^2-\lambda_2^2-\ldots$ must be invariant.

Put $$3\lambda+\lambda_1+\lambda_2+\ldots = \omega, \quad \lambda_i = \mu_i-\frac{\omega}{3} \quad (i = 1, 2, \ldots, 6),$$

so that $$\lambda = \omega-\tfrac{1}{3}\Sigma\mu_i.$$

Then for the group in ω and the μ's, ω and $\frac{1}{3}\omega^2+\frac{1}{9}(\Sigma\mu_i)^2-\Sigma\mu_i^2$ are invariant.

Since $\frac{1}{9}(\Sigma\mu_i)^2-\Sigma\mu_i^2$ is a definite form, this involves that the μ's are transformed among themselves.

Hence the most general transformation of the kind required is one which changes

$$(u-\tfrac{1}{3}s)\,\omega+\Sigma(a_i-\tfrac{1}{3}u)\,\mu_i$$

into $$(u-\tfrac{1}{3}s)\,\omega+\Sigma(a_i-\tfrac{1}{3}u)\,\mu_i',$$

where the μ's undergo any linear transformation which (i) replaces integral values by integral values, and (ii) leaves the definite form $\frac{1}{9}(\Sigma\mu_i)^2-\Sigma\mu_i^2$ invariant.

and replacing a_i by the curve a_i' given by

$$a_i' = q_i u + q_{i1} a_i + \ldots + q_{i6} a_6 \quad (i = 1, \ldots, 6),$$

and so replacing $\lambda u + \lambda_1 a_1 + \ldots + \lambda_6 a_6,$

by $\lambda' u + \lambda_1' a_1 + \ldots + \lambda_6' a_6,$

where $\lambda' = \lambda p + \lambda_1 q_1 + \ldots + \lambda_6 q_6,$

$$\lambda_i' = \lambda p_i + \lambda_1 q_{1i} + \ldots + \lambda_6 q_{6i} \quad (i = 1, \ldots, 6),$$

making the hypothesis that, if this prove possible, the grade and genus of any curve shall remain unaltered. Then the order of any curve will remain unaltered, in particular a plane section will be replaced by a plane section, two intersecting (or non-intersecting) lines will be replaced by two intersecting (or non-intersecting) lines, and so any one of the seventy-two sets of six lines which form the rows of the double sixes will be replaced by a set of six lines also belonging to a row of a double six. If we put

$$J = \begin{vmatrix} p & p_1 & \ldots & p_6 & 0 \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ q_i & q_{i1} & \ldots & q_{i6} & 0 \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & 1 \end{vmatrix},$$

a matrix of eight rows and columns, and put

$$\Delta = \begin{vmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & \ldots & 0 & 0 \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & \ldots & -1 & 0 \\ 3 & 1 & \ldots & 1 & -1 \end{vmatrix},$$

also of eight rows and columns, and introduce a symbol ω which is not transformed, or $\omega' = \omega$, the two conditions that the order and grade of a curve remain unaltered, namely,

$$3\lambda + \Sigma\lambda_i = 3\lambda' + \Sigma\lambda_i', \qquad \lambda^2 - \Sigma\lambda_i^2 = \lambda'^2 - \Sigma\lambda_i'^2,$$

are expressed by the fact that the form

$$\Delta(\lambda, \lambda_1, \ldots, \lambda_6, \omega)(\lambda, \lambda_1, \ldots, \lambda_6, \omega),$$

which is $\lambda^2 - \Sigma\lambda_i^2 + \omega(3\lambda + \Sigma\lambda_i) - \omega^2,$

is unaltered for all values of ω. The transformation being

$$(\lambda', \lambda_1', \ldots, \lambda_6', \omega') = \bar{J}(\lambda, \lambda_1, \ldots, \lambda_6, \omega),$$

this is the same as $$J\Delta\bar{J} = \Delta,$$

which in explicit form gives the equations

$$3p+\sum_r p_r = 3, \quad 3q_i+\sum_r q_{ir} = 1,$$

$$p^2-\sum_r p_r^2 = 1, \quad pq_i-\sum_r p_r q_{ir} = 0 = q_i q_j-\sum_r q_{ir}q_{jr}, \quad q_i^2-\sum_r q_{ir}^2 = -1.$$

These equations lead to another set of equations. We have, in fact,

$$\Delta^{-1} = \begin{vmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & \ldots & 0 & 0 \\ . & . & \ldots & . & . \\ 0 & 0 & \ldots & -1 & 0 \\ 3 & -1 & \ldots & -1 & -1 \end{vmatrix},$$

which is the same as Δ, save for a change of sign of the 2nd, 3rd, ..., 6th elements of the last row, and the equation

$$J\Delta\bar{J} = \Delta,$$

gives $$\bar{J}^{-1}\Delta^{-1}J^{-1} = \Delta^{-1},$$

and hence $$\bar{J}\Delta^{-1}J = \Delta^{-1},$$

which, in explicit form, gives the equations

$$3p-\sum_i q_i = 3, \quad 3p_r-\sum_i q_{ir} = -1,$$

$$p^2-\sum_i q_i^2 = 1, \quad pp_r-\sum_i q_i q_{ir} = 0 = p_r p_s-\sum_i q_{ir}q_{is}, \quad p_r^2-\sum_i q_{ir}^2 = -1.$$

These express that the equations

$$3u'-s' = 3u-s, \quad u'^2-\sum_i a_i'^2 = u^2-\sum_i a_i^2$$

hold identically; the former is the fact that a plane section changes into a plane section; the latter has no obvious geometrical interpretation.

The actual transformations may be obtained by the fact remarked that they must change a row of a double six into a row of a double six, the elements being taken in arbitrary order in a row; this gives 72 (6!) transformations, which all, conversely, are at once seen (from the forms below) to satisfy the two conditions above. The cubic curve u associated with the row $a_1, a_2, \ldots, a_6$ is that which does not meet any of these; it is

given by $u \equiv c_{12}+a_1+a_2$. Taking as typical of the double sixes the forms (I), (II), (III) of § 2 above, representing respectively one, twenty, and fifteen double sixes, for the rows of which the associated cubic curves are respectively

$$\text{(I)} \begin{cases} c_{12}+a_1+a_2, \\ c_{12}+b_1+b_2, \end{cases} \qquad \text{(II)} \begin{cases} b_3+a+a_2, \\ b_3+c_{23}+c_{31}, \end{cases} \qquad \text{(III)} \begin{cases} c_{12}+a_1+b_1, \\ c_{12}+a_2+b_2, \end{cases}$$

we have, as representing the transformations, the seventy-two replacements of respectively a_1, a_2, ..., a_6, and u by

(1)	a_1,	a_2,	a_3,	a_4,	a_5,	a_6,	u,
(1)	$2u-s+a_1$,	$2u-s+a_2$,	$2u-s+a_3$,	$2u-s+a_4$,	$2u-s+a_5$,	$2u-s+a_6$,	$5u-2s$,
(20)	a_1,	a_2,	a_3,	$u-a_5-a_6$,	$u-a_6-a_4$,	$u-a_4-a_5$,	$2u-a_4-a_5-a_6$,
(20)	$u-a_2-a_3$,	$u-a_3-a_1$,	$u-a_1-a_2$,	$2u-s+a_4$,	$2u-s+a_5$,	$2u-s+a_6$,	$4u-s-a_1-a_2-a_3$,
(15)	a_1,	$2u-s+a_1$,	$u-a_2-a_3$,	$u-a_2-a_4$,	$u-a_2-a_5$,	$u-a_2-a_6$,	$3u-s+a_1-a_2$,
(15)	a_2,	$2u-s+a_2$,	$u-a_1-a_3$,	$u-a_1-a_4$,	$u-a_1-a_5$,	$u-a_1-a_6$,	$3u-s-a_1+a_2$,

coupled with permutations of the first six elements of a row among themselves.

The 72 (6!) transformations are obtainable by combining the permutation of a_1, a_2, ..., a_6 with the operation expressed by

$$a_1' = a_1, \quad a_2' = a_2, \quad a_3' = a_3, \quad a_4' = u-a_5-a_6, \quad a_5' = u-a_6-a_4,$$

$$a_6' = u-a_4-a_5, \quad u' = 2u-a_4-a_5-a_6.$$

This operation leaves a_1, a_2, a_3 unaltered, as also $u-a_4$, $u-a_5$, $u-a_6$, while changing the sign of $u-a_4-a_5-a_6$. Putting

$$\xi = u-a_4, \quad \eta = u-a_5, \quad \zeta = u-a_6, \quad \upsilon = u-a_4-a_5-a_6,$$

the conditions that $3u-s$, $u^2-\Sigma a_i^2$ should remain unaltered become, when a_1, a_2, a_3 remain unaltered, that $\xi+\eta+\zeta$ and $\frac{1}{2}\upsilon^2+\xi^2+\eta^2+\zeta^2$ should remain unaltered, as they do when $\xi' = \xi$, $\eta' = \eta$, $\zeta' = \zeta$, $\upsilon' = -\upsilon$.

If in the preceding group of equations we introduce the condition

$$u = \tfrac{1}{3}s,$$

the group becomes one in the six symbols a_1, a_2, a_3, a_4, a_5, a_6 with the same structure as before. It is in this form that the group is given by Prof. Burnside. (See the references given in the Introduction.)

It appears, however, that if we do not introduce this condition, the preceding equations lead to the consideration of another group, in seven variables, discovered by Prof. Burnside (*Proc. London Math. Soc.*, 1911,

p. 305). For putting

$$a_7 = 3u - s,$$

and

$$\sigma = a_1 + a_2 + \ldots + a_7 = 3u,$$

$$\sigma_2 = a_1^2 + a_2^2 + \ldots + a_7^2,$$

the group considered above leaves unaltered a_7, and

$$u^2 - (a_1^2 + a_2^2 + \ldots + a_6^2) - a_7^2,$$

of which the last, multiplied by 9, is

$$\sigma^2 - 9\sigma_2,$$

and it is generated, as we have remarked, by permutations of a_1, a_2, a_3, a_4, a_5, a_6, coupled with the transformation

$$a_1' = a_1, \quad a_2' = a_2, \quad a_3' = a_3, \quad a_7' = a_7, \quad a_4' = \tfrac{1}{3}\sigma - a_5 - a_6,$$

$$a_5' = \tfrac{1}{3}\sigma - a_6 - a_4, \quad a_6' = \tfrac{1}{3}\sigma - a_4 - a_5.$$

It is natural then to consider the group generated by this last transformation, coupled with permutations, among all of $a_1, \ldots, a_7$. This is the group in question.

We know that the bitangents of a plane quartic curve are obtainable by projecting the straight lines of a cubic surface from a point of this surface and adding thereto the projection from this point of the plane section of the surface made by the tangent plane thereat. We see that the process just employed, of treating $3u - s$ in the same way as $a_1, \ldots, a_6$, has thus a geometrical interpretation. And the fact obtained by Prof. Burnside, that the new group has 2.28 times as many transformations as the original, apparently corresponds to the two facts that the twenty-eight bitangents are symmetrical among themselves, while the group contains the transformation $a_i' = -a_i$, which does not enter into the geometrical figure.

7. As was indicated at beginning, the formulæ which have been given for the cubic surface may be interpreted also on the plane upon which the cubic surface is represented. For this we are to regard $a_1, a_2, \ldots, a_6$ as representing six points on this plane, and u as representing a general straight line upon this plane. Then, for instance, $u - a_i - a_j$ represents upon the plane the line through the points a_i, a_j; $2u$ represents a general conic upon the plane, and $2u - s + a_1$ represents a conic passing through all of $a_1, \ldots, a_6$ except a_1; $3u - s$ represents a cubic curve through the six fundamental points; $5u - 2s$ represents a quintic curve with a double point at each of the fundamental points; $2u - (a_4 + a_5 + a_6)$

represents a conic passing through the points a_4, a_5, a_6; $4u-s-(a_1+a_2+a_3)$ represents a quartic curve with a double point in each of a_1, a_2, a_3, passing also through a_4, a_5, a_6; $3u-s+a_1-a_2$ represents a cubic curve not passing through a_1, but passing through the other five points, with a double point at a_2; and so on.

For any other rational surface, representable upon the plane with r fundamental points, a similar treatment of the curves lying upon the surface, in terms of $r+1$ base curves, is possible. And it would seem to be worth while to set out in detail the formulæ for the better known of these.

8. The previous work refers to the general curves of coresidual systems lying on the cubic surface. I wish to refer now to certain results relating to special curves.

Herr Reye has remarked (*Math. Ann.*, Vol. LV, 1902) that the quadric surfaces drawn through the cubic curves of either system associated with a given double-six on the cubic surface (the cubics u, $5u-2s$ of the preceding notation) are linearly expressible by nine of them, and therefore are harmonically circumscribed to one quadric Σ, and has proved that the five points common to two cubics, one from each system, have the property that the pole, in regard to Σ, of the plain containing any three of these five points, lies on the line joining the other two, so that the tangential equation of Σ can be written as the sum of the squares of these five points. From this it follows that the polar line of a line a_i of the double-six, in regard to Σ, is the line b_i. Thus, Σ is Schur's quadric associated with the double-six.

The question then arises how many of the thirty-six Schur quadrics are linearly independent. It can be easily seen that the first polar quadric, in regard to the cubic surface, of any point whatever, is linearly expressible by the nine quadrics passing through the cubic curves associated with any double-six. All the Schur quadrics are thus harmonically inscribed in the four linearly independent first polar quadrics. Hence *the tangential equations of all thirty-six Schur quadrics are linear functions of at most six of them.*

It can further be proved that *the twenty-four intersections of a Schur quadric with the curve of intersection of the cubic surface and its Hessian are the points of contact of the Hessian with the twelve lines* $(a_1, b_1, \ldots, a_6, b_6)$ *of the double-six to which the Schur quadric belongs.*

In a recent paper (*Proc. London Math. Soc.*, Vol. IX, 1910, p. 148) I have proved that the centres of degenerate conics $\lambda^2\theta+2\lambda\phi+\psi=0$ lie on a conic. Another formulation of the same proof is by the remark that

all such conics are harmonically inscribed to the same conic, while a pair of straight lines harmonically inscribed to a conic intersect thereon. In partial correction of a remark made *loc. cit.*, p. 149, we have the theorem: *Considering quadric* $(n-1)$*-folds* $\phi_1, \phi_2, \ldots, \phi_s$ *in space of* n *dimensions, let* $s = 1+\frac{1}{2}n$ *or* $\frac{1}{2}(1+n)$*, according as* n *is even or odd; then the system of quadrics*

$$\lambda^s\phi+\lambda^{s-1}\phi_1+\ldots+\phi_s = 0$$

contains $(n+1)s$*, that is* $\frac{1}{2}(n+1)(n+2)$ *or* $\frac{1}{2}(n+1)^2$ *loci possessing nodes. Every quadric* $(n-1)$*-fold* Ω *passing through* $ns+1$ *of these nodes, that is through all but* $\frac{1}{2}n$ *or* $\frac{1}{2}(n-1)$ *of them, passes through the remainder; the number of such loci* Ω *which are linearly independent is* $\frac{1}{2}n$ *or* n*; these are all harmonically circumscribed to the original* $(n-1)$*-folds* $\lambda^s\phi+\ldots+\phi_s = 0$. For instance, in three dimensions, quadric surfaces $\lambda^2\phi+2\lambda\psi+\chi = 0$ contain eight cones; their vertices are a set of associated points through which pass three quadrics $\Omega_1, \Omega_2, \Omega_3$; these are all harmonically circumscribed to any quadric $\lambda^2\phi+2\lambda\psi+\chi = 0$.

NOTE ON SOME TRANSFORMATIONS OF A GENERAL KUMMER SURFACE

By H. F. Baker.

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It is an interesting question whether a surface be capable of an *infinite* discontinuous group of birational transformations. For a particular form of Kummer surface this was shewn to be the case by Humbert [Liouville's, J., t. VI, 1900, p. 372, *Compt. Rend.*, 30 Jan., 1897], and thereafter simpler cases, also of degenerate Kummer surfaces, were given by Painlevé [*Compt. Rend.*, 1898, t. 126, p. 512]. Dr. J. I. Hutchinson has published a note [*Amer. Math. Soc. Bulletin*, Vol. VII, 1901, p. 211] giving, if I understand it, a transformation of infinite period for a general Kummer surface, and his result has been quoted without comment by several authors.* Examining his formulæ, which are expressed by ϑ functions, in an algebraical way, I have been unable to verify them, and the present note is offered as a provisional correction. It suggests, however (in the addendum), an algebraical problem of transformation of the surface of which the results would be interesting to have.

Putting

$$P_\theta = y+\theta x-\theta^2, \quad P_{\theta\phi} = x\theta\phi+y(\theta+\phi)+z-e_{\theta\phi},$$

where θ, ϕ are two of the five roots of the fundamental quintic, and

$$e_{\theta\phi} = \theta\phi(\theta'+\phi'+\psi')+\theta'\phi'\psi',$$

where θ', ϕ', ψ' are the remaining roots, the singular tangent planes of the Kummer surface consist of the plane at infinity, the five planes $P_\theta = 0$, and the ten planes $P_{\theta\phi} = 0$. The singular points are the point $x = 0$, $y = 0$, $z = \infty$, the five points $x = -y/\theta = z/\theta^2 = \infty$, and the ten points $x = \theta+\phi$, $y = -\theta\phi$, $z = e_{\theta\phi}$. Denoting these respectively by (∞), (θ), $(\theta\phi)$, the plane at infinity contains the point (∞) and the five points (θ), the plane $P_\theta = 0$ contains the point (∞) and the five points (θ), $(\theta\phi)$, $(\theta\theta')$, $(\theta\phi')$, $(\theta\psi')$, and the plane $(\theta\phi)$ contains the six points (θ), (ϕ), $(\theta\phi)$, $(\theta'\phi')$, $(\phi'\psi')$, $(\psi'\theta')$. Thus, through any two singular points there pass two singular planes, and any two singular planes have two singular points common. Hence if we take a, b, c, a', b', c' to denote

* *E.g.*, Fano, *Rend. Lombardo*, XXXIX, 1906, § 7.

the six singular planes through an arbitrarily chosen singular point, the remaining singular points may be denoted by the fifteen combinations in pairs of the six symbols a, b, c, a', b', c'. Of these singular points, six such as (bc), (ca), (ab), $(b'c')$, $(c'a')$, $(a'b')$ lie on a singular plane, so that the ten singular planes other than the six fundamental ones correspond to the ten ways of dividing the six symbols into two sets of three. Taking now one of the $\frac{6!}{2!\,4!}\,\frac{4!}{2!\,2!} \div 3! =$ fifteen ways of dividing these six symbols into three pairs, say the pairs aa', bb', cc', and, supposing (∞) to be the singular points through which these six planes are drawn, we can form four tetrahedra, whose vertices are singular points, namely, (∞, aa', bb', cc'), $(bc, b'c', bc', b'c)$, $(ca, c'a', ca', c'a)$, $(ab, a'b', ab', a'b)$, and no face of any one of these tetrahedra is a singular plane. There are thus sixty Göpel tetrads, of which each vertex is a singular point, and no face is a singular plane, and the joining line of any two of the sixteen nodes is common to three of these tetrads.*

Returning to a particular tetrad, say that given by the four points

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* As regards notation, the following remarks may be made.

In the natural notation, for the points we have ; (∞), *i.e.* $(0, 0, \infty)$, (a_1), (a_2), (c), (c_1), (c_2) and (a_1a_2), ..., denoting respectively $(1, -a_1, a_1^2, 0)$, ... and $(a_1+a_2, -a_1a_2, e_{a_1a_2})$, ... and for the planes ; ∞, P_{a_1}, ..., $P_{a_1a_2}$, ..., denoting respectively the plane at ∞,

$$y + a_1x - a_1^2, \ \ldots, \quad xa_1a_2 + y\,(a_1+a_2) + z - e_{a_1a_2}, \quad \ldots,$$

where $$e_{a_1a_2} = a_1a_2\,(c + c_1 + c_2) + cc_1c_2.$$

This being so, we may identify the notation used above with this natural notation, by supposing the planes

$$\infty, \ P_{a_1}, \ P_{a_2}, \ P_c, \ P_{c_1}, \ P_{c_2}, \ P_{a_1a_2}, \ P_{ca_1}, \ \ldots,$$

to be respectively $$a, \ b, \ c, \ a', \ b', \ c', \ abc, \ aba', \ \ldots,$$

where abc is an abbreviation for $(abc, a'b'c')$, ... And then we may change this into a biliteral notation for the planes by supplying the symbol a or suppressing it as the case may be, so writing, for the planes,

$$aa, \ ab, \ ac, \ aa', \ ab', \ ac', \ bc, \ ba', \ \ldots,$$

and, for the points, by supposing the points

$$\infty, \ a_1, \ a_2, \ c, \ c_1, \ c_2, \ (a_1a_2), \ (ca_1), \ \ldots,$$

to be respectively $$aa, \ ab, \ ac, \ aa', \ ab', \ ac', \ bc, \ a'b, \ \ldots.$$

If, then, we introduce $d = a'b'c'$, $d' = abc$, the array of points

$$\left.\begin{array}{cccc|cccc} aa' & ab' & ac' & ad' & c & c_1 & c_2 & a_1a_2 \\ ba' & bb' & bc' & bd' & ca_1 & c_1a_1 & c_2a_1 & a_2 \\ ca' & cb' & cc' & cd' & ca_2 & c_1a_2 & c_2a_2 & a_1 \\ da' & db' & dc' & dd' & c_1c_2 & c_2c & cc_1 & \infty \end{array}\right\}, \qquad \text{(I)}$$

whereby we may make an identification in the notation for points with that used by Humbert.

∞, (c), (a_1a_2), (c_1c_2), or in the notation of the footnote,

$$(\infty),\ (aa'),\ (bc),\ (b'c'),$$

which we denote respectively by

$$D,\quad A,\quad C,\quad B,$$

and putting, respectively,

$$\sigma X = P_{a_1a_2} - P_{c_1c_2},\qquad \sigma Y = P_c + \alpha_1\alpha_2,\qquad \sigma Z = P_c + \gamma_1\gamma_2,$$
$$\sigma T = \gamma_1\gamma_2 P_{a_1a_2} - \alpha_1\alpha_2 P_{c_1c_2},$$

where σ is a factor of proportionality, and $\alpha_1 = a_1 - c$, $\gamma_1 = c_1 - c$, ..., the equation of the Kummer surface is

$$\xi^2 + C^2\eta^2 + A^2\zeta^2 - 2\delta\eta\zeta - 2(\alpha_1+\alpha_2)\xi\zeta + 2(\gamma_1+\gamma_2)\xi\eta$$
$$-4(\alpha_1\alpha_2 - \gamma_1\gamma_2)^2 XYZT = 0, \qquad (1)$$

wherein $\xi = XT + pYZ$, $\eta = YT + qZX$, $\zeta = ZT + rXY$,

$$p = (\alpha_1-\gamma_2)(\alpha_2-\gamma_1)(\alpha_1-\gamma_1)(\alpha_2-\gamma_2),\quad q = \alpha_1\alpha_2,\quad r = \gamma_1\gamma_2,$$
$$C = \gamma_2 - \gamma_1,\quad A = \alpha_2 - \alpha_1,\quad \delta = (\alpha_1+\alpha_2)(\gamma_1+\gamma_2) - 2(\alpha_1\alpha_2 + \gamma_1\gamma_2).$$

If every symbol of the last array be multiplied by a_1a_2, where, as symbols, $a_1a_2cc_1c_2 = 1$, it becomes

$$\left.\begin{array}{cccc} c_1c_2 & cc_2 & cc_1 & \infty \\ ca_2 & c_1a_2 & c_2a_2 & a_1 \\ ca_1 & c_1a_1 & c_2a_1 & a_2 \\ c & c_1 & c_2 & a_1a_2 \end{array}\right\}, \qquad (II)$$

to be identified with the planes

$$\begin{array}{cccc} aa' & ab' & ac' & ad', \\ ba' & bb' & bc' & bd', \\ ca' & cb' & cc' & cd', \\ da' & db' & dc' & dd'. \end{array}$$

Any point of array (I) then lies on the planes of the same row and column (other than itself) of the square (II), and conversely. This is Humbert's rule, and enables us to make an identification of his notation for planes with our natural notation.

We notice further that if, in the symbols for the bitangents of a plane quartic, we exclude those of a double six or Steiner system,

$$\left(\begin{array}{cccccc} ua & ub & uc & gh & hf & fg \\ bc & ca & ab & vf & vg & vh \end{array}\right),$$

the remaining sixteen bitangents are those of the square array (obtained by combining the letters u, a, b, c with v, f, g, h), namely,

$$\left|\begin{array}{cccc} uv & uf & ug & uh \\ av & af & ag & ah \\ bv & bf & bg & bh \\ cv & cf & cg & ch \end{array}\right|,$$

Thus Humbert's notation may be regarded as a particular case of this notation.

This can also be put into the form

$$X^2(u\tau^2+w'\tau+v')+X(w'\tau^2+v\tau+u')+v'\tau^2+u'\tau+w=0, \qquad (2)$$

where $\tau = pYZ/T$ and u, v, w, u', v', w' are functions of Y, Z given by

$$u=\frac{1}{p^2}\left(\frac{C^2q^2}{Y^2}+\frac{A^2r^2}{Z^2}-\frac{2\delta qr}{YZ}\right), \qquad v=\frac{2}{p}\left[\mu-\delta\left(\frac{qZ}{Y}+\frac{rY}{Z}\right)\right],$$

$$w=C^2Y^2+A^2Z^2-2\delta YZ\,;$$

$$u'=2[(\gamma_1+\gamma_2)Y-(\alpha_1+\alpha_2)Z], \qquad v'=1,$$

$$w'=\frac{2}{p}\left[\frac{q(\gamma_1+\gamma_2)}{Y}-\frac{r(\alpha_1+\alpha_2)}{Z}\right],$$

and $$\mu=p+q(\gamma_2+\gamma_1)^2+r(\alpha_2+\alpha_1)^2-2(\alpha_1\alpha_2+\gamma_1\gamma_2)^2.$$

The form (2), which is symmetrical in regard to X and τ, shews at once that the equation is unaltered by the transformation

$$X'=\tau, \quad Y'=Y, \quad Z'=Z, \quad \tau'=X,$$

that is $$X'=\frac{pYZ}{T}, \quad Y'=Y, \quad Z'=Z, \quad T'=\frac{pYZ}{X},$$

or, say also, by

$$X'=\frac{p}{T}, \quad Y'=\frac{1}{Z}, \quad Z'=\frac{1}{Y}, \quad T'=\frac{p}{X}, \qquad (\vartheta_1)$$

and hence the form (1) shews that the equation is also unaltered by the transformations

$$X'=\frac{1}{Z}, \quad Y'=\frac{q}{T}, \quad Z'=\frac{1}{X}, \quad T'=\frac{q}{Y}, \qquad (\vartheta_2)$$

$$X'=\frac{1}{Y}, \quad Y'=\frac{1}{X}, \quad Z'=\frac{r}{T}, \quad T'=\frac{r}{Z}. \qquad (\vartheta_3)$$

Combining these together we have, for the transformation $\vartheta_2\vartheta_3$ or $\vartheta_3\vartheta_2$,

$$X'=T, \quad Y'=qZ, \quad Z'=rY, \quad T'=qrX, \quad (\vartheta_2\vartheta_3=\vartheta_3\vartheta_2)$$

and similarly for $\vartheta_3\vartheta_1=\vartheta_1\vartheta_3$ and $\vartheta_1\vartheta_2=\vartheta_2\vartheta_1$; and for $\vartheta_1\vartheta_2\vartheta_3$,

$$X'=\frac{p}{X}, \quad Y'=\frac{q}{Y}, \quad Z'=\frac{r}{Z}, \quad T'=\frac{pqr}{T}. \qquad (\vartheta_1\vartheta_2\vartheta_3)$$

Again, the form (2) shews that the equation is unaltered by three transformations of the form

$$X'=\frac{\lambda X+\rho}{\nu X+\sigma}, \quad \tau'=\frac{\lambda\tau+\rho}{\nu\tau+\sigma}, \quad Y'=Y, \quad Z'=Z,$$

wherein λ, ρ, ν, σ, are functions of the coefficients u, v, ..., that is of Y and Z. And, in fact, if for a moment $X=\alpha$, $X=\beta$, $X=\gamma$, $X=\delta$, denote the four values of X for which the quadratic equation in τ given by (2) has equal roots, namely, the roots of the quartic in X

$$(w'X^2+vX+u')^2 = 4(uX^2+w'X+v')(v'X^2+u'X+w),$$

then these three transformations are those which interchange two of the four values α, β, γ, δ of X, at the same time interchanging the other two. Such a transformation is, for instance, given by

$$\frac{(X'-\alpha)(X-\alpha)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{(X'-\delta)(X-\delta)}{(\delta-\beta)(\delta-\gamma)}$$

or

$$X' = \frac{X(\alpha\delta-\beta\gamma)+(\alpha+\delta)\beta\gamma-\alpha\delta(\beta+\gamma)}{X(\alpha+\delta-\beta-\gamma)+\beta\gamma-\alpha\delta},$$

and interchanges β and γ and also α and δ. In our case the tangent lines drawn from the vertex D to the plane quartic curve which is the section of the Kummer surface by a plane through the edge DA are evidently the intersections of this plane with the four singular planes of the surface which pass through D and do not intersect in DA; these intersect two of them in DB and two of them in DC. The pair which intersect in DB, since B is (c_1, c_2), are evidently P_{c_1} and P_{c_2}; those which intersect in DC are similarly P_{a_1}, P_{a_2}. By an elementary calculation we find these to be

$$X = \frac{(a_1\gamma_1)(a_1\gamma_2)}{\alpha_1}Y, \qquad X = \frac{(a_2\gamma_1)(a_2\gamma_2)}{\alpha_2}Y,$$

through DC, and

$$X = -\frac{(a_1\gamma_1)(a_2\gamma_1)}{\gamma_1}Z, \qquad X = -\frac{(a_1\gamma_2)(a_2\gamma_2)}{\gamma_2}Z,$$

through DB, where $(a_1\gamma_1)$ denotes $a_1-\gamma_1$, ... From these we compute the corresponding transformations to be

$$X' = \frac{-H_1X+p(AZ-CY)}{X\left(\frac{Cq}{Y}-\frac{Ar}{Z}\right)+H_1}, \quad Y'=Y, \quad Z'=Z, \quad \tau'=(\ldots)\tau, \qquad (U_{DA})$$

$$X' = \frac{H_2X-p(AZ+CY)}{X\left(\frac{Cq}{Y}+\frac{Ar}{Z}\right)-H_2}, \quad Y'=Y, \quad Z'=Z, \quad \tau'=(\ldots)\tau, \qquad (V_{DA})$$

$$X' = \frac{p(rY^2-qZ^2)X+pYZ(PZ+QY)}{X(PrY+QqZ)-p(rY^2-qZ^2)}, \quad Y'=Y, \quad Z'=Z, \quad \tau'=(\ldots)\tau, \qquad (W_{DA})$$

where $H_1 = (\alpha_1\gamma_1 - \alpha_2\gamma_2)(\alpha_1\gamma_1)(\alpha_2\gamma_2)$, $H_2 = (\alpha_1\gamma_2 - \alpha_2\gamma_1)(\alpha_1\gamma_2)(\alpha_2\gamma_1)$,

$$P = (\alpha_1+\alpha_2)(\alpha_1\alpha_2+\gamma_1\gamma_2) - 2\alpha_1\alpha_2(\gamma_1+\gamma_2),$$

$$Q = (\gamma_1+\gamma_2)(\alpha_1\alpha_2+\gamma_1\gamma_2) - 2\gamma_1\gamma_2(\alpha_1+\alpha_2),$$

and in each case the transformation for τ is the same as that for X.

It is easy to verify that

$$U_{DA}\, V_{DA} = V_{DA}\, U_{DA} = W_{DA}.$$

Now the edge DA is also an edge of two other tetrads $DAB'C'$, $DAB''C''$ given by (∞), (c), (c_1a_1), (c_2a_2) and (∞), (c), (c_1a_2), (c_2a_1). The former is obtained from the original $DABC$ by interchange of the roots c_2 and a_1. Using then X', Y', Z', T' to denote what X, Y, Z, T become by this interchange, we find that, with

$$\begin{aligned} m &= \alpha_2\gamma_2 - \alpha_1\gamma_1, \qquad p_2 = (\alpha_1\gamma_2)(\alpha_2\gamma_1), \\ X' &= mX - p_2(CY - AZ), \\ Y' &= \gamma_2(\alpha_2\gamma_1)\, Y + \alpha_2(\alpha_1\gamma_2)\, Z, \\ Z' &= \gamma_1(\alpha_1\gamma_2)\, Y + \alpha_1(\alpha_2\gamma_1)\, Z, \\ T' &= p_2(ArY - CqZ) \qquad\qquad + mT, \end{aligned}$$

with the inverse formulæ

$$\begin{aligned} X &= (q-r)\, X' + p_2\,[(\alpha_1\gamma_1)\, Y' - (\alpha_2\gamma_2)\, Z'], \\ Y &= \qquad\quad \alpha_1(\alpha_2\gamma_1)\, Y' - \alpha_2(\alpha_1\gamma_2)\, Z', \\ Z &= \qquad\quad -\gamma_1(\alpha_1\gamma_2)\, Y' + \gamma_2(\alpha_2\gamma_1)\, Z', \\ T &= \qquad\quad p_2[-\alpha_1\gamma_1(\alpha_2\gamma_2)\, Y' + \alpha_2\gamma_2(\alpha_1\gamma_1)\, Z'] + (q-r)\, T', \end{aligned}$$

where, as before, $(\alpha_1\gamma_2) = \alpha_1 - \gamma_2$, ...

By this transformation the original equation of the Kummer surface $f(X, Y, Z, T) = 0$ takes a form $F(X', Y', Z', T') = 0$; corresponding to the transformation $\omega = (\vartheta_1\vartheta_2\vartheta_3)$ of the original, which was $X_1 = p/X$, $Y_1 = q/Y$, $Z_1 = r/Z$, $T_1 = pqr/T$, there will hold for $F(X', Y', Z', T') = 0$ a transformation obtained from $(\vartheta_1\vartheta_2\vartheta_3)$ by interchange of the letters c_2 and a_1, namely

$$X_1' = p_2AC/X', \quad Y_1' = -\alpha_2\gamma_2/Y', \quad Z_1' = -\alpha_1\gamma_1/Z', \quad T_1' = p_2ACqr/T'.$$

Denoting this by ω' and writing the transformation from the original to the second tetrad in the form $(X') = J(X)$, so that in particular $(X_1') = J(X_1)$, we infer that the original equation $f(X, Y, Z, T) = 0$ is

unaltered by the transformation

$$(X_1) = J^{-1}\omega' J\,(X).$$

and it is in this way that Dr. Hutchinson obtains a transformation of the surface other than ω in the paper referred to at the outset. I find, however, by actual computation that

$$\omega U_{DA} = J^{-1}\omega' J = U_{DA}\omega.$$

In the same way, by considering the tetrad $DAB''C''$, obtained from $DAB'C'$ by interchange of the roots a_1, a_2, we obtain another transformation ω'' for this, and a consequent transformation $K^{-1}\omega'' K$ for the original equation, where K is the transformation by which we pass from the tetrad $DABC$ to $DAB''C''$, and is obtainable from J by interchange of a_1 and a_2. We see, however, at once that ω is unaffected by this interchange, while U_{DA} is changed into V_{DA}. Thus we similarly have

$$\omega V_{DA} = K^{-1}\omega'' K.$$

From a combination of the two transformations $J^{-1}\omega' J$ and $K^{-1}\omega'' K$, Dr. Hutchinson obtains a transformation in which Y, Z are unchanged. This is, however,

$$\omega U_{DA}.\omega V_{DA} = U_{DA}\omega.\omega V_{DA} = U_{DA}V_{DA},$$

that is

$$W_{DA},$$

and is of period 2. Unless I have mistaken Dr. Hutchinson's paper, it is this transformation which is stated by him to be of infinite period.

Addendum.

The preceding remarks, of interest in themselves in connexion with the question of an infinite discontinuous group of transformations for the general Kummer surface, raise irresistibly the problem of determining the complete group generated by the transformations such as those specified when considered for all the possible Göpel tetrads. The investigation is not difficult, but it is laborious, and I have not carried it through. I add, however, some results which I have obtained, in the hope that a reader of this note may be found willing to complete the task.

For the original tetrad there are no transformations obtainable by symmetry from ϑ_1, ϑ_2, ϑ_3 other than those put down. We notice in passing, however, that each of the three $\vartheta_2\vartheta_3$, $\vartheta_3\vartheta_1$, $\vartheta_1\vartheta_2$ is a projective transformation, and so is presumably one of the group of the known sixteen projective transformations of the Kummer surface. This group

is, then, to be expected to arise as a sub-group of the complete group now demanded.

But, beside the three U_{DA}, V_{DA}, W_{DA}, there will arise, in connexion with this one tetrad, three such transformations for every one of the other five edges of the tetrad, the product of each such three being unity. The eighteen such transformations, however, appear to be easily reducible to six of them, namely, to the three sets of the first two of those arising for the edges DA, DB, DC. For I find that if H_1 denote any one of the three belonging to the edge BC, then each of $\vartheta_2 H_1 \vartheta_2$ and $\vartheta_3 H_1 \vartheta_3$ belongs to the set U_{DA}, V_{DA}, W_{DA}, and similarly, if H_2 be one of the three belonging to the edge CA, each of $\vartheta_3 H_2 \vartheta_3$ and $\vartheta_1 H_2 \vartheta_1$ belongs similarly to DB, and if H_3 belong to AB, each of $\vartheta_1 H_3 \vartheta_1$ and $\vartheta_2 H_3 \vartheta_2$ belongs to DC.

We have previously put down the three belonging to the edge DA, namely, those denoted by U_{DA}, V_{DA}, W_{DA}. The first two of those belonging to the edge DB are found to be

$$X' = X, \quad Y' = \frac{Y\lambda_1 - AZ - X}{Y\left[\frac{p}{qX} + \frac{Ar}{qZ}\right] - \lambda_1}, \quad Z' = Z, \quad T' = \frac{T\lambda_1 - pZ - ArX}{T\left[\frac{A}{qX} + \frac{1}{qZ}\right] - \lambda_1};$$

$$X' = X, \quad Y' = \frac{Y\lambda_2 + AZ - X}{Y\left[\frac{p}{qX} - \frac{Ar}{qZ}\right] - \lambda_2}, \quad Z' = Z, \quad T' = \frac{T\lambda_2 - pZ + ArX}{T\left[-\frac{A}{qX} + \frac{1}{qZ}\right] - \lambda_2},$$

where

$$\lambda_1 = \frac{\alpha_1\alpha_2 + \gamma_1\gamma_2 - \alpha_1(\gamma_1 + \gamma_2)}{\alpha_1}, \qquad \lambda_2 = \frac{\alpha_1\alpha_2 + \gamma_1\gamma_2 - \alpha_2(\gamma_1 + \gamma_2)}{\alpha_2},$$

Similarly the first two of those belonging to the edge DC are found to be

$$X' = X, \quad Y' = Y, \quad Z' = \frac{Z\mu_1 - CY + X}{Z\left(\frac{Cq}{rY} - \frac{p}{rX}\right) - \mu_1}, \quad T' = \frac{T\mu_1 + pY - CqX}{T\left(\frac{C}{rX} - \frac{1}{rY}\right) - \mu_1};$$

$$X' = X, \quad Y' = Y, \quad Z' = \frac{Z\mu_2 + CY + X}{-Z\left(\frac{Cq}{rY} + \frac{p}{rX}\right) - \mu_2}, \quad T' = \frac{T\mu_2 + pY + CqX}{-T\left(\frac{C}{rX} + \frac{1}{rY}\right) - \mu_2},$$

where

$$\mu_1 = \frac{\alpha_1\alpha_2 + \gamma_1\gamma_2 - \gamma_1(\alpha_1 + \alpha_2)}{\gamma_1}, \qquad \mu_2 = \frac{\alpha_1\alpha_2 + \gamma_1\gamma_2 - \gamma_2(\alpha_1 + \alpha_2)}{\gamma_2}.$$

For purposes of verification we add the following two belonging to

the edge BC,

$$X'=X, \quad Y'=\frac{H_1Y+CqX-AT}{Y\left(\frac{Apr}{T}-\frac{Cp}{X}\right)-H_1}, \quad Z'=\frac{H_1Z-ArX+CT}{Z\left(\frac{Ap}{X}-\frac{Cpq}{T}\right)-H_1}, \quad T'=T,$$

$$X'=X, \quad Y'=\frac{H_2Y-CqX-AT}{Y\left(\frac{Apr}{T}+\frac{Cp}{X}\right)-H_2}, \quad Z'=\frac{H_2Z-ArX-CT}{Z\left(\frac{Ap}{X}+\frac{Cpq}{T}\right)-H_2}, \quad T'=T;$$

If these be respectively denoted for an instant by ψ_1, ψ_2 it will be found that $\mathfrak{I}_2\psi_1\mathfrak{I}_2=\mathfrak{I}_3\psi_1\mathfrak{I}_3$ is one of those associated with the edge DA, and $\mathfrak{I}_2\psi_2\mathfrak{I}_2=\mathfrak{I}_3\psi_2\mathfrak{I}_3$ is another of those associated with the edge DA. Incidentally this involves $\mathfrak{I}_2\mathfrak{I}_3\psi_1=\psi_1\mathfrak{I}_2\mathfrak{I}_3$, and so on; in other words, any one of the three transformations associated with the edges AD or BC is commutable with the projective transformation $\mathfrak{I}_2\mathfrak{I}_3$. A similar statement holds for each of the other two pairs of opposite edges.

Also, for purposes of verification, we add that a transformation associated with the edge CA is

$$X'=\frac{\lambda_2X+\frac{1}{q}(AT-pY)}{X\left(\frac{1}{Y}-\frac{Ar}{T}\right)-\lambda_2}, \qquad Y'=Y,$$

$$Z'=\frac{-\lambda_2Z+\frac{1}{q}(T-ArY)}{Z\left(\frac{A}{Y}-\frac{p}{T}\right)+\lambda_2}, \qquad T'=T;$$

if this be denoted by ψ it will be found that $\mathfrak{I}_1\psi\mathfrak{I}_1=\mathfrak{I}_3\psi\mathfrak{I}_3$ is one of the transformations associated with the edge DB.

And, finally, we add that a transformation associated with AB is

$$X'=\frac{\mu_1X+\frac{1}{r}(pZ-CT)}{X\left(\frac{Cq}{T}-\frac{1}{Z}\right)-\mu_1},$$

$$Y'=\frac{\mu_1Y+\frac{1}{r}(T-CqZ)}{Y\left(\frac{C}{Z}-\frac{p}{T}\right)-\mu_1}, \qquad Z'=Z, \qquad T'=T,$$

If this be denoted by χ we find $\mathfrak{I}_1\chi\mathfrak{I}_1=\mathfrak{I}_2\chi\mathfrak{I}_2$ is one of those associated with DC.

In the case of all the transformations associated with DA, $\frac{pYZ}{T}$ has the same transformation as X; for those assoicated with DB, DC a similar remark holds. For those associated with BC, $\frac{XT}{pY}$ has the same transformation as Z, for CA, $\frac{YT}{qX}$ the same transformation as Z, and for AB $\frac{ZT}{rX}$ the same as Y.

Coming next to the changes due to a change of tetrahedron, we have seen that another possible Göpel tetrad is that one obtained from the one we have considered by the interchange of the roots a_1 and c_2. We have already put down the formulæ for the change to this tetrahedron. For this new tetrahedron there will be a transformation, say ϕ_1, analogous to ϑ_1, of which the form is obtained from that of ϑ_1 by the interchange of these roots. This leads to a transformation of the surface referred to the original tetrahedron, namely, $J^{-1}\phi_1 J$. We can easily calculate, however, that

$$J^{-1}\phi_1 J = U_{DA}\vartheta_1.$$

By symmetry from this we must necessarily have

$$J\vartheta_1 J^{-1} = U'_{DA}\phi_1$$

where U'_{DA} is what U_{DA} becomes regarded as a transformation calculated for the new tetrad; the corresponding transformation for the original tetrad, being $J^{-1}U'_{DA}J$, is

$$J^{-1}U'_{DA}J = \vartheta_1 J^{-1}\phi_1 J = \vartheta_1 U_{DA}\vartheta_1 = U_{DA},$$

since

$$U_{DA}\vartheta_1 = \vartheta_1 U_{DA}.$$

In fact, U_{DA} is

$$X' = \frac{H_1 X - p(AZ - CY)}{X\left(\frac{Ar}{Z} - \frac{Cq}{Y}\right) - H_1}, \qquad Y' = Y, \qquad Z' = Z,$$

$$\frac{pYZ}{T'} = \frac{H_1\frac{pYZ}{T} - p(AZ - CY)}{\left(\frac{Ar}{Z} - \frac{Cq}{Y}\right)\frac{pYZ}{T} - H_1},$$

or

$$T' = \frac{-H_1 T + p(ArY - CqZ)}{\left(\frac{C}{Z} - \frac{A}{Y}\right)T + H_1} = \frac{H_1 T - p(ArY - CqZ)}{\left(\frac{A}{Y} - \frac{C}{Z}\right)T - H_1},$$

and ϑ_1 is

$$X' = \frac{p}{T}, \quad Y' = \frac{1}{Z}, \quad Z' = \frac{1}{Y}, \quad T' = \frac{p}{X}.$$

Thus $U_{DA}\vartheta_1$ is

$$X' = \frac{H_1\frac{p}{T} - p\left(\frac{A}{Y} - \frac{C}{Z}\right)}{\frac{p}{T}(ArY - CqZ) - H_1}, \qquad Y' = \frac{1}{Z}, \qquad Z' = \frac{1}{Y},$$

$$T' = \frac{H_1\frac{p}{X} - p\left(\frac{Ar}{Z} - \frac{Cq}{Y}\right)}{\frac{p}{X}(AZ - CY) - H_1},$$

and $\vartheta_1 U_{DA}$ is

$$X' = p\frac{H_1 - \left(\frac{A}{Y} - \frac{C}{Z}\right)T}{p(ArY - CqZ) - H_1 T}, \qquad Y' = \frac{1}{Z}, \qquad Z' = \frac{1}{Y}$$

$$T' = p\frac{H_1 - X\left(\frac{Ar}{Z} - \frac{Cq}{Y}\right)}{p(AZ - CY) - H_1 X},$$

verifying that, as stated, $\vartheta_1 U_{DA} = U_{DA}\vartheta_1$.

Then the equation $J^{-1}\phi_1 J = U_{DA}\vartheta_1$,

combined with an equation previously referred to, namely,

$$J^{-1}\phi_1\phi_2\phi_3 J = U_{DA}\vartheta_1\vartheta_2\vartheta_3,$$

lead to $J^{-1}\phi_2\phi_3 J = \vartheta_2\vartheta_3$,

where ϕ_2, ϕ_3 correspond, for the new tetrahedron, to ϑ_2, ϑ_3 respectively.

SOME TWO-DIMENSIONAL PROBLEMS IN ELECTROSTATICS AND HYDRODYNAMICS

By W. M. PAGE.

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IN the following paper some problems relating to the potentials of infinite cylinders of various cross-sections are discussed. The method of solution is a development of the usual Schwarzian method of mapping the cross-section of the cylinder upon the x axis in a second plane. The Schwarzian method has hitherto been used almost exclusively for solving problems connected with cylinders whose cross-sections are bounded by straight lines.

The development of the method used below gives a means of dealing with cylinders whose cross-sections contain curved as well as straight lines.

The first part of the paper contains a discussion of the potentials of single cylinders of various shapes.

At the end of the paper will be found the case of an infinite grating consisting of cylinders of oval section.

Maxwell,* Lamb†, and J. J. Thomson‡ have given the solution for a grating in which the cross-section is small compared with the distance between successive cylinders; but the solution given below applies to a grating in which the section of the cylinder may bear any relation we please to the distance between neighbouring cylinders.

The paper was originally written, for the greater part, in 1907, and arose out of an attempt to find coefficients of reflection and transmission for electric waves falling on a grating consisting of cylinders of section comparable with the distance between them.

[Lamb† and J. J. Thomson‡ have given the solution when the section is small.]

It was suggested, in the first instance, by Prof. Lamb's paper in *Proc.*

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* Maxwell, *Elec. and Mag.*, § 203.

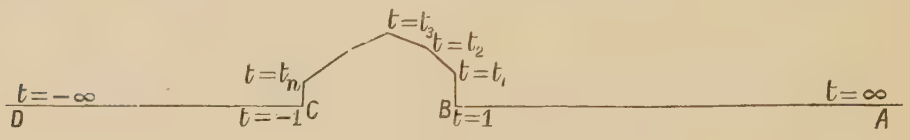
† Lamb, *Proc. London Math. Soc.*, 1898.

‡ J. J. Thomson, *Rec. Researches*, § 359.

London Math. Soc., 1898, and I desire to acknowledge my indebtedness. More recently there has appeared in the *Phil. Mag.* for August, 1910, a paper by Prof. Lamb and Mr. G. Cook dealing with oval cylinders of finite size, to which reference should be made.

We will first proceed to shew how certain figures in the z plane, made up partly of planes and partly of curved surfaces, may be mapped on the real axis of another plane, which we will call the "t" plane.

We consider the curves as the limits of equiangular polygons. Let us consider the case of a curved boss on an infinite plane.



z-plane

FIG. 1.

We take the curve as the limiting shape of an equiangular polygon of n sides.

Take B, C as $t = 1$, $t = -1$ respectively, and let the values of t corresponding to the corners of the polygon be t_1, t_2, t_3, ..., t_n.

Then the relation which transforms this figure in the z plane into the real axis in the t plane is

$$A\frac{dz}{dt} = \frac{[(t-t_1)(t-t_2)(t-t_3)\dots(t-t_n)]^{1/n}}{(t^2-1)^{\frac{1}{2}}}.$$

By giving values to t_1, ..., t_n we get a polygon in the z plane, and we can, theoretically, deduce the lengths of the sides of this polygon in terms of these values of t_1, ..., t_n.

If we took, for instance,

$$t_1 = \cos\alpha, \quad t_2 = \cos 3\alpha, \quad t_3 = \cos 5\alpha, \quad \dots, \quad t_n = \cos(2n-1)\alpha,$$

where

$$2n\alpha = \pi,$$

we should expect to get a semicircle in the z plane.

With these values of t,

$$(t-t_1)(t-t_2)\dots(t-t_n) \equiv \frac{1}{2^n}\left[(t+\sqrt{t^2-1})^n + (t-\sqrt{t^2-1})^n\right].$$

Hence we have to determine the limiting value of

$$\tfrac{1}{2}\left[(t+\sqrt{t^2-1})^n + (t-\sqrt{t^2-1})^n\right]^{1/n}.$$

If $|t|>1$, we may put

$$t = \cosh\mu \qquad (\mu \text{ complex if } t \text{ complex})$$

and the limit we seek is that of

$$\tfrac{1}{2}\left[e^{n\mu}+e^{-n\mu}\right]^{1/n}$$

or of

$$\tfrac{1}{2}e^{\mu}\left[1+e^{-2n\mu}\right]^{1/n}$$

which is

$$\tfrac{1}{2}e^{\mu} \text{ or } \tfrac{1}{2}(t\pm\sqrt{t^2-1}).$$

Hence if $t > 1$ the above expression has a definite limit.

But if $t < 1$ we have to find the limiting value of $\frac{1}{2}[2\cos n\mu]^{1/n}$, where $t = \cos\mu$. This limit is quite indefinite.

The reason of this indeterminateness may be seen from the following considerations :—

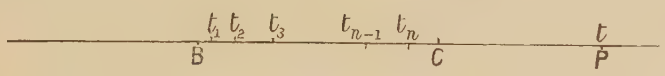


FIG. 2.

Finding the above limit is geometrically equivalent to finding the geometric mean of the distances of a point P in the t plane from the points $t = t_1$, $t = t_2$, ..., $t = t_n$ in the real axis.

If the point P is outside the region BC this geometric mean has a definite and continuous value; but if P lies within the region $B\ C$, the value of the mean changes discontinuously as P moves slightly, and, finally, when the points t_1, ..., t_n are indefinitely close, the geometric mean cannot be said to have a value.

The same difficulty occurs in all other figures into which a curve enters.

Now if the boss on the infinite plane is a semicircle of radius A, the required relation between z and t is known to be

$$\frac{dz}{dt} = \frac{A}{\sqrt{t^2-1}}(t+\sqrt{t^2-1}).$$

But this is exactly the value we obtained as a limiting value from the polygon, supposing t to be greater than unity.

Hence it was suggested by Mr. Richmond that it might be found effective to discard the limiting value of $[(t-t_1), \ldots, (t-t_n)]^{1/n}$, and to replace it by the expression $\frac{1}{2}(t+\sqrt{t^2-1})$ when t lies between $t_1, t_2, \ldots, t_n$.

This amounts to adopting the transformation

$$A\frac{dz}{dt}=\frac{\frac{1}{2}(t+\sqrt{t^2-1})}{(t^2-1)^{\frac{1}{2}}},$$

and seeing what consequences follow from this transformation. The replacement of the indefinite limit $[(t-t_1), \ldots, (t-t_n)^{1/n}$ by the expression $\frac{1}{2}(t+\sqrt{t^2-1})$ will be made in various cases into which a curve enters.

We thus start off with a given relation between z and t, and afterwards find the curve in the z plane to which it gives rise.

When we have a relation transforming the figure in the z plane into the real axis in the t plane, we can go on to find the electric field when a conductor in the shape of the figure is charged, or to find the motion when a fluid is circulating round the figure.

The first case we will consider is that of a cylindrical conductor whose section is of the following form :—

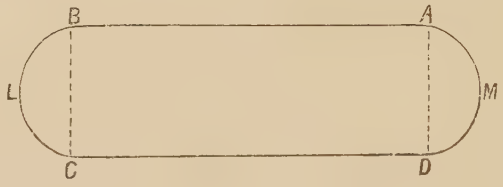


FIG. 3.

The sides AB, CD of the conductor are plane, while the ends BLC, AMD are curves whose shape will be determined later.

$ABCD$ is a rectangle.

Cylindrical Conductor, whose Section consists of Two Parallel Lines joined by Curves.

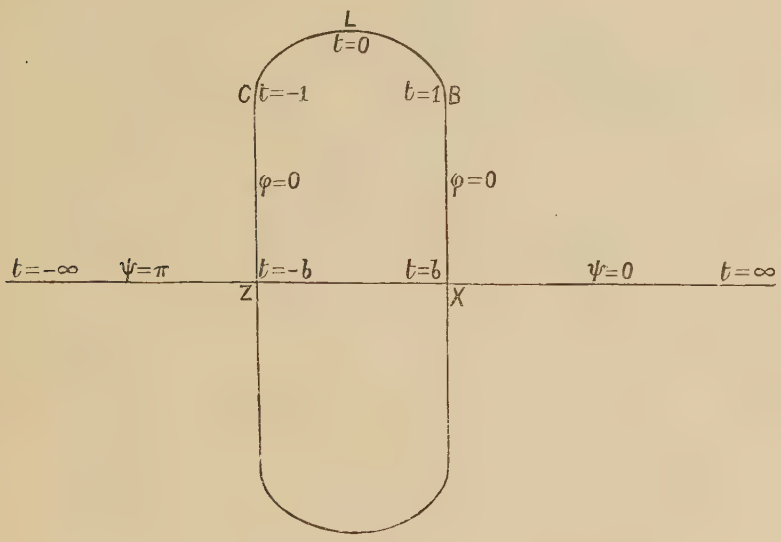
We consider BLC as the limit of a polygon of n sides whose angles are all equal to $\pi-\frac{\pi}{n}$.

Let $t_1, t_2, t_3, \ldots, t_n$ be the values of t at the vertices of this polygon.

Then $$A\frac{dz}{dt}=\frac{[(t-t_1)(t-t_2)\ldots(t-t_n)]^{1/n}}{\sqrt{t^2-b^2}},$$

X, Z are $t=\pm b$; and B, C are $t=\pm 1$.

We take $t_1=\cos\frac{\pi}{2n}$, $t_2=\cos\frac{3\pi}{2n}$, $t_3=\cos\frac{5\pi}{2n}$, $\ldots$,



z-plane

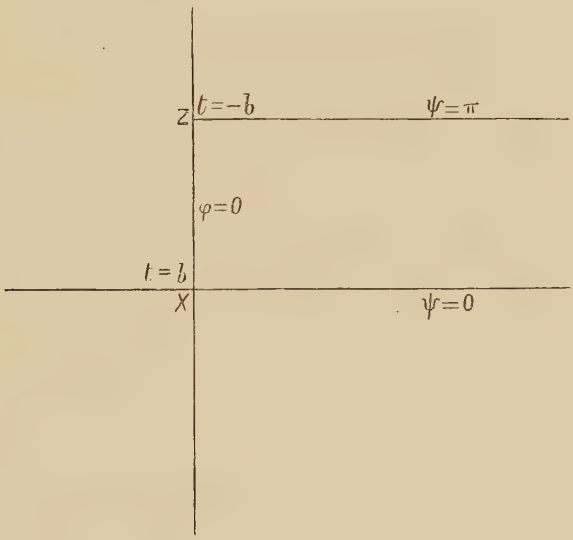


FIG. 4. w-plane

and replace $\underset{n=\infty}{\text{Lt}} [(t-t_1) \dots (t-t_n)]^{1/n}$ by $\frac{1}{2}(t+\sqrt{t^2-1})$.

Thus
$$2A \frac{dz}{dt} = \frac{t+\sqrt{t^2-1}}{\sqrt{t^2-b^2}},$$

and $$B\frac{dw}{dt} = \frac{1}{\sqrt{t^2-b^2}};$$

therefore $t = b\cosh w$, since $t = b$ when $w = 0$,

$t = -b$ when $w = \iota\pi$.

[If we take $b = 1$, so that there is no straight part XB, we get

$$2Az = t+\sqrt{t^2-1} = e^w,$$

giving the case of a circular cylinder of radius $\frac{1}{2A}$].

Therefore $$2A\frac{dz}{dw} = b\cosh w+\sqrt{b^2\cosh^2 w-1}.$$

If we put $b = \sec a$ and neglect the constant in z, this becomes

$$\frac{dz}{dw} = \cosh w+\sqrt{\cosh^2 w-\cos^2 a},$$

the required relation between z and w.

We now find the shape of the curved end; and the surface density.

It is convenient to change our axes.

Put $\frac{\pi}{2}-a$ for a, $\iota\frac{\pi}{2}+\iota w$ for w, ιz for z.

ψ is now the potential function.

The figure in the z-plane becomes

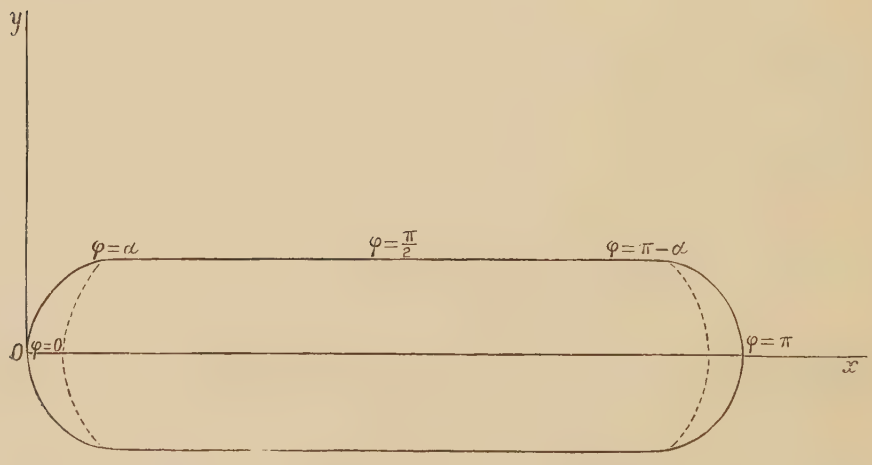


FIG. 5.

The relation between z and w becomes

$$\frac{dz}{dw} = \sin w + \iota\sqrt{\sin^2 a - \sin^2 w}.$$

The surface of the conductor is $\psi = 0$.

Thus along the surface of the cylinder

$$\frac{dz}{d\phi} = \sin\phi + \iota\sqrt{\sin^2 a - \sin^2\phi}.$$

The values of the stream function ϕ, at special points are shewn in the diagram.

From $\phi = 0$ to $\phi = a$,

$$x = 1 - \cos\phi,$$

$$y = \int_0^\phi \sqrt{\sin^2 a - \sin^2\phi}\, d\phi;$$

therefore along the curved surface

$$s = \phi \sin a.$$

The curve is more pointed than a circular arc which has the same extremities and which subtends an angle $2a$ at the centre.

The straight side of the cylinder is tangential to the curved part at $\phi = a$, so that there are no sharp corners.

Over the straight portion of the boundary, from $\phi = a$ to $\phi = \frac{\pi}{2}$,

$$x = 1 - \cos\phi + \int_a^\phi \sqrt{\sin^2\phi - \sin^2 a}\, d\phi,$$

$$y = \int_0^a \sqrt{\sin^2 a - \sin^2\phi}\, d\phi.$$

Again, since for the curved surface $s = \phi \sin a$, it follows that over the curved surface the surface density is constant, and equal to $\frac{1}{4\pi \sin a}$.

Fig. 5 is drawn to scale, and represents the case of $a = \frac{\pi}{4}$. The dotted line shews for comparison a circular arc having the same extremities as the curve, and subtending an angle of 90° at the centre.

The figure of the above conductor is derived by replacing the limiting value of $[(t-t_1)\ldots(t-t_n)]^{\frac{1}{2}}$ by the expression which holds for the known case of a circle.

Instead of taking a circle as the known case we may take an ellipse, and derive a differently shaped conductor.

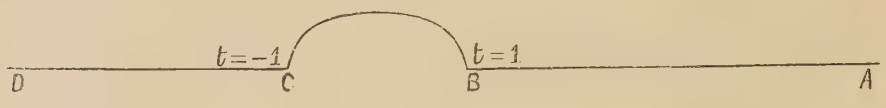


FIG. 6.

Thus suppose we have a boss BC in the shape of a semi-ellipse of axes $2c\cosh\gamma$, $2c\sinh\gamma$, cut off by the major axis. Then the relation between z and t is

$$C\frac{dz}{dt}=\frac{[(t-t_1)\dots(t-t_n)]^{1/n}}{\sqrt{t^2-1}}.$$

But the solution for this case is known to be

$$\frac{dz}{dt}=\frac{c\cosh a\sqrt{t^2-1}+c\sinh at}{\sqrt{t^2-1}}.$$

Therefore for some values of $t_1, t_2, \dots, t_n$ the expression

$$[(t-t_1)(t-t_2)\dots(t-t_n)]^{1/n},$$

must take a limiting form $\quad At+B\sqrt{t^2-1}.$

We may thus obtain the solution for a conductor whose curved ends are of different shape from those found on p. 319.

Thus as a solution of the problem on p. 316 take

$$\frac{dz}{dt}=\frac{At+B\sqrt{t^2-1}}{\sqrt{t^2-b^2}},$$

$$t=b\cosh w.$$

Then $$\frac{dz}{dw}=A\cosh w+B\sqrt{\cosh^2 w-\cos^2 a}.$$

Changing our variables as on p. 318, this becomes

$$\frac{dz}{dw}=A\sin w+\iota B\sqrt{\sin^2 a-\sin^2 w},$$

where ψ is now the potential function.

The conductor is $\psi=0$.

Along the curved part ϕ ranges from 0 to a.

Thus the curve is given by

$$x = A(1-\cos\phi),$$

$$y = B\int_0^{\phi} \sqrt{\sin^2\alpha - \sin^2\phi}\, d\phi,$$

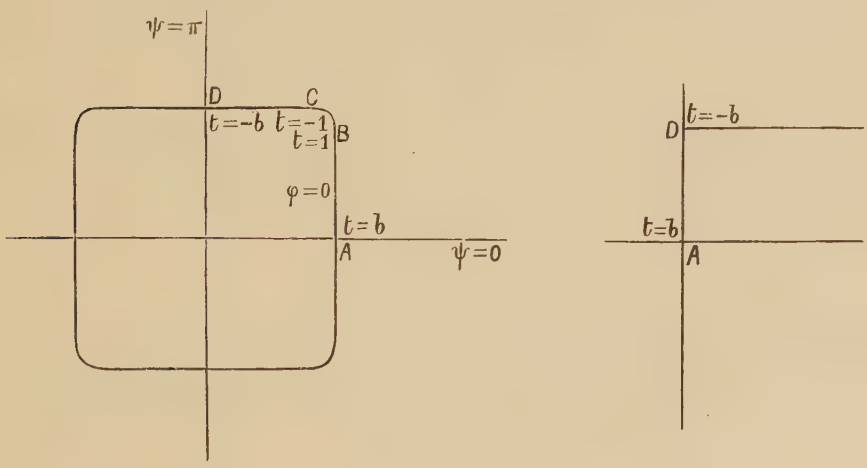
$$\left(\frac{ds}{d\phi}\right)^2 = (A^2-B^2)\sin^2\phi + B^2\sin^2\alpha,$$

which is the same curve as on p. 319, with the ordinates altered in a given ratio.

By giving different values to A and B we get a series of curves.

The surface density over the curved ends is no longer constant.

Cylindrical Conductor of Square Section with Rounded Edges.



z-plane FIG. 7. w-plane

The curved portion of the section lies between the points B and C. We take these to be $t = \pm 1$. Therefore

$$A\frac{dz}{dt} = \frac{[(t-t_1)(t-t_2)\ldots(t-t_n)]^{1/n}}{\sqrt{t^2-b^2}}.$$

We suppose the curved portion BC to be the limit of n sides of a polygon of $4n$ sides.

We thus replace the limiting value of the numerator by $\sqrt{t+\sqrt{t^2-1}}$.

This, save for a constant factor, is equal to $\sqrt{t+1}+\sqrt{t-1}$.

Hence we assume $$A\frac{dz}{dt} = \frac{\sqrt{t+1}+\sqrt{t-1}}{\sqrt{t^2-b^2}}.$$

Also $$\frac{dw}{dt} = \frac{1}{\sqrt{t^2-b^2}},$$

whence $$t = b\cosh w.$$

Therefore $$A\frac{dz}{dw} = \sqrt{b\cosh w+1}+\sqrt{b\cosh w-1}.$$

Put $$b = \operatorname{cosec} a,$$

and write $$\iota\frac{\pi}{2}+\iota w \text{ for } w, \quad \iota z \text{ for } z.$$

Then $$\frac{dz}{dw} = \sqrt{\sin a-\sin w}+\iota\sqrt{\sin a+\sin w}.$$

The conductor is now given by $\psi = 0$. Therefore along the conductor

$$\frac{dz}{d\phi} = \sqrt{\sin a-\sin\phi}+\iota\sqrt{\sin\phi+\sin a}.$$

The conductor is placed with reference to the axes as shewn in the following diagram.

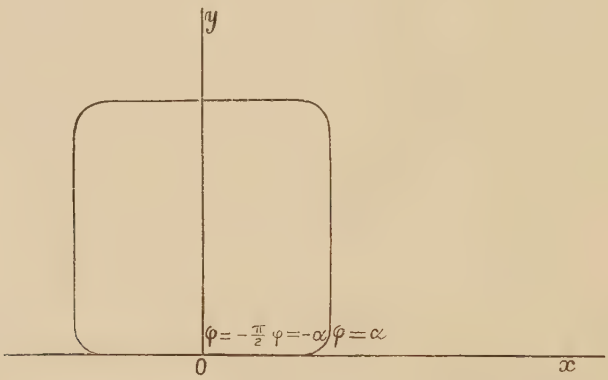


FIG. 8.

Along the curved portion we have

$$x = \int_{-\frac{1}{2}\pi}^{\phi}\sqrt{\sin a-\sin\phi}\,d\phi,$$

$$y = \int_{-a}^{\phi}\sqrt{\sin\phi+\sin a}\,d\phi;$$

therefore $$s = \phi\sqrt{2\sin\alpha},$$

the arc being measured from the middle point of the curve.

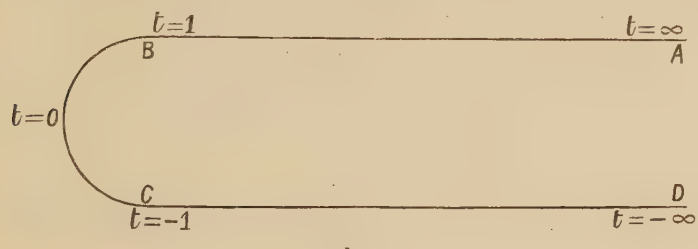
The surface density over the curved portion is constant and equal to

$$\frac{1}{4\pi\sqrt{2\sin\alpha}}.$$

As before, we could get different curves for the corners by taking $At+B\sqrt{t^2-1}$ instead of $t+\sqrt{t^2-1}$ when replacing the expression

$$\mathrm{Lt}\,[(t-t_1)\ldots(t-t_n)]^{1/n}.$$

Cylinder, whose Cross-Section reaches to Infinity, the Finite End being Rounded.



z-plane.

FIG. 9.

The sides BA, CD are straight, and of infinite length. The curve reaches from B to C. We take $t = \pm 1$ for these points.

The solution may be derived from that on p. 317, viz.,

$$A\frac{dz}{dt} = \frac{t+\sqrt{t^2-1}}{\sqrt{t^2-b^2}},$$

by making b infinite. The t in the denominator can be neglected in comparison with b, and the infinite constant b may be supposed absorbed by A.

We thus get for this case

$$\frac{dz}{dt} = a(t+\sqrt{t^2-1}).$$

Also $$w = t.$$

Therefore the relation between z and w is

$$\frac{dz}{dw} = a(w+\iota\sqrt{1-w^2}).$$

The conductor is $\psi = 0$.

Over the curved end, $\psi = 0$ and $0 < \phi < 1$.

Therefore for the curve

$$x = \frac{a}{2}\phi^2,$$

$$y = a\int_0^{\phi} \sqrt{1-\phi^2}\, d\phi,$$

$$s = a\phi.$$

Therefore the curve is a *cycloid* having its cusps at B and C.

Over the cycloid the surface density is constant and equal to $\frac{1}{4\pi a}$.

Over the flat part the surface density is $\frac{1}{4\pi a\,(\phi+\sqrt{\phi^2-1})}$; ϕ being given in terms of x by the relation

$$x = \frac{a}{2}\,\phi^2 + \int_1^{\phi} \sqrt{\phi^2-1}.$$

Also $$y = a\int_0^1 \sqrt{1-\phi^2}\, d\phi.$$

By taking $\frac{dz}{dw} = a(Aw+\iota B\sqrt{1-w^2})$ we can get different curves.

Infinite Wedge formed by Two Semi-Infinite, Perpendicular Planes with the Edge Rounded off.

We suppose the curve BC to be the limit of n sides of a polygon of $4n$ sides.

Hence we assume

$$\frac{dz}{dt} = [(t-t_1)(t-t_2)\ldots(t-t_n)]^{\frac{1}{2n}},$$

and replace $[(t-t_1)\ldots(t-t_n)]^{\frac{1}{2n}}$ by

$$[\tfrac{1}{2}t+\tfrac{1}{2}\sqrt{t^2-1}]^{\frac{1}{2}} \quad \text{or} \quad \tfrac{1}{2}\sqrt{t+1}+\tfrac{1}{2}\sqrt{t-1}.$$

Thus we take the relation between z and t to be

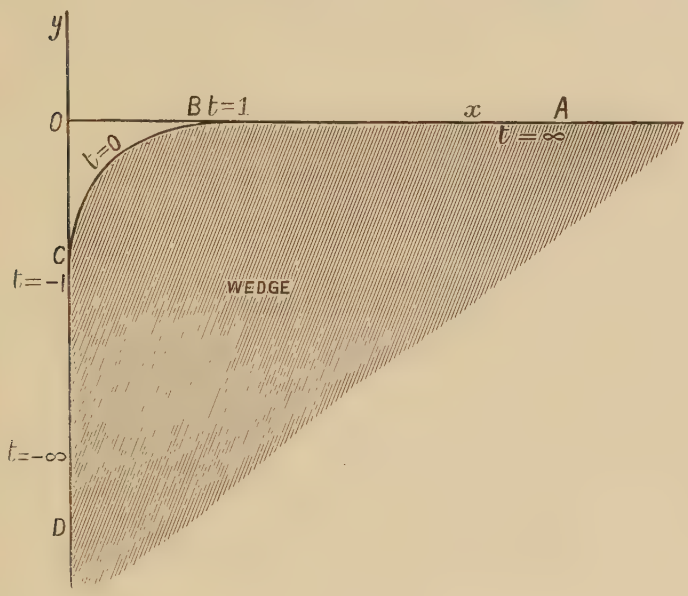
$$z = (t+1)^{\frac{3}{2}}+(t-1)^{\frac{3}{2}}.$$

Also $$w = t.$$

Therefore the relation between z and w is

$$z = (w+1)^{\frac{3}{2}}+(w-1)^{\frac{3}{2}}.$$

We take
$$z = (t+1)^{\frac{3}{2}} + (t-1)^{\frac{3}{2}}, \quad \text{if} \quad t > 1;$$
$$z = (t+1)^{\frac{3}{2}} - \iota(1-t)^{\frac{3}{2}}, \quad \text{if} \quad 1 > t > -1;$$
$$z = -\iota(-1-t)^{\frac{3}{2}} - \iota(1-t)^{\frac{3}{2}}, \quad \text{if} \quad t < -1.$$



z-plane

FIG. 10.—The wedge is the shaded portion. AB, CD are straight lines. The edge is rounded off into the curve BC.

The conductor is given by $\psi = 0$.
Along the flat part from B to A,
$$x = (\phi+1)^{\frac{3}{2}} + (\phi-1)^{\frac{3}{2}}, \quad y = 0.$$

For the curve
$$x = (\phi+1)^{\frac{3}{2}},$$
$$y = -(1-\phi)^{\frac{3}{2}}.$$

Therefore the curve is the *four-cusped hypocycloid**
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2,$$
having cusps at B and C.

The surface density over the curve is constant and equal to $\frac{1}{4\pi}\frac{\sqrt{2}}{3}$.

* I am informed that the case of a four-cusped hypocycloid was dealt with in the Mathematical Tripos, Part II, 1896 (Divs. VII, VIII, Qn. 4).

Over the flat part, the surface density is $\frac{1}{4\pi}\,\frac{2}{3}\,\frac{1}{\sqrt{\phi+1}+\sqrt{\phi-1}}$.

If we had taken $At+B\sqrt{t^2-1}$ instead of $t+\sqrt{t^2-1}$, we should have obtained curves of different shape rounding off the edge.

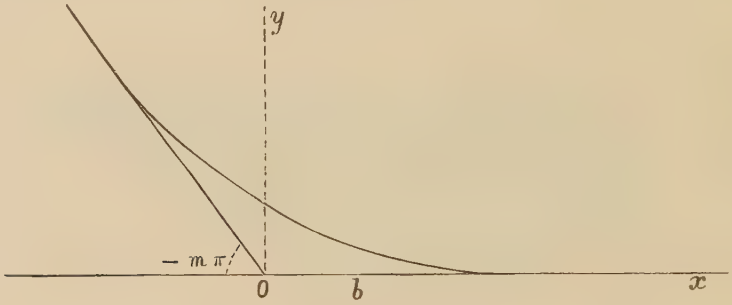


FIG. 11.

If the wedge has any angle α, from 0 to 2π, and the edge is rounded off by a hypocycloidal arc from one cusp to the next, the relation between z and w is

$$\frac{dz}{dw} = (1-m^2)\,b\left[\sqrt{\frac{w+1}{2}}+\sqrt{\frac{w-1}{2}}\right]^{2m}$$

$$= (1-m^2)\,b\,[w+\sqrt{w^2-1}]^m,$$

or

$$z = b\,[w+\sqrt{w^2-1}]^m\,[w-m\sqrt{w^2-1}].$$

Grating constructed of Conducting Cylinders of Oval Section.

Let *CDHL* represent the section of one of the cylinders. *DE*, *CB*, *BA* are lines of symmetry, and, in the electric problem, are lines of force.

We map the boundary *EDCBA* in the z-plane on the real axis in the t-plane.

As before, we consider *CD* as the limit of n sides of a polygon of $4n$ sides, and find that the appropriate relation to assume between z and t is

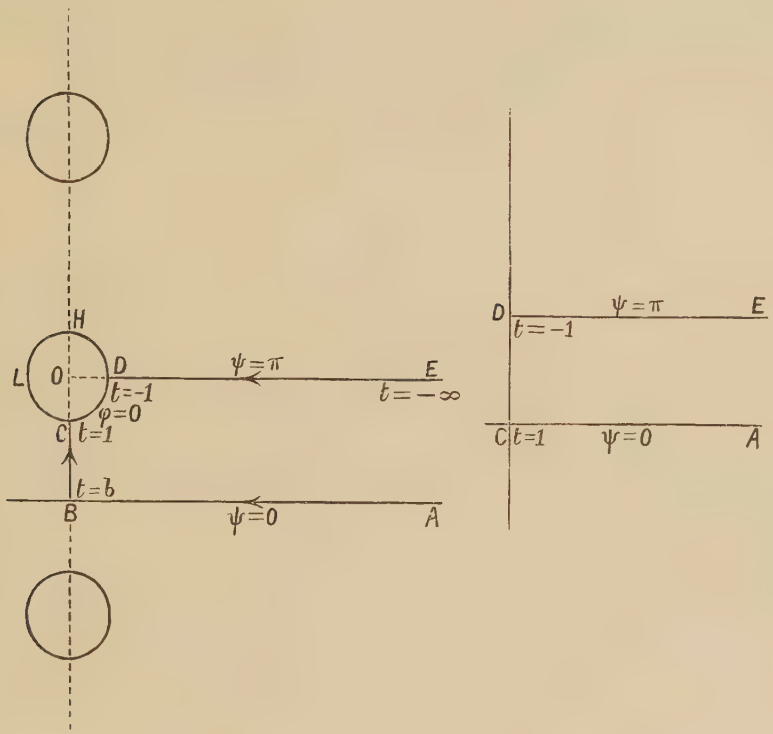
$$A\,\frac{dz}{dt} = \frac{\sqrt{t+\sqrt{t^2-1}}}{\sqrt{t-b}},$$

or

$$A\,\frac{dz}{dt} = \frac{\sqrt{t+1}+\sqrt{t-1}}{\sqrt{t-b}}.$$

Also $$\frac{dw}{dt} = \frac{1}{\sqrt{t^2-1}};$$

therefore $$\cosh w = t.$$



z-plane FIG. 12. w-plane

Therefore the relation between z and w is

$$A\frac{dz}{dw} = \frac{\sqrt{t^2-1}}{\sqrt{t-b}}\sqrt{2t+2\sqrt{t^2-1}}$$

$$= \frac{\sinh w}{\sqrt{\cosh w - b}}\sqrt{2\cosh w + 2\sinh w},$$

or $$A\frac{dz}{dw} = \frac{e^{2w}-1}{\sqrt{e^{2w}-2be^w+1}}.$$

This relation may be integrated in the form

$$A(z+C) = \sqrt{u^2-2bu+1} - \log u + b\log\frac{u-b+\sqrt{u^2-2bu+1}}{\sqrt{b^2-1}}$$

$$+\log\frac{1-bu+\sqrt{u^2-2bu+1}}{\sqrt{b^2-1}},$$

where $u = e^w$.

The curve is given by $\phi = 0$.

The shape of the curve may be obtained by integrating the relation between z and t.*

It will be found that

$$OD = \sqrt{2(b+1)} - (b-1)\log\frac{\sqrt{b+1}+\sqrt{2}}{\sqrt{b-1}},$$

$$OC = (b+1)\left(\frac{\pi}{2} - K - \tfrac{1}{2}\sin 2K\right),$$

$$BC = (b+1)(K+\tfrac{1}{2}\sin 2K) - (b-1)\frac{\pi}{2},$$

where $$\sin^2 K = \frac{b-1}{b+1}. \quad [A \text{ is taken to be unity.}]$$

Fig. 12 is drawn to scale and represents the case of $b = 3$. We have thus obtained a formula giving the field due to a charged grating composed of parallel cylindrical wires of oval section, the size of the wires bearing any relation we please to the distance between consecutive wires.

We get a very similar relation for the solution of the problem in which a liquid is flowing through the grating. It will be noticed that in the above work there is no limitation to the thickness of the wires. They may be of any size we please compared with the distance between them.

Maxwell† and Lamb† have given the solution for the case in which the thickness of the wires is supposed small compared with the distance between them.

It would be interesting to attempt to find the coefficients of reflexion and transmission of electric waves for a grating in which the wires are of considerable thickness.

J. J. Thomson† and Lamb† have found these coefficients for a grating composed of thin wires.

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* We could, of course, obtain different curves by taking

$$At + B\sqrt{t^2-1} \text{ instead of } t+\sqrt{t^2-1}.$$

† *Loc. cit.*, p. 313.

A THEORY OF BINARY QUADRATIC ARITHMETICAL FORMS WITH COMPLEX INTEGRAL COEFFICIENTS

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THIS paper was partly suggested by a very interesting memoir on the same subject by Mr. Hurwitz, which is contained in Vol. 25 of the *Acta Mathematica* (pp. 231–290). It will be seen, however, that my procedure differs from his in several important ways. In the first place, by the adoption of a geometrical method at the outset, it is shown that we can set up a definition of a reduced form which leads at once to the conclusion that the number of reduced forms is finite; we thus avoid the necessity of considering the possibility of points of condensation. Secondly, the fact that the roots of a reduced form are expressible as pure recurrent chain-fractions appears as a corollary, instead of being a definition. Thirdly (although this is a minor point), I have been able to content myself with the ordinary definition of proper equivalence ($\alpha\delta-\beta\gamma=1$) without introducing Kronecker's further restriction, namely, $(\alpha, \beta, \gamma, \delta) \equiv (1, 0, 0, 1)$ or $(0, -1, 1, 0)$ to modulus $(1+i)$. Of course, in certain connections, Kronecker's restriction is essential, or at least convenient, but not in the theory dealt with here. Finally, I have taken as the standard form $ax^2+bxy+cy^2$, and not $ax^2+2bxy+cy^2$. This seems a small matter, but it is really very important; and it is quite time that explicit binomial or multinomial coefficients should be banished from every discussion of arithmetical forms.

1. Let the plane of the complex variable z be ruled into unit squares by means of the parallel lines

$$\left.\begin{aligned} x &= m+\tfrac{1}{2} \\ y &= n+\tfrac{1}{2} \end{aligned}\right\} \quad (m, n = 0, \pm 1, \pm 2, \ldots),$$

and

so that each integral point $m+ni$ is at the centre of a unit square.

The unit square which has its centre at the origin will be called U. By definition, the points *belonging* to U are (i) all the points within its

contour, and (ii) all the points on its lowest and left-hand sides, with the exception of $\frac{-1+i}{2}$ and $\frac{1-i}{2}$.

If μ is any complex integer, other than 0, $U(\mu)$ will mean the unit square which has its centre at μ; and the points belonging to it are defined in a manner analogous to that employed for U. The result is that every point z belongs to one and only one unit square. The centre of this square may be denoted by $E(z)$, or more briefly by $[z]$. In particular

$$[\tfrac{1}{2}(-1-i)] = 0, \quad [\tfrac{1}{2}(-1+i)] = i, \quad [\tfrac{1}{2}(1+i)] = 1+i, \quad [\tfrac{1}{2}(1-i)] = 1.$$

If we invert the contour of U with respect to the circle $x^2+y^2 = 1$, we obtain a quatrefoil which I shall call Q. In the same way for each unit square $U(\mu)$ there is a corresponding quatrefoil $Q(\mu)$ obtained by inverting the contour of $U(\mu)$ with respect to the unit circle which has its centre at μ.

If μ, ν are any two integral points it will be convenient to say that they are *disjoint*, if $U(\mu)$ does not include any part of $Q(\nu)$; in this case $U(\nu)$ does not include any part of $Q(\mu)$. If the contrary is the case, μ and ν will be said to be *conjoint*.

Any two points z, z' will be said to be disjoint or conjoint, according as $[z]$ and $[z']$ are so.

2. Take any quadratic form given by

$$f = ax^2+bxy+cy^2 = (a, b, c),$$

with complex integral coefficients. Its discriminant D is given by

$$D = b^2-4ac,$$

which is supposed not to be a square.

Let ω_1, ω_2 be the roots of

$$az^2+bz+c = 0,$$

and let H, K be their representative points. We proceed to prove the following theorem :—

It is possible to find a form properly equivalent to f, such that if $\bar{\omega}_1$, $\bar{\omega}_2$ are the transforms of ω_1, ω_2, then $\bar{\omega}_1$ belongs to U, and $\bar{\omega}_2$ is not inside Q.

The points H, K will belong to two definite unit squares $U(\mu)$, $U(\nu)$, which may possibly coincide. If we apply the substitution $(z: z+\mu)$ the new form

$$f_1 = f(z+\mu) = (a, b_1, c_1)$$

has one root belonging to U. Let H_1, K_1 be the points representing the roots: then, if

$$\dots OH_1 . OK_1 \geqslant 1,$$

which implies that $|a| \leqslant |c_1|$, f_1 has the property stated. If not, we apply the "inversion" ($z : -z^{-1}$), and thus obtain

$$f_2 = (c_1, -b_1, a),$$

with root-points H_2, K_2, such that H_2 is not inside Q, and moreover

$$H_2K_2 = \frac{H_1K_1}{OH_1 . OK_1} > HK.$$

Suppose now that H_2 belongs to $U(\mu_1)$: then the "translation" ($z : z+\mu_1$) converts f_2 into

$$f_3 = (c_1, b_2, c_2),$$

with root-points H_3, K_3 of which H_3 belongs to U, and

$$H_3K_3 = H_2K_2 > HK.$$

If now $|c_1| \leqslant |c_2|$, it follows that $OH_3 . OK_3 \geqslant 1$, and f_3 is such as we require. If not, we again invert, and obtain

$$f_4 = (c_2, -b_2, c_1),$$

with roots H_4, K_4, such that H_4 is not inside Q, and

$$H_4K_4 = \frac{H_3K_3}{OH_3 . OK_3} > H_3K_3.$$

Now let H_4 be within $U(\mu_2)$: then the translation ($z : z+\mu_2$) leads to

$$f_5 = (c_2, b_3, c_3),$$

with root-points H_5, K_5, of which H_5 belongs to U, while

$$H_5K_5 = H_4K_4 > H_3K_3.$$

If $|c_2| \leqslant |c_3|$, our object is attained: if not, we repeat the process, so as to reach a new form f_7, and so on.

Now consider the sequence of forms

$$f_1, f_3, f_5, f_7, \dots,$$

each of these has a root, properly equivalent to ω_1, which belongs to U, and moreover

$$H_1K_1 < H_3K_3 < H_5K_5 < \dots,$$

that is to say,

$$\left|\frac{\sqrt{D}}{a}\right| < \left|\frac{\sqrt{D}}{c_1}\right| < \left|\frac{\sqrt{D}}{c_2}\right| < \dots,$$

where a, c_1, c_2, ... are the first coefficients of $f_1, f_3, f_5, \ldots$. Hence

$$|a|, \quad |c_1|, \quad |c_2|, \quad \ldots$$

is a diminishing sequence of positive real integers, which must eventually stop.

Let f_{2h+1}, or (c_h, b_{h+1}, c_{h+1}), be the last form at which we arrive: then if H_{2h+1}, K_{2h+1} are its root-points, we must have

$$OH_{2h+1}.OK_{2h+1} \geqslant 1,$$

because otherwise we could apply an inversion followed by a translation so as to increase the distance between the root-points. This proves the theorem. It should be observed that we may take either root of the original form as ω_1, so, in general, we can get two equivalent forms with the property stated.

3. We now introduce a definition of a reduced form.

A form is reduced when one of its roots belongs to U, and the other is not inside Q.

It will be noticed that if (a, b, c) is reduced, so, generally speaking, is its inverse $(c, -b, a)$. The only exception is when the second root of (a, b, c) lies on a part of the contour of Q which inverts into one of the sides of U which belong to $U(1)$, $U(i)$, respectively.

What we have proved is that every form is equivalent to at least one reduced form (a, b, c) with $|a| \leqslant |c|$. The next step is to show that the number of reduced forms is finite. Suppose that (a, b, c) is a reduced form with root-points H, K. The shortest distance of a point on the contour of Q from a point on the contour of U is

$$\sqrt{2} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}};$$

hence
$$|\omega_2 - \omega_1| = \left|\frac{\sqrt{D}}{a}\right| \geqslant \frac{1}{\sqrt{2}};$$

and therefore
$$|a| \leqslant |\sqrt{2D}|, \quad N(a) \leqslant 2|D|, \tag{1}$$

where $N(a)$ means the norm of a. Similarly,

$$|c| \leqslant |\sqrt{2D}|, \quad N(c) \leqslant 2|D|, \tag{2}$$

and finally, since
$$b^2 = D + 4ac,$$

$$|b|^2 \leqslant |D| + 4|ac| \leqslant 9|D|,$$

whence
$$|b| \leqslant 3|\sqrt{D}|, \quad N(b) \leqslant 9|D|. \tag{3}$$

It follows that each of the integers a, b, c can have only a finite number of values, and the number of reduced forms is accordingly finite. The limiting conditions (1), (2), (3) are necessary but not sufficient.

4. Since every form is equivalent to a reduced form, we have only to consider the equivalence or non-equivalence of reduced forms.

Let (a_0, b_0, a_1) be any reduced form: suppose ω_1 the root that belongs to U, and that $U(\mu)$ is the unit square to which the other root ω_2 belongs. In the first place let us assume that 0, μ are disjoint. Then the translation $(z: z+\mu)$ brings ω_2 to a point belonging to U, and ω_1 to a point which is outside Q. Now apply the inversion $(z: -z^{-1})$, the resulting form will be reduced. The combination of the translation and the inversion is

$$(z: \mu - z^{-1}),$$

and the new reduced form is

$$(a_{-1}, b_{-1}, a_0),$$

adjacent to (a_0, b_0, a_1) on the left hand.

Of the two roots of a reduced form, I shall say that the one belonging to U is the minor root, and the other the major root. Thus the transformation of (a_0, b_0, a_1), or f, in the present case, leads to an adjacent form $f_{-1} = (a_{-1}, b_{-1}, a_0)$, the minor root of which is the transform of the minor root of f.

Suppose further that $-\omega_2^{-1}$, $-\omega_1^{-1}$, the roots of $(a_1, -b_0, a_0)$ are disjoint. Then, if $-\omega_1^{-1}$ belongs to $U(\mu')$, the inversion $(z: z^{-1})$ followed by the translation $(z: z+\mu')$, or, which is the same thing, the transformation

$$[z: (\mu' - z)^{-1}],$$

changes f into a reduced form (a_1, b_1, a_2) adjacent to it on the right. Hence

If the roots of a reduced form are disjoint, and also those of the inverse form, there are dexter and sinister reduced forms adjacent to it. In each case the minor root of the adjacent form is the transform of the minor root of the original form.

5. It remains, then, to consider reduced forms which fail to satisfy one or other of the foregoing conditions. These I shall call *critical* forms. The major root of a critical form belongs to $U(\mu)$, where 0, μ are conjoint.

On drawing a figure, we see that if 0, μ are conjoint, and μ is not inside Q, μ can have only a limited number of values: these are, in fact,

given by

$$\mu = \pm 1 \pm i, \quad \pm 1 \pm 2i, \quad \pm 2 \pm i, \quad \pm 2, \quad \pm 2i.$$

On account of the symmetry of the axes, there are only three really distinct cases to consider, namely,

$$\mu = 1+i, \quad 1+2i, \quad 2.$$

CASE I: $\mu = 1+i$.—For a critical form we have ω_1 belonging to U and not outside $Q(1+i)$: moreover ω_2 is not outside the quadrilateral $ABCD$ (Fig. 1). Apply the translation $(z: z+1+i)$: then the quadrilateral takes up the position shown by the dotted contour, while the domain of ω_1 (*i.e.*, the transform of the original ω_1) is the area $PQRSTU$, or W. Now invert these domains: that of ω_1 becomes W', and that of ω_2 becomes that part of the quadrant $x \leqslant \frac{1}{2}$, $y \geqslant -\frac{1}{2}$, which is not inside Q.

To bring ω_1 from its present position into U, we must apply one of three translations, according to the part of W' to which ω_1 belongs: these are $(z: z-i)$, $(z: z+1-i)$, $(z: z+1)$ respectively. Now each of these translations moves the domain of ω_2 away from Q: hence the result is a reduced form.

The final result may be expressed in the following way. If f is a reduced form with $[\omega_2] = 1+i$, then one, and only one, of the transformations

$$\left.\begin{aligned} &\text{(i)} \quad z = 1+i-\frac{1}{z'} \\ &\text{(ii)} \quad z = 1+i-\frac{1}{z'-i} \\ &\text{(iii)} \quad z = 1+i-\frac{1}{z'+1-i} \\ &\text{(iv)} \quad z = 1+i-\frac{1}{z'+1} \end{aligned}\right\} \tag{4}$$

will convert f into a reduced form. If the first transformation is the right one, f is normal: if not, it is critical. The minor root of f becomes the minor root of the new form: and in each case we shall say that the new form is adjacent to f on the left hand. Thus, when $[\omega_2] = 1+i$, we can always find one, and only one, form adjacent to f on the left hand.

CASE II: $\mu = 1+2i$.—Suppose f is a critical form with $[\omega_2] = 1+2i$. Then it can be seen from Fig. 2, p. 336, that the domain of ω is a pentagon, and that of ω_1 a triangle cut off by $Q(1+2i)$ from U. After applying the translation $(z: z+1+2i)$, followed by an inversion, or, in other words,

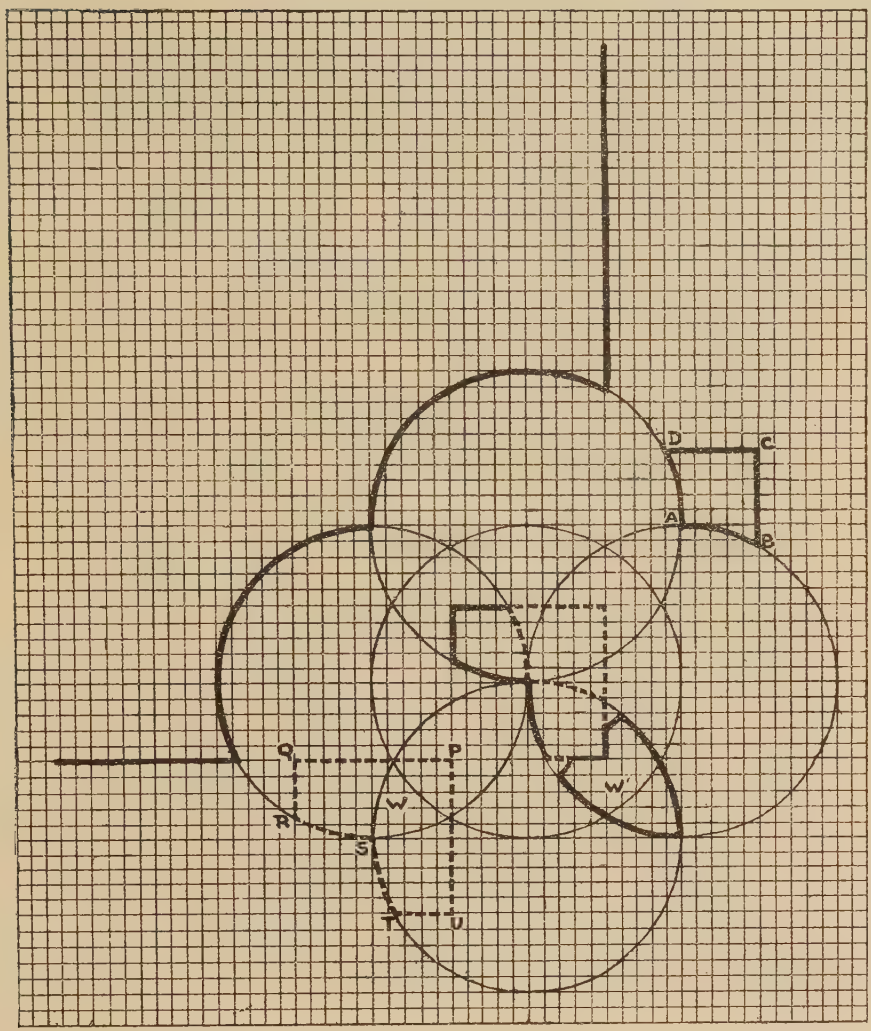

FIG. 1.

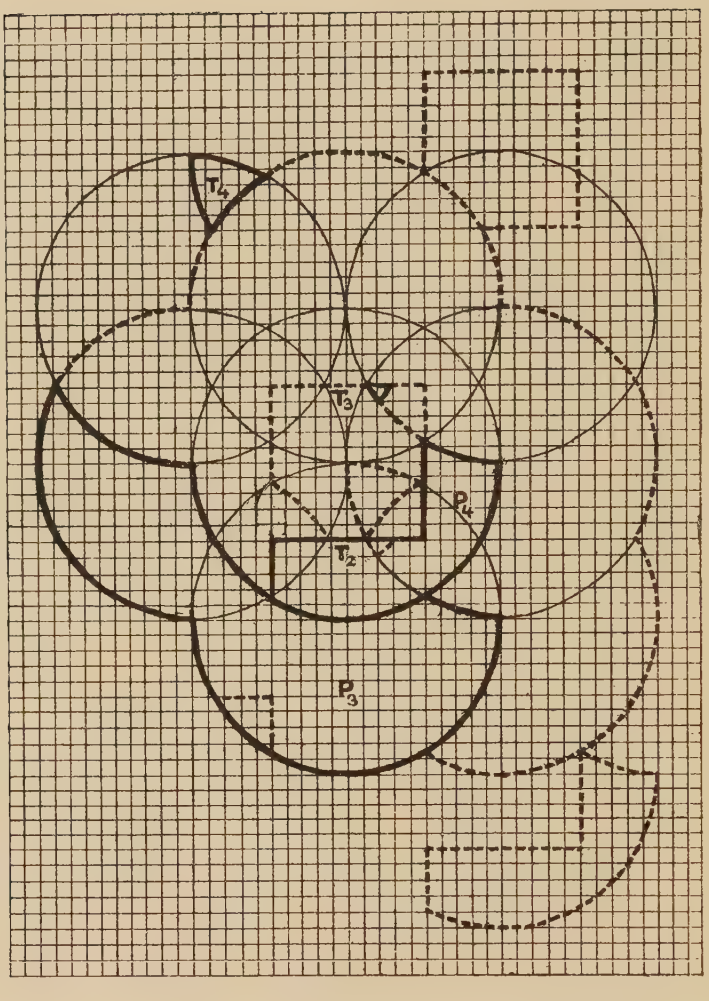

FIG. 2.

the substitution $(z : 1+2i-z^{-1})$,

the new domain of ω_2 is the space not inside a cinquefoil C, and that of ω_1 is a small triangle marked T_2 in the figure. The translation $(z : z-i)$ brings T_2 to T_3, and the result is generally a reduced form. It is not so, however, if after the preceding substitution ω_2 lies between C and $Q(-i)$: in this case, after the last substitution, the domain of ω_2 is a pentagon marked P_3 in the figure, having two sides on the contour of Q. If we now invert, the result is that T_3 becomes T_4 which is outside Q and inside $U(-1+2i)$, also P_3 becomes the pentagon marked P_4 in the figure.

Finally, if we apply $(z : z-1+2i)$ this brings T_4 within U and carries P_4 outside Q : so the resulting form is reduced, with a single exception, namely, when

$$\bar{\omega}_2 = \tfrac{1}{2}\{1-(2-\sqrt{3})i\},$$

in which case we must further apply $(z : z+1)$. Omitting this special case, and observing that one side of T_2 belongs to U, we conclude that—

If $[\omega_2] = 1+2i$, one, and only one, of the substitutions

$$\left.\begin{array}{ll} \text{(i)} & z = 1+2i-\dfrac{1}{z'} \\ \text{(ii)} & z = 1+2i-\dfrac{1}{z'-i} \\ \text{(iii)} & z = 1+2i-\dfrac{1}{-i-\dfrac{1}{z'-1+2i}} \end{array}\right\} \quad (5)$$

will convert f into a reduced form. In each case, as before, this will be called the form adjacent to f on the left hand.

Case III : $\mu = 2$.—On referring to Fig. 3 it will be seen that for a critical form f of this category the domain of ω_2 is a quadrilateral cut off from $U(2)$ by Q, and that of ω_1 a quadrilateral cut off from U by $Q(2)$. After applying $(z : 2-z^{-1})$ the domain of ω_1 becomes the quadrilateral marked F in the figure, and that of ω_2 becomes that part of the half-plane $x \leqslant \frac{1}{2}$ which is not inside Q. To bring ω_1 inside U, we must apply $(z : z+1)$, the effect of which is to move ω_2 away from Q_0 : so the resulting form is reduced. Thus—

If $[\omega_2] = 2$, one, and only one, of the substitutions

$$\left.\begin{array}{ll} \text{(i)} & z = 2-\dfrac{1}{z'} \\ \text{(ii)} & z = 2-\dfrac{1}{z'+1} \end{array}\right\} \quad (6)$$

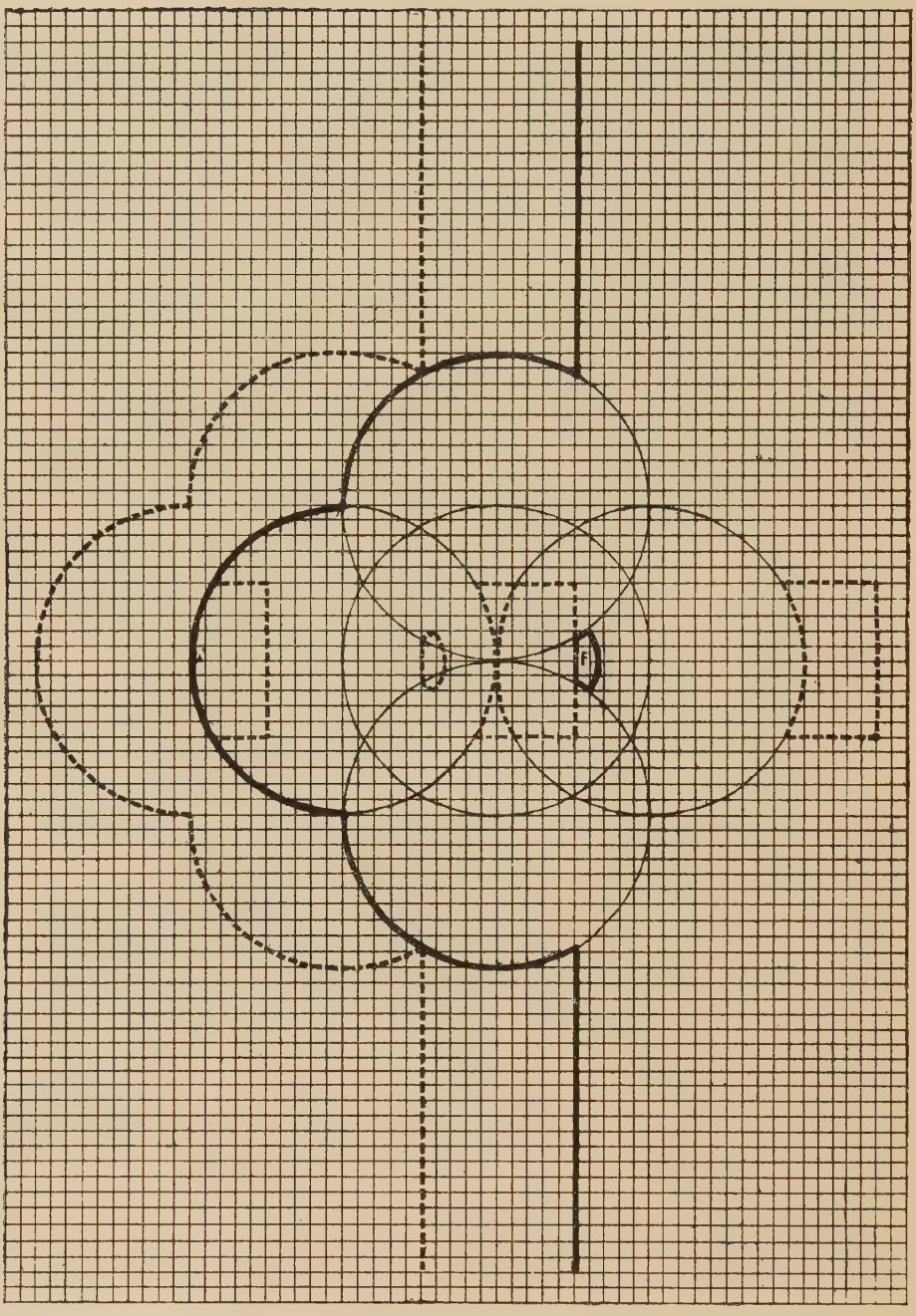

FIG. 3.

will convert f into a reduced form. As before, we say that this is adjacent to f on the left hand.

Since precisely similar arguments apply to the other cases of critical forms, we conclude that every reduced form has one, and only one, reduced form adjacent to it on the left hand.

6. Consider, now, $(c, -b, a)$, the inverse of a reduced form (a, b, c). This is, in general, reduced, and will have a determinate form f' adjacent to it on the left hand. The inverse of f' will be called the form adjacent to f on the right. It follows now that (in general) every reduced form has one, and only one, form adjacent to it on the left and right respectively. The case which we postpone for the present is when $(c, -b, a)$ has one root-point on one of the lines $x = \frac{1}{2}$, $y = \frac{1}{2}$.

Let f be any reduced form, f_1 the adjacent form to the right, f_2 the adjacent to f_1 on the right, and so on. Then since the number of reduced forms is finite, we must arrive at an identity

$$f_r = f_{r+s}.$$

But identical forms have the same adjacent form on the left: hence

$$f_{r-1} = f_{r+s-1}, \quad f_{r-2} = f_{r+s-2}, \quad \ldots;$$

and, finally,

$$f = f_s.$$

Thus f is a member of a cycle of reduced forms

$$(f, f_1, f_2, \ldots, f_{s-1}, f, f_1, \ldots),$$

which is periodic and such that if f_h is any one of them, f_{h-1} and f_{h+1} are its adjacents to the left and right respectively.

Suppose, in particular, that f_h is critical to the left: then f_{h-1} is derived from it by one of the special substitutions given by (4), (5), (6) above. The next substitution will be of the form $(z' : \mu' - z^{-1})$, and consequently if μ_i is the last partial quotient in the special substitution, this is changed into $\mu_i + \mu'$. A consideration of Figs. 1–3 shows that $\mu_i + \mu'$ cannot vanish: hence, whether or not critical forms occur, we have a determinate sequence of substitutions

$$z = \mu_1 - \frac{1}{\mu_2} - \frac{1}{\mu_3} - \ldots - \frac{1}{\mu_h} - \frac{1}{z'}, \tag{7}$$

which give us all the reduced forms of the period, together with certain interpolated or link forms, if critical reduced forms occur. Moreover, each minor root is transformed into another minor root.

Let the period of f obtained by substitutions of type (7) be

$$(f, f_{-1}, f_{-2}, \ldots, f_{-s+1}):$$

and let S_{-r} be the substitution which changes f_{-r+1} into f_{-r}. Then, if we put

$$S_{-1}S_{-2}\ldots S_{-s} = S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

S converts f into f_{-s}, that is, into itself. Moreover, since the transform of the minor root is the minor root,

$$\omega_1 = \frac{a\omega_1+\beta}{\gamma\omega_1+\delta}$$

or

$$\gamma\omega_1^2+(\delta-\alpha)\omega_1-\beta = 0.$$

Suppose, now, that $f = (a, b, c)$ and is primitive: that is to say, let a, b, c have no common divisor. Then we must have

$$\gamma = au, \quad \delta-\alpha = bu, \quad -\beta = cu,$$

where u is some integer. Hence

$$Du^2 = (\delta-\alpha)^2+4\beta\gamma = (\delta+\alpha)^2-4;$$

so if we put

$$\delta+\alpha = t,$$

(t, u) is an integral solution of

$$t^2-Du^2 = 4, \tag{8}$$

and

$$S = \begin{pmatrix} \frac{1}{2}(t-bu), & -cu \\ au, & \frac{1}{2}(t+bu) \end{pmatrix}. \tag{9}$$

We have also

$$S^{-1} = \begin{pmatrix} \frac{1}{2}(t+bu), & cu \\ -au, & \frac{1}{2}(t-bu) \end{pmatrix},$$

and each of these is a fundamental automorph of f. Written out in full

$$\left.\begin{aligned} S(z) &= \mu_1-\cfrac{1}{\mu_2-\cfrac{1}{\mu_3-\ldots-\cfrac{1}{\mu_h-\cfrac{1}{z}}}} \\ S^{-1}(z) &= \cfrac{1}{\mu_h-\cfrac{1}{\mu_{h-1}-\ldots-\cfrac{1}{(\mu_1-z)}}} \end{aligned}\right\}, \tag{10}$$

where $h = s$ or $h > s$, according as the period does not, or does, contain critical forms.

Hence also we have pure recurring chain-fraction expressions for the roots of f: namely,

$$\left.\begin{aligned} \omega_2 &= \mu_1-\cfrac{1}{\mu_2-\ldots-\cfrac{1}{\mu_h-\cfrac{1}{\mu_1-\ldots}}} \\ \omega_1 &= \cfrac{1}{\mu_h-\cfrac{1}{\mu_{h-1}-\ldots-\cfrac{1}{\mu_1-\cfrac{1}{\mu_h-\ldots}}}} \end{aligned}\right\}. \tag{11}$$

7. At this stage it may be convenient to give a numerical example. Suppose then, that

$$D = 5+4i.$$

We have, for a reduced form (a, b, c),

$$N(a), \quad N(c) < 2\sqrt{41} < 13,$$

$$N(b) < 9\sqrt{41} < 59{\cdot}6\,;$$

hence $|b| < 8$, and moreover $b \equiv 1 \pmod 2$. We take all values of b satisfying these conditions: for each of them we calculate $\frac{1}{4}(b^2-D)$, factorise it in all possible ways, rejecting all those products ac for which $N(a)$ or $N(c) \geqslant 13$, and finally calculate approximately the roots of the remaining forms so as to find out which are reduced. The result is that there are twenty-four reduced forms, namely,

$$\pm(1, \pm 3, 1-i), \ \pm(1-i, \pm 3, 1), \ \pm(i, \pm 3, -1-i),$$

$$\pm(1+i, \pm 3, -i), \ \pm[1+i, \pm(3+2i), 1+i], \ \pm[1-i, \pm(3+2i), -1+i].$$

The major root of $(1, -3, 1-i)$ is in $U(3)$: so applying $(z : 3-z^{-1})$, we get $(1-i, -3, 1)$, of which the major root is in $U(1+2i)$. This form is not critical, and $(z : 1+2i-z^{-1})$ converts it into $(-1+i, -3-2i, 1-i)$ with its major root in $U(-1-2i)$. This transforms into $(-1, -3, -1+i)$, and so on. Finally, we have a period of six forms, and arranging them in the opposite order, the period is

$$P = (1, -3, 1-i) \sim (1-i, -3-2i, -1+i) \sim (-1+i, -3, -1)$$
$$\sim (-1, -3, -1+i) \sim (-1+i, -3-2i, 1-i) \sim (1-i, -3, 1)$$
$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} P.$$

Another period is

$$\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix} P = (1, 3, 1-i) \sim (1-i, 3+2i, -1+i) \sim (-1+i, 3, -1)$$
$$\sim (-1, 3, -1+i) \sim (-1+i, 3+2i, 1-i) \sim (1-i, 3, 1)$$
$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} P.$$

Another period is

$$Q = (i, 3, -1-i) \sim (-1-i, 3, i) \sim (-1-i, 3+2i, -1-i)\,;$$

or, in the opposite order,

$$Q = (i,\ 3,\ -1-i) \sim (-1-i,\ 3+2i,\ -1-i) \sim (-1-i,\ 3,\ i).$$

The transformations $\begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}$, $\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$, $\begin{pmatrix} 0, & i \\ i, & 0 \end{pmatrix}$ change Q into three other equivalent periods, and the reduced forms are now exhausted.

Now the periods P, Q are equivalent: in fact,

$$\begin{pmatrix} 1, & -2 \\ 1, & -1 \end{pmatrix} (1,\ -3,\ 1-i) = (-1-i,\ 3+2i,\ -1-i);$$

the reason of this, as we shall see later on, is that the equation

$$t^2-(5+4i)\,u^2 = 4i$$

admits of solution. Hence, finally, for $D = 5+4i$ there is only one class, and its signature is I, 1.

The automorph of $(1,\ -3,\ 1-i)$ is

$$z = \frac{(10+7i)\,z-6+2i}{(4+2i)\,z-2+i},$$

and hence $\qquad t = 8+8i, \qquad u = 4+2i,$

is a fundamental solution of

$$t^2-(5+4i)\,u^2 = 4.$$

The automorph of $(i,\ 3,\ -1-i)$ is

$$z = \frac{(10+7i)\,z-2-6i}{(2-4i)\,z-2+i},$$

leading to $t = 8+8i$, $u = -(4+2i)$, which is practically the same as before.

The major root of $(1,\ -3,\ 1-i)$ has for its chain-fraction expansion

$\omega_2(1,\ -3,\ 1-i)$

$$= 3-\frac{1}{(1+2i)}-\frac{1}{(-1-2i)}-\frac{1}{(-3)}-\frac{1}{(-1-2i)}-\frac{1}{(1+2i)}-\frac{1}{3}-\ldots,$$

and similarly

$$\omega_2(i,\ 3,\ -1-i) = 3i-\frac{1}{(2-i)}-\frac{1}{(2-i)}-\frac{1}{3i}-\ldots.$$

A fundamental solution of

$$t^2-(5+4i)\,u^2 = 4i,$$

is, in fact, $$t = 3-i, \quad u = 1-i;$$

and from this we can deduce the substitution

$$\begin{pmatrix} 1, & -2i \\ 1+i, & 3-2i \end{pmatrix},$$

which converts $(1, -3, 1-i)$ into $(-i, -3, 1+i)$.

8. There are several points illustrated by this particular example. In the first place, if (a, b, c) is any reduced form, we may apply to it the three elementary transformations

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}, \quad \begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix},$$

and thus obtain $(-c, -b, -a)$, $(c, -b, a)$, $(-a, b, -c)$ equivalent to it. Each of these may, or may not, belong to the period of (a, b, c). If ω_1, ω_2 are the roots of (a, b, c), the roots of the other forms are respectively $(\omega_2^{-1}, \omega_1^{-1})$, $(-\omega_2^{-1}, -\omega_1^{-1})$ and $(-\omega_1, -\omega_2)$.

We have also the reduced form

$$(-ia, b, ic),$$

the roots of which are $(i\omega_1, i\omega_2)$. This may, or may not, be equivalent to (a, b, c). If it is, we shall have

$$\frac{a\omega_1+\beta}{\gamma\omega_1+\delta} = i\omega_1,$$

with $$a\delta-\beta\gamma = 1.$$

From this we deduce successively

$$i\gamma\omega_1^2+(\delta i-a)\omega_1-\beta = 0,$$

$$i\gamma = ua,$$

$$\delta i-a = ub,$$

$$-\beta = uc,$$

$$Du^2 = (\delta i-a)^2+4i\beta\gamma$$

$$= (\delta i+a)^2-4i;$$

consequently, if we put $$t = \delta i+a,$$

we shall have $$t^2-Du^2 = 4i. \tag{12}$$

This is impossible unless $(i \mid D) = 1$; that is to say, no prime divisor

of D can have a norm of the type $8n+5$. Suppose that (t, u) is actually an integral solution of (12), then the substitution

$$\begin{pmatrix} -\frac{1}{2}i(t+bu), & cu \\ iau, & \frac{1}{2}(t-bu) \end{pmatrix}$$

converts (a, b, c) into $(-ia, b, ic)$.

We conclude from this that, if D has a prime divisor whose norm is of the type $8n+5$, there must be at least two classes. For instance, if $D = 5$, the reduced forms are

$$\pm(1, \pm 3, 1), \quad \pm(i, \pm 3, -i);$$

the distinct periods are derived from

$$P = (1, -3, 1), \quad Q = (i, -3, -i),$$

and the signature is II, 1. The simplest forms are

$$(1, 1, -1), \quad \text{character } (f \mid 1+2i) = +,$$
$$(i, 1, i), \quad \text{character } (f \mid 1+2i) = -.$$

Fermat's theorem on the decomposition of a real prime into the sum of two squares, when it is of the type $4n+1$, is so celebrated that it may be worth while to record the analogous theorem connected with the determinants $\pm 4i$. It is as follows:

If μ is a complex prime, the norm of which is not of the type $(8n+5)$, then the equations

$$x^2+iy^2 = \mu, \quad x^2-iy^2 = \mu,$$

each admit of an infinite number of solutions, which are all connected by means of the solutions of $t^2+4iu^2 = 1$.

For instance,
$$\begin{aligned} 3 &= 1^2+i(1-i)^2 = 1^2-i(1+i)^2 \\ &= (3-4i)^2+i(5-i)^2 \\ &= (17-24i)^2+i(29-5i)^2, \end{aligned}$$

and so on: the general formula being

$$3 = \{t_n-(2+2i)u_n\}^2+i\{2u_n+(1-i)t_n\}^2,$$

where t_n, u_n are obtained in the usual way from

$$t_n+2u_n\sqrt{-i} = [3+(2+2i)\sqrt{-i}]^n.$$

As an example of the actual reduction of a prime to this form, let us take $\mu = 7+8i$.

First of all, we have to solve

$$\lambda^2 \equiv -4i \pmod{28+32i}.$$

Now $$0 \equiv (7+8i)(-1+i) \equiv -15-i \pmod{7+8i},$$

so putting $\lambda = 2x$, we have

$$x^2 \equiv 15 \pmod{7+8i}.$$

The congruence $$x^2 \equiv 15 \pmod{113}$$

is satisfied by $x \equiv \pm 44$; hence we find that $\lambda \equiv 88 \pmod{7+8i}$ will do. Since $88 = -2+6.15$, we may put $\lambda = -2-6i$, and thus we get the form

$$f = (7+8i,\ -2-6i,\ i),$$

with discriminant $-4i$. We find that

$$\begin{pmatrix} 0, & -1 \\ 1, & -3+i \end{pmatrix} f = (i,\ 0,\ 1),$$

whence, conversely, $$\begin{pmatrix} -3+i, & 1 \\ -1, & 0 \end{pmatrix} (i,\ 0,\ 1) = f;$$

and therefore $$7+8i = 1^2+i(3-i)^2,$$

from which all the other representations can be derived.

In the case of a real prime, we can make use of the real representations $\mu = x^2 \pm 2y^2$, which are equivalent to $\mu = x^2+i(y \pm yi)^2$; but complex primes have to be treated as above.

9. Allusion has already been made to the case where a form has its minor root on the boundary of U. More generally, let us consider the case of a form (a, b, c) with one root on the boundary of a unit square; such a form we may call a limiting form.

Let us write

$$a = \alpha_1+\alpha_2 i, \quad b = \beta_1+\beta_2 i, \quad c = \gamma_1+\gamma_2 i,$$
$$D = d_1+d_2 i, \quad \sqrt{D} = \delta_1+\delta_2 i,$$

where $$\delta_1 = \sqrt{\tfrac{1}{2}(\sqrt{d_1^2+d_2^2}+d_1)}, \quad \delta_2 = \sqrt{\tfrac{1}{2}(\sqrt{d_1^2+d_2^2}-d_1)},$$

the roots being chosen so that δ_1, δ_2 are real, and $2\delta_1\delta_2 = d_2$.

One root of f, then, is given by

$$\omega = \frac{(\delta_1-\beta_1)+(\delta_2-\beta_2)\,i}{2(\alpha_1+\alpha_2 i)} = x+yi,$$

and hence $$x = \frac{1}{2(\alpha_1^2+\alpha_2^2)}[-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_1\delta_1+\alpha_2\delta_2].$$

Now, if d_2 is not zero, neither δ_1 nor δ_2 is rational, and x cannot be rational either. So the first condition for a limiting form is that D is a real integer; and hence either $\delta_2 = 0$ or $\delta_1 = 0$, according as D is positive or negative.

Suppose first that $\delta_2 = 0$: then x is rational if $\alpha_1 = 0$. We have now a limiting form if

$$\frac{1}{2\alpha_2^2}(-\alpha_2\beta_2) = m+\tfrac{1}{2},$$

where m is a real integer. The form is

$$f = [\alpha_2 i,\ \beta_1-(2m+1)\alpha_2 i,\ \gamma_1+\gamma_2 i],$$

with the condition

$$D = \{\beta_1-(2m+1)\alpha_2 i\}^2-4\alpha_2 i(\gamma_1+\gamma_2 i),$$

which, since D is real, gives

$$D = \beta_1^2-(2m+1)^2\alpha_2^2+4\alpha_2\gamma_2,$$

$$(2m+1)\beta_1\alpha_2+2\alpha_2\gamma_1 = 0.$$

Therefore we must have

$$\beta_1 = 2n, \qquad \gamma_1 = -(2m+1)n,$$

and the form is

(i) $$f = \{\alpha_2 i,\ 2n-(2m+1)\alpha_2 i,\ -(2m+1)n+\gamma_2 i\},$$

with $$D = 4n^2-(2m+1)^2\alpha_2^2+4\alpha_2\gamma_2.$$

Conversely, when D can be thrown into this shape, there is a limiting form, with roots given by

$$\omega = m+\tfrac{1}{2}+\frac{(2n\pm\sqrt{D})\,i}{2\alpha_2},$$

each of which lies on $x = m+\frac{1}{2}$.

The other cases are quite similar: thus we have

(ii) $D = -\Delta < 0$,

$$\Delta = 4n^2-(2m+1)^2\alpha_1^2+4\alpha_1\gamma_1,$$

$$f = [\alpha_1,\ -(2m+1)\alpha_1+2ni,\ \gamma_1-(2m+1)ni],$$

$$\omega = m+\tfrac{1}{2}-\frac{(2n\pm\sqrt{\Delta})\,i}{2\alpha_1}.$$

(iii) $D > 0$, $\quad D = 4m^2-(2n+1)^2\alpha_1^2-4\alpha_1\gamma_1,$

$$f = [\alpha_1,\ 2m-(2n+1)\alpha_1 i,\ \gamma_1-(2n+1)mi],$$

$$\omega = \frac{-2m \pm \sqrt{D}}{2\alpha_1} + (n+\tfrac{1}{2})\, i.$$

(iv) $D = -\Delta < 0$, $D = 4m^2-(2n+1)^2\alpha_2^2-4\alpha_2\gamma_2,$

$$f = [\alpha_2 i,\ (2n+1)\alpha_2+2mi,\ (2n+1)m+\gamma_2 i],$$

$$\omega = \frac{-2m \pm \sqrt{\Delta}}{2\alpha_2} + (n+\tfrac{1}{2})\, i.$$

For a reduced limiting form we must have $m = -1$ in cases (i), (ii), and $n = -1$ in cases (iii), (iv). The further conditions of reduction are respectively

$$(2n \pm \sqrt{D})^2 \leqslant \alpha_2^2, \quad (2n \pm \sqrt{\Delta})^2 \leqslant \alpha_1^2,$$

$$(2m \pm \sqrt{D})^2 \leqslant \alpha_1^2, \quad (2m \pm \sqrt{\Delta})^2 \leqslant \alpha_2^2.$$

The simplest example is when $D = 3$. In this case there are sixteen reduced forms, namely,

$$\pm(1,\ -2+i,\ -i), \quad \pm(-1,\ -2-i,\ -i),$$

$$\pm(i,\ -2+i,\ -1), \quad \pm(-i,\ -2-i,\ -1),$$

and their opposites; all of these are limiting, and all properly equivalent. A similar result holds good for $D = -3$; in fact, we have only to multiply each coefficient by i to obtain a set of reduced forms for $D = -3$, and so in general.

10. It appears from the foregoing that when, by the rules which have been explained, we have constructed the period P to which a given reduced form belongs, there may be 1, 3, or even 7 other periods properly equivalent to P; the last case occurring only when $t^2-Du^2 = 4i$ admits of integral solutions (and not always then). Now when we have two or more equivalent periods we can join them up into a single period; but it is really more advantageous to keep them distinct, because we thus obtain the primary automorphs more quickly, and obtain the fundamental chain-fraction expression for $\sqrt{D}$ [or $\frac{1}{2}\sqrt{D}$, when $D \equiv 0 \pmod 4$]. For instance, when $D = 4(3+2i)$, there are two (and I believe only two) distinct periods

$$P = (1,\ -4-2i,\ 2i) \sim (2i,\ -4-2i,\ 1),$$

$$Q = (i,\ -4-2i,\ 2) \sim (2,\ -4-2i,\ i);$$

the equation $t^2-Du^2 = 4i$ has no integral solutions, and each of the periods P, Q is associated with three other ones, obtained by applying

$\begin{pmatrix} 0, & i \\ i, & 0 \end{pmatrix}$, $\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$, and $\begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}$. We have

$$\omega_2(1, -4-2i, 2i) = 4+2i-\frac{1}{(1-2i)}-\frac{1}{(4+2i)}-\ldots,$$

and hence $$\sqrt{3+2i} = 2+i-\frac{1}{\underset{*}{(1-2i)}}-\frac{1}{\underset{*}{(4+2i)}}-\frac{1}{(1-2i)}-\ldots,$$

which may be contrasted with Mr. Hurwitz's expansion (*l.c.*, p. 262). We also arrive at once at $(3-3i, 1-2i)$ as a primary solution of

$$t^2-(3+2i)u^2 = 1.$$

Methods similar to those employed by Mr. Hurwitz may be applied to show that two periods cannot be properly equivalent unless they are associated in one of the ways that have been indicated.

11. It ought, perhaps, to have been remarked before that not every complex integer can be the discriminant of a complex form. We must have, in fact,

$$D \equiv 0, -1, 1, 2i \pmod 4,$$

and correspondingly $$b = 0, i, 1, 1+i \pmod 2.$$

Conversely, if any one of these conditions is satisfied, we have at least one form of determinant D: namely,

$$D = 4\mu, \qquad f = (1, 0, -\mu),$$
$$D = 4\mu-1, \quad f = (1, i, -\mu),$$
$$D = 4\mu+1, \quad f = (1, 1, -\mu),$$
$$D = 4\mu+2i, \quad f = (1, 1+i, -\mu).$$

I conclude by giving a table of results obtained for some of the smaller values of D. In each case, F denotes the "simplest" form of its class.

$$D = 3 \quad (\mathrm{I}, 1).$$
$$F = (1, -i, -1),$$
$$P = (1, -2+i, -i) \sim (-i, -2+i, 1),$$

with seven associates.

$$(2+i)^2-3.1^2 = 4i.$$

$$D = 4i \quad (\mathrm{I}, 1).$$

$$F = (1, 0, -i),$$

$$P = (1, -2-2i, i) \sim (i, -2-2i, 1),$$

with three associates.

$$3^2 - 4i(1-i)^2 = 1.$$

$$D = 1+4i \quad (\mathrm{I}, 1).$$

$$F = (1, -1, -i),$$

$$P = (1, -1-2i, -1) \sim (-1, -1-2i, 1),$$

with three associates.

$$i^2 - (1+4i)i^2 = 4i.$$

$$D = 5 \quad (\mathrm{II}, 1).$$

$$F_1 = (1, -1, -1), \qquad (f_1 | 1+2i) = +$$

$$P_1 = (1, -3, 1),$$

with three associates.

$$F_2 = (i, 1, i), \qquad (f_2 | 1+2i) = -$$

$$P_2 = (i, 3, -i) \sim (-i, 3, i),$$

with three associates.

$$3^2 - 5.1^2 = 4.$$

$$D = 4+2i \quad (\mathrm{II}, 1).$$

$$F_1 = (1, -1-i, -1), \qquad (f_1 | 1+2i) = +$$

$$P_1 = (1, -3-i, 1+i) \sim (1+i, -3-i, 1),$$

with three associates.

$$F_2 = (i, 1+i, i), \qquad (f_2 | 1+2i) = -$$

$$P_2 = (i, -3-i, 1-i) \sim (1-i, -3-i, i),$$

with three associates.

$$(5-i)^2 - (4+2i)(2-i)^2 = 4.$$

$$D = 4+4i \quad (\mathrm{I}, 1).$$

$$F = (1,\ 0,\ -1-i),$$

$$P = (1,\ -2,\ -i) \sim (-i,\ -2,\ 1),$$

with three associates.

$$(1-2i)^2-(1+i)(2i)^2 = 1.$$

$$D = 5+4i \quad (\mathrm{I}, 1).$$

$$F = (1,\ -1,\ -1-i),$$

$$P = (1,\ -3,\ 1-i) \sim (1-i,\ -3-2i,\ -1+i)$$
$$\sim (-1+i,\ -3,\ -1) \sim (-1,\ -3,\ -1+i)$$
$$\sim (-1+i,\ -3-2i,\ 1-i) \sim (1-i,\ -3,\ 1),$$

with three associates.

$$(3-i)^2-(5+4i)(1-i)^2 = 4i.$$

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THE IRREDUCIBILITY OF LEGENDRE'S POLYNOMIALS

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1. It has for some time been suspected that Legendre's polynomial $P_n(x)$—divided by x when n is odd—is irreducible; *i.e.*, that it is not the product of two factors with rational coefficients. The object of this paper is to shew that this is certainly the case when any of the numbers $n-1$, n, $n+1$, $n+2$, $\frac{1}{2}(n+1)$, $\frac{1}{2}(n+2)$ are primes; and again when n or $n-1$ is a power of 2.

2. The proof depends upon the following lemma. Let $f(x)$ be a polynomial with integral coefficients, having no factor common to them all. If the first r coefficients of $f(x)$ be divisible by a prime p, whilst the first of them is not divisible by p^2, $f(x)$ is either irreducible, or contains an irreducible factor of the same type, and therefore of degree r at least.

For, if $f(x)$ be reducible, it is the product of two polynomials with integral coefficients. Let then

$$\begin{aligned} f(x) &= a_0x^{m+n}+a_1x^{m+n-1}+\ldots \\ &= (b_0x^m+b_1x^{m-1}+\ldots)(c_0x^n+c_1x^{n-1}+\ldots), \end{aligned}$$

so that

$$\begin{aligned} a_0 &= b_0c_0, \\ a_1 &= b_0c_1+b_1c_0, \\ a_2 &= b_0c_2+b_1c_1+b_2c_0, \end{aligned}$$

and so on. By supposition a_0 is not divisible by p^2, so that one of b_0c_0 is prime to p; let this be c_0.

As a_0 is divisible by p, b_0 is so; as a_1, b_0 are divisible by p, b_1c_0 is so, and as c_0 is prime to p, b_1 is divisible by p. In this way it follows that the first r coefficients $b_0b_1\ldots b_{r-1}$ are all divisible by p, whilst b_0 is not divisible by p^2.

$f(x)$ has thus a factor of the same type; if this be reducible it contains another factor of this type, and so on. Hence $f(x)$ has an irreducible factor of the type. If we read "last" for "first" in the enunciation of this lemma it is still true.

The particular case where the first coefficient of $f(x)$ is $= 1$, the last $= \pm p$, and the rest are divisible by p, was used by Eisenstein to prove the irreducibility of $(x^p-1)/(x-1)$ when p is prime. See Mathews, *Theory of Numbers*, pp. 186–188.

3. The polynomial $P_n(x)$ is of the form

$$\frac{n+1.n+3 \ldots 2n-1}{2.4 \ldots n} x^n - \frac{n-1.n+1 \ldots 2n-3}{2.4 \ldots n-2 \times 2} x^{n-2} + \frac{n-3.n-1 \ldots 2n-5}{2.4 \ldots n-4 \times 2.4} x^{n-4} - \ldots,$$

when n is even.

It is $$\frac{n+2.n+4 \ldots 2n-1}{2.4 \ldots n-1} x^n - \frac{n.n+2 \ldots 2n-3}{2.4 \ldots n-3 \times 2} x^{n-2} + \frac{n-2.n \ldots 2n-5}{2.4 \ldots n-5 \times 2.4} x^{n-4} - \ldots,$$

when n is odd.

When reduced to its lowest terms each coefficient is a fraction whose denominator is a power of 2; and they all become integral when multiplied by the greatest power of 2 which divides $n!$. Let this integralised form be called $R_n(x)$; its coefficients have no common factor. For, since $P_n(1) = 1$, their sum is a power of 2, so that no odd number divides them all, and the first at least is odd.

4. To apply the lemma to this function $R_n(x)$ we use the following properties of its coefficients. Here, and for the rest of this paper, p is used to denote an odd prime.

(1) If $E(n/p)$ be odd and $< p$, the last coefficient of $R_n(x)$ is divisible by p, but not by p^2. If n be even, the last $p+1$ coefficients are divisible by p; if n be odd, the last $p-1$ coefficients are so.

(2) If $E(2n/p)$ be odd and $< p$, the first coefficient of $R_n(x)$ is divisible by p, but not by p^2; and if r be the remainder when $2n+1$ is divided by p, the first r coefficients are divisible by p. In both cases, the zero coefficients of x^{n-1}, x^{n-3}, ... are to be reckoned in.

5. To prove the first: the last coefficient of $P_n(x)$ is $\frac{1.3 \ldots n-1}{2.4 \ldots n}$, when n is even, and $\frac{3.5 \ldots n}{2.4 \ldots n-1}$, when n is odd.

If $n = pt+r$, t and r being $< p$, and t odd, the multiples of p which occur in the numerator are p, $3p$, $5p$, ..., tp; those which occur in the denominator are $2p$, $4p$, ..., $(t-1)p$. Note that when $r = 0$, n must be odd. Hence, when t is odd, the last coefficient of $R_n(x)$ is divisible by p, but not by p^2.

When t is even, n must be even when $r = 0$; the numerator contains $p, 3p, \ldots, (t-1)p$, and the denominator $2p, 4p, \ldots, tp$. The last coefficient is thus prime to p when t is even.

Again, when n is even, the last, last but one, last but two, &c., of the coefficients of $P_n(x)$ are, neglecting sign,

$$\frac{1.3 \ldots n-1}{2.4 \ldots n}, \text{ zero}, \quad \frac{3.5 \ldots n+1}{2.4 \ldots n-2\times 2}, \text{ zero}, \quad \ldots .$$

If p^m be the highest power of p which divides the first of these numerators, it divides $p \,.\, p+2 \,. \ldots \, n-1$; and this again divides the first $p+1$ numerators.

Now, none of the denominators are divisible by p^m, for they are all divisors of the first, which contains p^{m-1} only. Hence p divides the first $p+1$ of these coefficients; *i.e.*, the last $p+1$ of $R_n(x)$.

When n is odd, the corresponding coefficients are

$$\frac{3.5 \ldots n}{2.4 \ldots n-1}, \text{ zero}, \quad \frac{5.7 \ldots n+2}{2.4 \ldots n-3\times 2}, \text{ zero}, \quad \ldots,$$

and the same argument shews that the first $p-1$ are divisible by p; *i.e.*, that the last $p-1$ coefficients of $R_n(x)$ are so.

To prove the second property: the first coefficient of $P_n(x)$ is the same, except for sign and a power of 2, as the last coefficient of $P_{2n}(x)$, they are, in fact, $\frac{2n!}{2^n.n!\,n!}$ and $\frac{2n!}{2^{2n}.n!\,n!}$. Hence, by what has been done, the leading coefficient of $R_n(x)$ is divisible by p, but not by p^2, when $E(2n/p)$ is odd and $< p$.

Let $2n = pt+r$, t being odd, and t, r less than p, so that the leading coefficient of $R_n(x)$ is divisible by p. Then r is odd, and each of $2n-1$, $2n-3$, ..., $2n-r+2$ is prime to p. The numerators of the coefficients of $P_n(x)$ are

$$2n-1 \,.\, 2n-3 \,. \ldots \, m; \quad \text{zero}; \quad 2n-3 \,.\, 2n-5 \,. \ldots \, m-2; \quad \text{zero} \ldots .$$

m being $n+1$ or $n+2$ as n is even or odd. If p^s be the greatest power of p which divides the first, it divides $2n-r \,.\, 2n-r-2 \,. \ldots \, m$, and this again divides the first $r+1$ numerators. None of the denominators are divisible

by p^s; for the first is not so, and the rest are divisors of the first. Hence p divides the first $r+1$ coefficients of $R_n(x)$, and $r+1 \equiv 2n+1 \bmod p$. The second property is proved.

6. Taken in conjunction with the lemma of § 2, these results shew that—

(1) When $E(n/p)$ is odd and $< p$, $R_n(x)$, if reducible, contains an irreducible factor whose degree is at least $p+1$ when n is even, and at least $p-1$ when n is odd.

(2) When $E(2n/p)$ is odd and $< p$, and r is the least positive residue of $2n+1$, there is an irreducible factor of degree r at least.

The first four cases mentioned in § 1 follow at once, thus

If $n = p+1$, $E(n/p) = 1$, there is an irreducible factor of degree $p+1$ or n, so that P_n is irreducible.

If $n = p$, $E(n/p) = 1$, there is an irreducible factor of degree $n-1$, and P_n/x is irreducible.

If $n = p-1$, $E(2n/p) = 1$, $2n+1 = 2p-1 \equiv p-1 \bmod p$. There is an irreducible factor of degree n, P_n is irreducible.

If $n = p-2$, $E(2n/p) = 1$, $2n+1 = 2p-3 \equiv p-3 \bmod p$. There is an irreducible factor of degree $p-3$ or $n-1$, and P_n/x is irreducible.

7. When $n = 2p-1$,

$E(n/p) = 1$, and the last $p-1$ coefficients are divisible by p,

$E(2n/p) = 3$, the first $p-1$ coefficients are so divisible.

If there is an irreducible factor with both these properties, its degree is $2p-2$, or $n-1$, and P_n/x is irreducible. If not, there are two distinct irreducible factors each of degree $p-1$, and hence there cannot be one of degree $> p-1$. Now Tchébicheff has proved that there is always a prime between p and $2p-2$; if this be q, $E(n/q) = 1$, and $R_n(x)$ has an irreducible factor of degree $q-1$ at least, and this is $> p-1$. Hence this alternative is inadmissible, and P_n/x is irreducible.

Again, when $n = 2p-2$, $E(n/p) = 1$, $E(2n/p) = 3$, also

$$2n+1 \equiv p-3 \bmod p.$$

The first $p-3$ and the last $p+1$ coefficients are divisible by p: there is either a single factor of degree $2p-2$ or n, so that P_n is irreducible; or else P_n is the product of irreducible factors of degrees $p-3$, $p+1$. As in the last paragraph we may shew that this is inadmissible, and P_n is irreducible.

8. For the cases n or $n-1$ a power of 2, we may start from the definition

$$P_n(x) = \frac{1}{\pi}\int_0^\pi [x+y\cos\phi]^n\, d\phi \quad (y^2 = x^2-1).$$

When expanded, this is of the form

$$x^n + a_2 x^{n-2}y^2 + a_4 x^{n-4}y^4 + \ldots + a_n y^n,$$

when n is even; the coefficient a_r being the product of a binomial coefficient, and $\frac{1.3.5\ldots r-1}{2.4.6\ldots r}$. The latter, on reduction, is a fraction whose denominator is a power of 2, namely the highest which divides $r!$. This is always $< 2^r$, and if r be a power of 2 it is 2^{r-1}.

Put $$n = 2m, \qquad z = \frac{4x^2}{x^2-1},$$

so that $$x^2 = \frac{z}{z-4}, \qquad y^2 = \frac{4}{z^2-4}.$$

The equation $P_n(x) = 0$ then becomes

$$f(z) = z^m + 2^2 a_2 z^{m-1} + 2^4 a_4 z^{m-2} + \ldots + 2^m a_m = 0,$$

an equation whose coefficients are all even after the first; when m is a power of 2 the last is not divisible by 4. Hence $f(z)$ is irreducible.

If now $P_n(x)$ have a rational factor $g(x^2)$, $g\left(\frac{z}{z-4}\right) = 0$, has a root in common with $f(z) = 0$. This is impossible since $f(z)$ is not reducible.

If $P_n(x)$ has an irreducible factor of the form $\alpha(x^2)+x\beta(x^2)$ it is also divisible by $\alpha(x^2)-x\beta(x^2)$; their product will then divide $P_n(x)$. Its constant term is the square of the last coefficient of $\alpha(x^2)$, which is not zero; hence it is not $= P_n(x)$, whose last term is not square, and must be a divisor of $P_n(x)$ of form $g(x^2)$, which has been shown not to exist.

A precisely similar proof applies to the case where $n-1$ is a power of 2.

As an indication of the extent to which these tests are applicable, it

may be stated that, for $n < 100$, P_n or P_n/x is certainly irreducible in all but eleven cases; viz., for $n = 26, 34, 49, 50, 55, 63, 75, 76, 86, 91, 92$. It is, of course, not asserted that these are reducible, or even that they are probably so.

9. In conclusion, it may be shewn that R_n has in any case an irreducible factor of degree $> \frac{2}{3}n$. For, suppose $n = 3t+r$, $r < 3$. There is a prime between $2t+1$ and $4t$. If $2t+3$ be prime, there is an irreducible factor of degree $2t+2$ at least; if $4t-1$ be prime, there is one of degree at least $6t+2r+1-(4t-1)$, or $2t+2+2r$, and if there be a prime between $2t+3$ and $4t-1$, the limit is higher still. Hence there is always an irreducible factor of degree $2t+2$ at least, and this is greater than $\frac{2}{3}n$.

ON A CERTAIN SERIES OF FOURIER

By W. H. Young, Sc.D., F.R.S.

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1. Apart from the practical importance which the detailed study of a particular series of Fourier may possess, such a study may be of theoretical interest in two ways—first, in the light it throws on the properties of the general Fourier series by showing what the possibilities are, and by suggesting theorems; secondly, in virtue of the fact that such a series, considered in conjunction with the general series, gives rise to definite convergence properties of series involving the coefficients of the Fourier series, whether these be numerical or themselves trigonometrical series connected with the general Fourier series. Of primary importance in this connexion are the two series, respectively sine and cosine series, whose coefficients are the reciprocals of the natural numbers in order. The sums of both these series are, of course, well known, and the fact that the sine series converges boundedly is classical; it is, moreover, a particular case of the theorem* that the Fourier series of functions of bounded variation converge boundedly. So little have these series, however, been studied that it was only quite recently† that the fact was enunciated that the partial summations of the sine series are all positive or all negative, according as x is positive or negative in the interval $(-\pi, \pi)$. The present paper is devoted to the cosine series, and more especially to the mode in which it diverges in the neighbourhood of the origin. Thus, it is shown that the chasm function, as well as the peak function, has the value of $+\infty$ at the origin, so that the divergence is what I have called‡ uniform diver-

* W. H. Young, "On the Integration of Fourier Series," § 3, 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 352, 353.

† Dunham Jackson, "Ueber eine trigonometrische Summe," 1911, *Rend. di Palermo*, xxxii, pp. 257–262, where references to Fejér will be found.

‡ W. H. Young, "On Uniform and Non-Uniform Convergence and Divergence of a Series of Continuous Functions and the Distinction of Right and Left," § 6, 1907, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 36.

genee. Moreover, the ratio of the sum-function to the n-th partial summation has the unique double limit unity as $n \to \infty$ and the point approaches the origin.

As an illustration of the application which may be made of these results to the general theory of Fourier series, a property of the coefficients of the cosine terms of these series is deduced which holds good for an extremely wide class of series. As is well known, the coefficients of the cosine terms and those of the sine terms behave in various respects differently from one another. The property here obtained has reference precisely to one of these points of difference. The series whose typical term is a sine-coefficient converges when each term is divided by the integer denoting its place in the series, but it is known * that this is not true for the cosine-coefficients in the general case. It is not difficult to see that, if the Fourier series has for its function one which remains summable when raised to some positive power greater than unity, the series in question converges; this follows, in fact, from the known sufficient condition for the convergence of the allied series of the Fourier series, coupled with a necessary condition that a function should be the integral of one which remains summable when raised to such a positive power. It is shown in the present paper that this series converges if the function be such that it remains summable when multiplied by the logarithm of its modulus, a condition which, of course, includes the condition for convergence above given as a very particular case. In the proof of this result the following inequality is required, and is, accordingly, proved :—

$$uv \leqslant u \log u + e^{v-1}, \quad (0 < u).$$

This is deducible by limiting process from the inequality

$$(p+1)uv \leqslant u^{1+p} + pv^{1+1/p},$$

which I have already † had occasion more than once to employ in the theory of Fourier series. Once the statement has been made, it can be proved still more easily by a direct process. In the general theory of classes of summable functions and elsewhere, this inequality, and others akin to it, are bound to play a prominent and increasingly important part.

It will be seen from perusal of the paper that the reasoning employed

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* W. H. Young, "On the Nature of the Successions formed by the Coefficients of a Fourier Series," §§ 1 and 4, 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 344–348.

† W. H. Young, "On a Class of Parametric Integrals and their Application in the Theory of Fourier Series," § 7, 1911, *Proc. Roy. Soc.*, Ser. A, Vol. 85, p. 407.

is of a delicate nature as soon as the more obvious properties of the series, such as the distribution of the maxima and minima of the partial summations, have been disposed of. It is not impossible, therefore, that the corresponding discussion of the nature of the divergence of the most general Fourier series of the type here considered, that is of one whose function is continuous in the extended sense, periodic and monotone towards the origin, which is the sole infinity, may not require essentially more subtle considerations; they will necessarily be, however, at least in part, of a different character to those here employed. From this point of view, therefore, as well as from others, the publication of the present account seems desirable. A further remark may be added: the notion of extended continuity, in which infinite values are allowed, produces in the theory of integration no sensible simplification in the statement of results; a point at which there is continuity in the extended sense behaves for purposes of integration like a discontinuity. In the theory of series, on the other hand, I have more than once emphasized the inevitableness of the concept. Its importance is once more illustrated by the results here obtained.

2. Let

$$S_n(x) = \cos x + \tfrac{1}{2}\cos 2x + \ldots + \frac{1}{n}\cos nx, \quad (0 \leqslant x \leqslant \pi). \tag{1}$$

The greatest value of $S_n(x)$ is obviously at the origin, and since

$$\begin{aligned} S_n'(x) &= -\sin x - \sin 2x - \ldots - \sin nx \\ &= -\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\left(\cos\tfrac{1}{2}x - \cos\frac{2n+1}{2}x\right) \\ &= -\operatorname{cosec}\tfrac{1}{2}x \sin\frac{nx}{2}\sin\frac{(n+1)x}{2}, \end{aligned} \tag{2}$$

we see that the maxima of $S_n(x)$ are at the points 0, $2\pi/n$, $4\pi/n$, ..., and the minima at $2\pi/(n+1)$, $4\pi/(n+1)$, ...

Now taking $\lambda < k$,

$$\begin{aligned} &S_n[2k\pi/(n+1)] - S_n[2\lambda\pi/(n+1)] \\ &\quad = \int_{2\lambda\pi/(n+1)}^{2k\pi/(n+1)} S_n'(x)\,dx \\ &\quad = -\frac{2}{n+1}\int_{\lambda\pi}^{k\pi}\left(\sin x \cot\frac{x}{n+1} - \cos x\right)\sin x\,dx \\ &\quad = -\frac{2}{n+1}\int_{\lambda\pi}^{k\pi}\sin^2 x \cot\frac{x}{n+1}\,dx, \end{aligned} \tag{3}$$

which is certainly negative, since $k \leqslant \frac{1}{2}(n+1)$, so that the cotangent in the integrand is positive.

Thus the minimal values of $S_n(x)$ decrease as we move to the right, so that the least value of $S_n(x)$ is at the point $[2k\pi/(n+1)]$, where k is the greatest integer in $\frac{1}{2}(n+1)$. Thus if n is odd, the least value is $S_n(\pi)$, while if n is even, it is $S_n\left(\pi - \frac{\pi}{n+1}\right)$.

In the former case we have

$$S_n(\pi) = -1 + \tfrac{1}{2} - \ldots - \frac{1}{n} > -1.$$

In the latter case,

$$\begin{aligned} S_n\left(\pi - \frac{\pi}{n+1}\right) &\equiv S_n(\pi - p), \text{ say,} \\ &= -\cos p + \tfrac{1}{2}\cos 2p - \ldots + \frac{1}{n}\cos np \\ &= -\left(1 - \frac{1}{n}\right)\cos p + \left(\tfrac{1}{2} - \frac{1}{n-1}\right)\cos 2p - \ldots \\ &\qquad + (-)^{\frac{1}{2}n}\left(\frac{2}{n} - \frac{1}{\frac{1}{2}n+1}\right)\cos \tfrac{1}{2}np. \end{aligned}$$

Since $$1 - \frac{1}{n} > \tfrac{1}{2} - \frac{1}{n-1} > \frac{1}{3} - \frac{1}{n-2} > \ldots > \frac{2}{n} - \frac{1}{\frac{1}{2}n+1},$$

and $$\cos p \geqslant \cos 2p \geqslant \ldots \geqslant \cos \tfrac{1}{2}np,$$

we have $$S_n\left(\pi - \frac{\pi}{n+1}\right) > -\left(1 - \frac{1}{n}\right)\cos p > -1.$$

Thus, in any case, $$S_n(x) > -1, \quad (0 \leqslant x \leqslant \pi). \tag{4}$$

3. If we take any interval $(0, e)$, we know that, for all values of n, such that

$$\frac{2\pi}{n+1} < e, \tag{5}$$

the least value of $S_n(x)$ in the interval is at the point $2\lambda\pi/(n+1)$, where

$$\frac{2\lambda\pi}{n+1} \leqslant e < \frac{2(\lambda+1)\pi}{n+1}. \tag{6}$$

Thus we have only to show that, as n increases indefinitely, the value of

$S_n(x)$ at this point is greater than a certain positive quantity, depending on e in such a way that, by choosing e small enough, the quantity is as large as we please, to prove that $S_n(x)$ diverges uniformly to $+\infty$ at the origin.

Now regarding k in (3) as giving the point already considered at which $S_n(x)$ assumes its least value in $(0, \pi)$, we know that the first term on the left of (3) is greater than -1; therefore it is only necessary for the proof of our statement to shew that the integral on the right, taken with sign $+$ instead of $-$, is as large as we please.

Now the integral may be written

$$\frac{2k}{n+1}\int_{\lambda\pi/k}^{\pi}(\sin kx)^2\cot\frac{kx}{n+1}\,dx \geqslant \frac{2k}{n+1}\int_{e_1}^{\pi}(\sin kx)^2\cot\frac{kx}{n+1}\,dx,$$

where
$$e_1 = e\left(1+\frac{e}{2\pi-e}\right),$$

since $k \geqslant \frac{1}{2}n$, and therefore

$$\lambda\pi/k \leqslant 2\lambda\pi/n \leqslant \frac{2\lambda\pi}{n+1}\left(1+\frac{1}{n}\right) \leqslant e\left(1+\frac{1}{n}\right) \leqslant e\left(1+\frac{1}{-1+2\pi/e}\right),$$

using (6). Since $n \leqslant 2k \leqslant n+1$, our integral

$$\geqslant \frac{n}{n+1}\int_{e_1}^{\pi}\tfrac{1}{2}(1-\cos 2kx)\cot\frac{x}{2}\,dx.$$

But, by the theorem of Riemann-Lebesgue,* since $\cot\frac{1}{2}x$ is summable from e_1 to π,

$$\operatorname*{Lt}_{n\to\infty}\frac{n}{n+1}\int_{e_1}^{\pi}\cot\tfrac{1}{2}x\cos 2kx\,dx = 0.$$

But, by (5), $n \to \infty$, when $e \to 0$; therefore our integral

$$\geqslant \left(1-\frac{e}{2\pi}\right)\log\operatorname{cosec}\tfrac{1}{2}e + E,$$

where E has the unique limit zero, when e approaches zero.

This shows that our integral is as great as we please, which, as already pointed out, proves the statement that $S_n(x)$ diverges uniformly to $+\infty$ at the origin.

4. Write

$$\sigma_{m,n}(x) = S_{m+n}(x) - S_m(x) = \sum_{r=m+1}^{m+n} r^{-1}\cos rx, \quad (0 \leqslant x \leqslant \pi).$$

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* See, for example, Hobson's *Theory of Functions of a Real Variable*, p. 674.

Differentiating,

$$\sigma'_{m,n}(x) = \sum_{r=m+1}^{m+n} (-\sin rx)$$

$$= \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x \left\{ \cos(2m+1)\frac{x}{2} - \cos(2m+2n+1)\frac{x}{2} \right\}$$

$$= -\sin\frac{nx}{2}\sin\frac{(2m+n+1)x}{2} \Big/ \sin\frac{x}{2}$$

$$= -\sin\tfrac{1}{2}p(2m+1)x \sin\tfrac{1}{2}(p+1)(2m+1)x/\sin\tfrac{1}{2}x, \qquad (7)$$

where we have written $n = p(2m+1)$,

and we may, if we please, so choose our succession of n's, so that p is an odd integer. Thus $2m+n+1 = (2m+1)(p+1)$.

Since the origin is evidently a maximum of $\sigma_{m,n}(x)$, we see, from (7), that the maxima of $\sigma_{m,n}(x)$ are at the points $0, 2\pi/n, 4\pi/n, \ldots$, and the minima at the points $2\pi/(2m+n+1), 4\pi/(2m+n+1), \ldots$, the extreme right-hand minimum point being at $x = \pi$.

5. Let us write y, in (7), for $\frac{1}{2}(2m+1)x$. Then we have

$$\sigma_{m,n}(x) - \sigma_{m,n}(\pi) = \frac{2}{2m+1}\int_{(m+\frac{1}{2})\pi}^{(m+\frac{1}{2})x} \sin py \sin(p+1)y \operatorname{cosec}\frac{y}{2m+1}\,dy. \qquad (8)$$

If, in the first place, the point $(m+\frac{1}{2})x$ lies at, or to the right of, the point $\frac{1}{2}\pi$, we use the Second Theorem of the Mean, and bring out the monotone decreasing factor $\operatorname{cosec} y/(2m+1)$ of the integrand. We thus have

$$\sigma_{m,n}(x) - \sigma_{m,n}(\pi) = \frac{1}{2m+1}\operatorname{cosec}\tfrac{1}{2}x \int_{(m+\frac{1}{2})x}^{x} [\cos y - \cos(2p+1)y]\,dy$$

$$= \frac{1}{\pi}\,\frac{\pi}{x(m+\frac{1}{2})}\,\frac{\frac{1}{2}x}{\sin\frac{1}{2}x}\,4\theta,$$

where θ is numerically $\leqslant 1$.

Thus, since $\frac{1}{2}x/\sin\frac{1}{2}x \leqslant \frac{1}{2}\pi$,

$$|\sigma_{m,n}(x) - \sigma_{m,n}(\pi)| \leqslant 4 \quad (\tfrac{1}{2}\pi \leqslant x \leqslant \pi). \qquad (9)$$

In the second place, if $x < \frac{1}{2}\pi/(m+\frac{1}{2})$, and therefore $y < \frac{1}{2}\pi$, the

function $\dfrac{\sin y}{\sin y/(2m+1)}$ is a monotone decreasing function of y up to $y=\frac{1}{2}\pi$. Thus dividing the interval of integration in (8) into two parts at the point $y=\frac{1}{2}\pi$, the second part gives us an integral which, as we have just seen, is numerically $\leqslant 4$, and the first part gives us an integral which may be written

$$\frac{2}{2m+1}\int_{(m+\frac{1}{2})x}^{\frac{1}{2}\pi}\frac{\sin y}{\sin y/(2m+1)}(\sin^2 py\cot y+\sin py\cos py)\,dy$$

$$=\frac{2}{2m+1}\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}$$

$$\times\left\{\int_{(m+\frac{1}{2})x}^{X}\sin^2 py\cot y\,dy+\frac{1}{2p}\{\sin^2 pX-\sin^2 p(m+\tfrac{1}{2})x\}\right\}\geqslant-\frac{1}{2p}.$$

Thus we have, finally, combining with (9),

$$\sigma_{m,n}(x)-\sigma_{m,n}(\pi)\geqslant-4-\frac{1}{2p}, \tag{10}$$

where $n=p(2m+1)$.

6. Hence $$S_m(x)-S_{m+n}(x)\leqslant 5+\frac{1}{2p},$$

since $\sigma_{m,n}(\pi)$ is numerically $\leqslant 1/(m+1)\leqslant 1$.

Now, let n increase without limit, $S_{m+n}(x)$ becomes in the limit $S(x)$, where, except at the origin,

$$S(x)=\tfrac{1}{2}\log\frac{1}{2(1-\cos x)}.$$

Combining this with a former result (§ 2), we have, since $p\to\infty$,

$$-1\leqslant S_m(x)\leqslant S(x)+5. \tag{11}$$

Hence $$-E\leqslant\int_E S_m(x)\,dx\leqslant 5E+\int_E S(x)\,dx,$$

which shows that $\int_E S_m(x)\,dx$ has the unique double limit zero, when $E\to 0$, $m\to\infty$.

It will be seen in § 9, that this last result, which is otherwise evident, is included in the more general statements of that article.

7. We shall now require the inequality referred to in the introduction. We know that, if $u \geqslant 0$ and $w \geqslant 0$,

$$(p+1)uw \leqslant u^{p+1}+pw^{1+1/p}.$$

Hence, subtracting u from both sides and dividing by p,

$$u\left(\frac{p+1}{p}w-\frac{1}{p}\right) \leqslant \frac{1}{p}\left[u(u^p-1)\right]+w^{1+1/p};$$

or, say,
$$uv \leqslant u\frac{u^p-1}{p}+\left(\frac{1+pv}{1+p}\right)^{1+1/p} \quad (v \geqslant -1/p), \tag{12}$$

where
$$pv = (p+1)w-1;$$

and therefore
$$w = (1+pv)/(1+p).$$

Now

$$\operatorname*{Lt}_{p\to 0}\frac{1}{\log\left(\frac{1+pv}{1+p}\right)^{1+1/p}} = \operatorname*{Lt}_{p\to 0}\frac{p/(1+p)}{\log\frac{1+pv}{1+p}} = \frac{0}{0} = \operatorname*{Lt}_{p\to 0}\frac{1/(1+p)^2}{\frac{v}{1+pv}-\frac{1}{1+p}} = \frac{1}{v-1}.$$

Therefore, letting p approach zero, the inequality (12) gives us in the limit the following:—

$$uv \leqslant u\log u+e^{v-1} \quad (0 \leqslant u, \ -\infty < v). \tag{13}$$

8. The preceding proof is of interest as showing how the inequality naturally suggested itself. The following proof is, however, more direct.

We have at once
$$v \leqslant e^{v-1};$$

and therefore
$$v-\log u \leqslant e^{v-1-\log u} \leqslant \frac{1}{u}e^{v-1},$$

whence
$$uv \leqslant u\log u+e^{v-1}.$$

9. We have already proved (§ 6), that

$$-1 \leqslant S_n(x) \leqslant S(x)+5. \tag{11}$$

Hence $e^{-\frac{1}{2}} \leqslant e^{\frac{1}{2}S_n(x)-1} \leqslant e^{\frac{3}{2}}e^{\frac{1}{2}S(x)} \leqslant e^{\frac{3}{2}}\left(\frac{1}{2(1-\cos x)}\right)^{\frac{1}{2}} \leqslant (\frac{1}{2}e^3\operatorname{cosec}\frac{1}{2}x)^{\frac{1}{2}}$.

This shews that $e^{\frac{1}{2}S_n(x)-1}$ is summable; moreover, if $u(x)$ is any positive function, such that $u \log u$ is summable, we have, by the inequality (13),

$$-\int_E \tfrac{1}{2}u(x)\,dx \leqslant \int_E u(x)\,\tfrac{1}{2}S_n(x)\,dx \leqslant \int_E u \log u\,dx + \int_E e^{\frac{1}{2}S_n(x)-1}\,dx$$

$$\leqslant \int_E u \log u\,dx + (\tfrac{1}{2}e^3)^{\frac{1}{2}} \int_E \frac{dx}{\sqrt{\sin \tfrac{1}{2}x}},$$

which shews that $\int_E u(x)\,S_n(x)\,dx$ has the unique double limit zero, as $n \to \infty$, $E \to 0$.

Again, if $f(x)$ is any function such that $|f(x)| \log |f(x)|$ is summable, and $u(x)$ denote either the function which is equal to $f(x)$ wherever positive and zero elsewhere, or the function which is equal to $|f(x)|$ wherever negative and zero elsewhere, $u(x) \log u(x)$ is evidently summable, and therefore $\int_E u(x)\,S_n(x)\,dx$ has the unique double limit zero, and therefore, since $f(x)$ is the difference of these two functions, $\int_E f(x)\,S_n(x)\,dx$ has the unique double limit zero.

10. Now the succession $f(x)\,S_n(x)$ converges, except at the origin, to the summable function $f(x)\,S(x)$. Therefore, Vitali's condition* having been just shown to be satisfied, the succession may be integrated term by term, and we have

$$\operatorname*{Lt}_{n\to\infty} \int_0^x f(x)\,S_n(x)\,dx = \int_0^x f(x)\,S(x)\,dx.$$

Hence, since $S_n(x)$ is an even function, we see that, provided $|f(x)| \log |f(x)|$ is summable, not only from 0 to π, but also from $(-\pi)$ to 0, the same equation holds when we change the upper limit of integration from x to $-x$. Hence,

$$\operatorname*{Lt}_{n\to\infty} \int_a^b f(x)\,S_n(x)\,dx = \int_a^b f(x)\,S(x)\,dx \quad (-\pi \leqslant a \leqslant b \leqslant \pi), \quad (14)$$

provided only $|f(x)| \log |f(x)|$ is summable from $-\pi$ to π.

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* See my paper on "Semi-Integrals and Oscillating Successions of Functions," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 308.

Taking the extreme limits of integration π and $-\pi$, and writing as usual,

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{r=1} (a_r \cos rx + b_r \sin rx),$$

(14) becomes

$$\operatorname*{Lt}_{n\to\infty} \left(\frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \log \frac{1}{2(1-\cos x)}\, dx. \qquad (15)$$

Thus, as stated in the introduction, in the case of the large class of functions $f(x)$ here considered, the series whose general term is the n-th Fourier cosine constant of $f(x)$ divided by n, converges, and its sum may be expressed by the right-hand side of (15).

CLOSED ORBITS OF EJECTION AND RELATED PERIODIC ORBITS. PAINLEVÉ'S THEOREM

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1. *Introduction.*

In the circular solutions of the problem of two bodies there is continuity between orbits in which the motion is in the forward direction and those in which it is retrograde, only when the orbits are of zero dimensions, or when they are infinitely great. But in the restricted problem of three bodies, the deviations from the circular forms of the orbits are such that, in some classes, there is geometrical continuity between those in which the motion is forward and those in which it is retrograde; and the limits between the two types are orbits passing through one of the finite bodies. If the infinitesimal body leaves one of the finite bodies, its orbit is called an *orbit of ejection;* and if it strikes a finite mass, *an orbit of collision.*

In certain cases orbits of ejection are also orbits of collision, or closed orbits of ejection. When the direction of collision is exactly opposite to that of ejection, and in the line passing through the two finite bodies, they are the limits of two classes of periodic orbits, in one of which the motion is direct, and in the other of which it is retrograde. The closed orbits of ejection are not themselves periodic orbits, even if the physical impossibility be disregarded and the problem be considered purely from the mathematical point of view; for, if the expressions for the coordinates are followed, in the sense of analytic continuity, beyond the values of t for which a collision occurs, they become complex, and never become real again for increasing real values of t. Those orbits in which the ejection and collision are not in opposite directions are not the limits of periodic orbits, or at least of orbits which re-enter after a single revolution.

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One object of the investigations of this paper is to determine the limiting forms of certain types of periodic orbits, and thus partially to prepare the way for the discussion of the evolution of the various classes of periodic orbits with varying values of the parameters upon which they depend, and to show the relations among these various classes. The existence of the closed orbits of ejection will be established, some of their properties will be derived, and it will be proved that each one in which the direction of ejection and collision is opposite, and in the line joining the finite bodies, is the limit of two series of periodic orbits whose existence will also be established. Related periodic orbits which re-enter only after many revolutions will be considered. Finally, it will be proved that, if one finite mass is sufficiently small, and if the initial conditions satisfy one determinable analytic condition, the orbit will be one of ejection. This is a form of the theorem conjectured by Painlevé.*

2. *Ejectional Orbits in the Two-Body Problem.*

As preliminary to the general problem, the special case in which there is only one finite mass will first be treated. Let the mass of the finite body be $1-\mu$, and let the units be so chosen that the gravitational constant is unity. Then the motion of the infinitesimal body projected along the fixed ξ-axis satisfies the differential equation

$$\frac{d^2\xi}{dt^2} = \mp\frac{1-\mu}{\xi^2}, \tag{1}$$

where the sign is $-$ or $+$ according as the motion is in the positive or negative direction from the origin.

Suppose $\xi = \xi_0$, $d\xi/dt = \xi' = \xi_0'$ at $t = t_0$. Then the first integral of (1) is

$$\left(\frac{d\xi}{dt}\right)^2 = \xi'^2 = \pm\frac{2(1-\mu)}{\xi} \mp \frac{2(1-\mu)}{\xi_0} + \xi_0'^2 = \pm\frac{2(1-\mu)}{\xi} + c_1. \tag{2}$$

If c_1 is negative, $|\xi|$ has a finite maximum for which ξ' vanishes; if c_1 is zero, ξ' approaches zero as $|\xi|$ becomes infinitely great; if c_1 is positive, ξ' is finite for $|\xi|$ infinite. It will be assumed that c_1 is negative in order to get orbits of ejection which are closed. Then, without loss of generality, it can be supposed that ξ_0 is the greatest value of ξ for projection in the positive direction, or the least for projection in the negative direction.

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* *Leçons sur la Théorie Analytique des Equations Différentielles*, p. 583.

Hence

$$\xi_0' = 0, \qquad c_1 = \mp \frac{2(1-\mu)}{\xi_0}. \tag{3}$$

With the initial values (3), the integral of (2) becomes

$$\sqrt{\xi_0\xi - \xi^2} - \frac{\xi_0}{2} \sin^{-1}\left(-1 + \frac{2\xi}{\xi_0}\right) = \sqrt{\frac{2(1-\mu)}{\pm\xi_0}}\, t + c_2 \tag{4}$$

$$= \sqrt{\frac{2(1-\mu)}{\pm\xi_0}}\, (t - t_0) + (\tfrac{1}{4} + j)\pi\xi_0,$$

where j is an integer.

Consider ξ as a function of t. Since the right members of equations (1) and (2) are regular for all values of t and all values of ξ, except $\xi = 0$, it follows that ξ is a regular function of t for all values of t except those for which ξ vanishes. These values of t are easily determined from (4), and are found to be

$$t_j - t_0 = j\pi\xi_0 \sqrt{\frac{\pm\xi_0}{2(1-\mu)}}, \tag{5}$$

where j takes all integral values.

The character of ξ as a function of t in the vicinity of $t = t_j$ is easily determined from (4). The left member is expansible as a power series in $\sqrt{\pm\xi} = \eta$, and the equation can be written in the form

$$F(\eta) = \sqrt{\xi_0\xi - \xi^2} - \frac{\xi_0}{2} \sin^{-1}\left(-1 + \frac{2\xi}{\xi_0}\right) - (\tfrac{1}{4} + j)\,\pi\xi_0 = \sqrt{\frac{2(1-\mu)}{\pm\xi_0}}\, (t - t_0).$$

It is found that

$$F(0) = -j\pi\xi_0, \quad \frac{\partial F}{\partial \eta}(0) = 0, \quad \frac{\partial^2 F(0)}{\partial \eta^2} = 0, \quad \frac{\partial^3 F(0)}{\partial \eta^3} = -4(\pm\xi_0)^{-\frac{1}{2}}.$$

Therefore η is expansible as a power series in $(t - t_j)^{\frac{1}{3}}$, starting with a term of the first degree in $(t - t_j)^{\frac{1}{3}}$. Since $\pm\xi = \eta^2$, it follows that ξ is expansible as a power series in $(t - t_j)^{\frac{1}{3}}$, starting with a term of the second degree in $(t - t_j)^{\frac{1}{3}}$. It is easily seen from (4) that $F(\eta)$ is an odd function of η. Therefore $F(\eta)$ is an odd series in $(t - t_j)^{\frac{1}{3}}$, and ξ is an even series in $(t - t_j)^{\frac{1}{3}}$. Since the only singularities are given by (5), the radius of the circle of convergence for the series for both η and ξ is

$$\pi\,|\xi_0|\,\sqrt{\frac{\pm\xi_0}{2(1-\mu)}}.$$

The form of the solution in the vicinity of $t = t_j$ being known, the

coefficients of the series can easily be found from (1) by the method of undetermined coefficients. It is convenient in the computation to let

$$\tau = (t-t_j)^{\frac{1}{3}}, \tag{6}$$

after which (1) becomes

$$\tau \frac{d^2\xi}{d\tau^2} - 2\frac{d\xi}{d\tau} = -\frac{9(1-\mu)}{\xi^2}\tau^5. \tag{7}$$

The solution of this equation with the initial value of ξ equal to zero has the form

$$\pm\xi = a_2\tau^2 + a_4\tau^4 + \ldots + a_{2n}\tau^{2n} + \ldots. \tag{8}$$

By direct substitution and comparison of coefficients, it is found that

$$(9)\quad \begin{cases} \pm\xi = c\tau^2 + a\tau^4 - \frac{3}{7}\frac{a^2}{c}\tau^6 + \frac{23}{63}\frac{a^3}{c^2}\tau^8 - \frac{1894}{4851}\frac{a^4}{c^3}\tau^{10} + \frac{3293}{7007}\frac{a^5}{c^4}\tau^{12}\ldots, \\ c = \frac{9}{2}(1-\mu)]^{\frac{1}{3}}, \\ a = \text{arbitrary constant.} \end{cases}$$

It is noticed from the first of these equations that the numerical coefficients of τ^{2n}, except for $n = 1$, are positive if n is even and negative if n is odd. This is a general property which will be proved because it will be needed in § 8, when the existence of the closed orbits of ejection is established. The coefficients of τ^{2n} $(n = 2, 3, \ldots)$, so far as they are given, involve a only in the form a^{n-1} as a factor. This is also a general property of the solution.

Suppose $a_4, a_6, \ldots, a_{n-2}$ have been computed, and that they have the properties which have just been enumerated. It follows from (7) and the form of the solution (9) that a_{2n} is defined by

$$[2n(2n-1)-4n-4]a_{2n} = \Sigma(-1)^{\kappa_1+\ldots+\kappa_n+1} A_{\kappa_1,\ldots,\kappa_n} a_{2j_1}^{\kappa_1} \ldots a_{2j_n}^{\kappa_n}, \tag{10}$$

where the $A_{\kappa_1,\ldots,\kappa_n}$ are positive integers. Consider the general term of the right member of (10) aside from the positive integers $A_{\kappa_1,\ldots,\kappa_n}$. Since it is the coefficient of τ^{2n-2}, as can easily be seen from (7), it follows that

$$(11)\quad \begin{cases} \kappa_1(2j_1-2)+\ldots+\kappa_n(2j_n-2) = 2n-2, \\ \text{or } \kappa_1(j_1-1)+\ldots+\kappa_n(j_n-1) = n-1. \end{cases}$$

Suppose $n-1$ is even. There are certain $\kappa_i(j_i-1)$ in which the κ_i are even. The numerical coefficients of the corresponding terms are all positive. There are certain $\kappa_\lambda(j_\lambda-1)$ which are odd, and in which there-

fore both κ_λ and $j_\lambda - 1$ are odd. The number of these terms is even since $n-1$ is even. The numerical coefficients of the corresponding terms are also positive. There are finally certain $\kappa_p(j_p-1)$ in which κ_p is odd, while j_p-1 is even. The numerical coefficients of these terms are negative, and they are the only terms having negative coefficients. Suppose $\kappa_1+\ldots+\kappa_n$ is even; then, since the κ_i and κ_p alone are odd and the number of κ_i is even, the number of κ_p is also even. The product of the coefficients of these terms is therefore positive. Since $(-1)^{\kappa_1+\ldots+\kappa_{n+1}}$ is negative, the numerical coefficients of the right number of (10) for these terms are negative. If $\kappa_1+\ldots+\kappa_n$ is odd, the number of κ_p is odd, and the products of the coefficients of these terms are negative. But in this case $(-1)^{\kappa_1+\ldots+\kappa_{n+1}}$ is positive. Therefore in all cases in which $n-1$ is even the numerical coefficients of the right members are negative. Since $2n(2n-1)-4n-4$ is positive for $n>2$, the a_{2n} for n odd are all negative.

By similar reasoning it can be shown that a_{2n} is positive if n is even. The alternation of the signs of the numerical coefficients of the series for ξ is therefore established as a general property.

The degree of α in $a_{2j_i}^{\kappa_i}$ $(i=1,\ldots,n)$ is, by hypothesis, $\kappa^i(j_i-1)$. Therefore the general term of (10) carries α to the degree

$$\kappa_1(j_1-1)+\ldots+\kappa_n(j_n-1),$$

which, by (11), equals $n-1$. The property that a_{2n} involves α as a factor to the degree $n-1$ is thus established.

The expression for ξ has a branch-point at $t=t_j$ where three branches unite. If $t-t_j=\rho e^{\sqrt{-1}(\phi+2n\pi)}$, the three distinct branches are

$$(12) \qquad \xi_1=\rho e^{\sqrt{-1}\frac{2}{3}\phi},\quad \xi_2=\rho e^{\sqrt{-1}\frac{2}{3}(\phi+2\pi)},\quad \xi_3=\rho e^{\sqrt{-1}\frac{2}{3}(\phi+4\pi)}.$$

If $t-t_j$ is real and negative, $\phi=\pi$ and the second of these expressions alone is real. If $t-t_j$ is real and positive, $\phi=0$, and the first of them alone is real. Consequently the analytic continuation of the real branch of the expression for ξ through $t=t_j$ leads to a complex value, and this value remains complex as t, remaining real, increases to $+\infty$. Therefore the motion is not periodic.

In case the orbit is not one of ejection, the branch-points of the functions which express the coordinates in terms of the time are not on the real axis. If the major axis is kept fixed, and if the eccentricity e is varied from zero to unity, an examination of the equations from which the character of the functions can be determined shows that the singularities

start from both positive and negative infinity on the lines

$$t_j - t_0 = j\pi\xi_0 \sqrt{\frac{\pm\,\xi_0}{2(1-\mu)}},$$

and approach the real axis as a limit as e approaches unity. At the singular points two branches of the function permute, except when, for $e = 1$, two of them have united on the real axis, and then three branches permute. If e increases beyond unity, each of the branch-points divides into two which move equally in opposite directions along the real axis, and for $e = \infty$, one of each pair unites with another of an adjacent pair.

3. *The Integral.*

Equation (2) holds for all values of t, and therefore when the series (9) is substituted in it the result is an identity in τ. The conditions that the coefficients of the various powers of τ shall be identical in the left and right members furnish severe tests of the accuracy of the computation of (9). The integral also gives the relation between the arbitrary α and the greatest distance ξ_0. It is found, by direct substitution, that

$$\alpha = \mp\, \tfrac{9}{10}\, \frac{1-\mu}{c\xi_0}. \tag{13}$$

Since ξ_0 is positive or negative according to the direction of projection, α is a negative quantity.

The interval from ejection to collision is found from (5), (9), and (13) to be

$$P = \frac{\pi(\pm\,\xi_0)^{\frac{3}{2}}}{\sqrt{2(1-\mu)}} = \frac{9\pi\sqrt{1-\mu}}{(-10\alpha)^{\frac{3}{2}}}. \tag{14}$$

For $\alpha = 0$, which corresponds to a parabolic orbit, P is infinite.

4. *Orbits of Ejection in Rotating Axes.*

In the problem of three bodies the motion of the infinitesimal body will be referred to rotating axes. In the demonstration of the existence of closed orbits of ejection in the problem of three bodies it will be necessary to use some of the properties of the orbits of ejection in that of two bodies. For this reason the orbits now under consideration will be referred to axes rotating uniformly with the period 2π.

Suppose the ejection takes place at $t = t_1$, and along the x-axis, where

the rectangular coordinates are denoted by x and y. Then x and y are given by the equations

$$x = +\xi\cos(t-t_1), \qquad y = -\xi\sin(t-t_1). \tag{15}$$

Those orbits which re-enter in the direction opposite to that of ejection are of greatest interest in the present connection. The condition that they shall have this property is that their period in t from ejection to collision shall be a multiple of 2π. This condition becomes, by virtue of (14),

$$(\pm\xi_0)^{\frac{3}{2}} = 2j\sqrt{2(1-\mu)} \quad \text{or} \quad (-10\alpha)^{\frac{3}{2}} = \frac{9\sqrt{1-\mu}}{2j}, \tag{16}$$

where j is a positive integer.

In Fig. 1 the curves for j equal to 1, 2, and 3 are shown, at least as

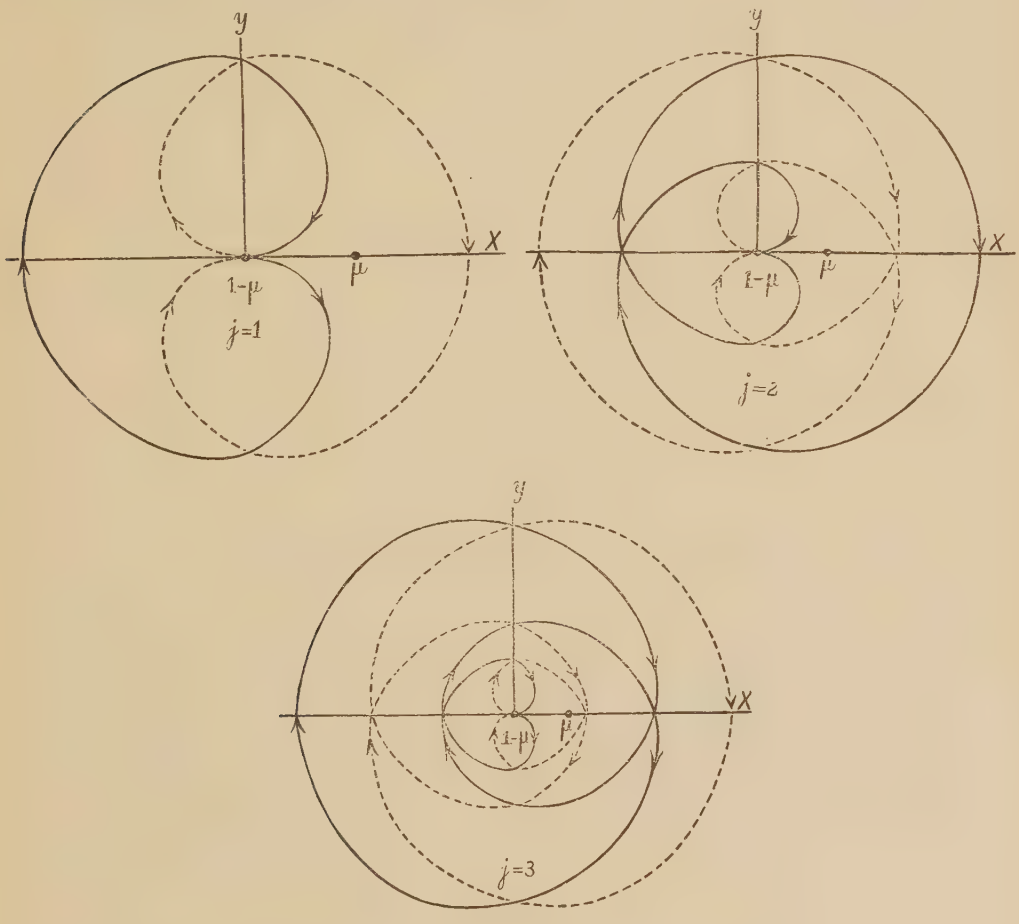

FIG. 1.

to general form, in full lines for ejection along the x-axis in the positive direction, and in dotted lines for ejection in the negative direction. These curves for the three values of j are not drawn to the same scale, for it follows from (16) that their linear dimensions are proportional to $j^{\frac{2}{3}}$. One of the important properties of all these curves is that they intersect the x-axis perpendicularly at their mid-points. The forms of the curves for larger values of j are readily inferred from those which are given.

5. *Ejectional Orbits in the Problem of Three Bodies.*

The differential equations of motion for the infinitesimal body when the finite masses describe circular orbits are, in canonical units,

$$(17)\qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = \dfrac{\partial U}{\partial x}, \\ \dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = \dfrac{\partial U}{\partial y}, \\ U = \frac{1}{2}(x^2+y^2) + \dfrac{1-\mu}{r_1} + \dfrac{\mu}{r_2}, \\ r_1^2 = (x+\mu)^2+y^2, \quad r_2^2 = (x-1+\mu)^2+y^2. \end{cases}$$

These equations have the integral

$$(18)\qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2U - C.$$

When μ is zero the problem reduces to that of two bodies which was treated in § 2. The singularity in the solution, whether μ is zero or not, comes from the fact that r_1 tends toward zero as a limit as t tends toward t_1. It is intuitionally clear that the mass μ will have only a slight influence on the motion of the infinitesimal body while r_1 is small, and it seems probable, therefore, that the nature of the singularity at $t = t_1$ is the same whether μ is distinct from zero or not. This is, indeed, the case, as was first proved by Levi-Civita in a very important memoir.*

It follows from (18) that $x'^2+y'^2$ tends toward infinity as r_1 tends toward zero, but that the limit of $r_1[x'^2+y'^2]$ for r_1 equal to zero is the

* "Sur la Résolution Qualitative du Problème restreint des Trois Corps," *Acta Mathematica*, Vol. xxx (1906), pp. 305–327.

finite quantity $2(1-\mu)$. If μ is zero, x and y are developable as power series in $(t-t_0)^{\frac{1}{3}}$, and this suggests defining an independent variable σ in terms of which x, y, and $t-t_1$ are expressible by series of the form

$$(19) \qquad \begin{cases} x+\mu = a_2\sigma^2+a_3\sigma^3+\ldots, \\ y = \beta_2\sigma^2+\beta_3\sigma^3+\ldots, \\ t-t_1 = 0+\gamma_3\sigma^3+\ldots \quad (\gamma_3 \neq 0). \end{cases}$$

Since the solutions of analytic differential equations in the vicinity of points for which they are regular are themselves regular, while in the vicinity of singular points the solutions are regular or not depending on supplementary circumstances, it is advantageous to choose such dependent variables that the equations shall be regular for $r_1 = 0$, $\sigma = 0$. This Levi-Civita has done, preserving the canonical form, with rare skill and elegance. His variables, which may be denoted here by p, q, u, and v, are related to the rectangular coordinates and their derivatives by

$$(20) \qquad \begin{cases} x+\mu+\sqrt{-1}\,y = (p+\sqrt{-1}\,q)^2, \\ x'+y-\sqrt{-1}\,(x+\mu+y') = \dfrac{u-\sqrt{-1}\,v}{2(p+\sqrt{-1}\,q)}, \\ dt = (p^2+q^2)d\sigma = \rho^2 d\sigma. \end{cases}$$

In these variables equations (17) become

$$(21) \qquad \begin{cases} \dfrac{dp}{d\sigma} = +\dfrac{\partial H}{\partial u}, \quad \dfrac{du}{d\sigma} = -\dfrac{\partial H}{\partial p}, \\ \dfrac{dq}{d\sigma} = +\dfrac{\partial H}{\partial v}, \quad \dfrac{dv}{d\sigma} = -\dfrac{\partial H}{\partial q}, \\ H = \frac{1}{8}\{(u+2\rho^2 q)^2+(v-2\rho^2 p)^2\}-\{1-\mu-C\rho^2+\frac{1}{2}\rho^6+\mu\rho^2 V\}, \\ V = \dfrac{1}{r_2}-p^2+q^2-\mu, \quad r_2^2 = 1-2(p^2-q^2)+(p^2+q^2)^2. \end{cases}$$

It follows from (20) that $p=q=0$ if $x+\mu=y=0$, and that u and v are finite for $r_1 = 0$. Therefore the form of H shows that the differential equations (21) are regular in the vicinity of $p = q = 0$. Consequently p and q are developable as power series in σ, vanishing with σ. It follows from (20) that x, y, and $t-t_1$, considered as functions of σ, have the form (19), and therefore x and y expressed in terms of τ, defined in (6), have the form

$$(22) \qquad \begin{cases} x+\mu = c\tau^2+a_3\tau^3+\ldots, \\ y = b_2\tau^2+b_3\tau^3+\ldots. \end{cases}$$

6. *Construction of the Solutions of Ejection.*

The character of the solutions in the vicinity of $(t-t_1)^{\frac{1}{3}} = \tau = 0$ having been found, they can be obtained without difficulty from (17). Upon using τ as the independent variable and expanding the expression for r_2, these equations become

$$(23)\quad \begin{cases} \tau\dfrac{d^2x}{d\tau^2} - 2\dfrac{dx}{d\tau} - 6\tau^3\dfrac{dy}{d\tau} = 9\tau^5 x - \dfrac{9(1-\mu)\tau^5(x+\mu)}{[(x+\mu)^2+y^2]^{\frac{3}{2}}} \\ \qquad\qquad + 9\mu\tau^5[2(x+\mu)+3(x+\mu)^2 - \tfrac{3}{2}y^2 + \ldots], \\ \tau\dfrac{d^2y}{d\tau^2} - 2\dfrac{dy}{d\tau} + 6\tau^3\dfrac{dx}{d\tau} = 9\tau^5 y - \dfrac{9(1-\mu)\tau^5 y}{[(x+\mu)^2+y^2]^{\frac{3}{2}}} \\ \qquad\qquad - 9\mu\tau^5 y[1+3(x+\mu)+\ldots]. \end{cases}$$

It will now be supposed that the line of ejection is along the x-axis. Then $(t-t_1)^{-\frac{1}{3}}\, dy/dt = \frac{1}{3}\tau^{-3}\, dy/d\tau = 0$. Therefore the initial conditions are

$$(24)\qquad x(0) = -\mu, \quad y(0) = \frac{1}{3\tau^3}\frac{dy}{d\tau} = 0, \quad \left[\frac{x+\mu-c\tau^2}{\tau^4}\right]_{\tau=0} = a.$$

With the initial conditions (24), the solutions have an important property of symmetry. Let them be written in the form

$$(25)\quad \begin{cases} x+\mu = \tau^2 f(\tau), & \dfrac{dx}{d\tau} = \tau\phi(\tau), \\ y \quad\;\; = \tau^4 g(\tau), & \dfrac{dy}{d\tau} = \tau^3\psi(\tau). \end{cases}$$

Now make the transformation of variables

$$(26)\quad \begin{cases} x = +x_1, \qquad y = -y_1, \qquad \tau = -\tau_1, \\ \dfrac{dx}{d\tau} = -\dfrac{dx_1}{d\tau_1}, \quad \dfrac{dy}{d\tau} = +\dfrac{dy_1}{d\tau_1}. \end{cases}$$

Equations (23) are not changed in form by this substitution. Therefore the solution of the transformed equations with the initial conditions

$$x_1(0) = -\mu, \quad y_1(0) = \frac{1}{3\tau_1^3}\frac{dy_1}{d\tau_1} = 0, \quad \left[\frac{x_1+\mu-c\tau_1^2}{\tau_1^4}\right]_{\tau_1=0} = a$$

are

$$\begin{cases} x_1+\mu = \tau_1^2 f(\tau_1), & \dfrac{dx_1}{d\tau_1} = \tau_1 \phi(\tau_1), \\ y_1 \quad\quad = \tau_1^4 g(\tau_1), & \dfrac{dy_1}{d\tau_1} = \tau_1^3 \psi(\tau_1), \end{cases}$$

where f, ϕ, g, and ψ are identical with the functions represented by the same symbols in (25). Therefore

$$\begin{cases} \tau^2 f(\tau) = +\tau_1^2 f(\tau_1) = +\tau^2 f(-\tau), & \tau\phi(\tau) = -\tau_1\phi(\tau_1) = +\tau\phi(-\tau), \\ \tau^4 g(\tau) = -\tau_1^4 g(\tau_1) = -\tau^4 g(-\tau), & \tau^3\psi(\tau) = +\tau_1^3\psi(\tau_1) = -\tau^3\psi(-\tau). \end{cases}$$

It follows that x and $dy/d\tau$ are even functions of τ, and that $dx/d\tau$ and y are odd functions of τ. The first equation of (22) contains only even powers of τ, and the second only odd powers starting with a term of the fifth degree as the lowest.

With the initial conditions (24), the solution of (23) is found to be

$$(27)\quad \begin{cases} \pm(x+\mu) = c\tau^2 + a\tau^4 - \left[\frac{3}{7}\dfrac{a^2}{c} + \frac{9}{14}\mu\right]\tau^6 \\ \qquad\qquad + \left[-\frac{1}{2}c + \frac{2}{6}\frac{3}{3}\dfrac{a^3}{c^2} + \frac{1}{2}\mu c - \frac{3}{5}\frac{a\mu}{6}\dfrac{}{c}\right]\tau^8 + \ldots, \\ \pm y = -c\tau^5 - a\tau^7 + \left[\frac{3}{7}\dfrac{a^2}{c} + \frac{9}{14}\mu\right]\tau^9 \\ \qquad\qquad + \left[\frac{1}{6}c - \frac{2}{6}\frac{3}{3}\dfrac{a^3}{c^2} - \frac{1}{5}\mu c\right]\tau^{11} + \ldots, \\ c = \left[\dfrac{9(1-\mu)}{2}\right]^{\frac{1}{3}}, \quad a = \text{arbitrary constant}, \end{cases}$$

where the positive or negative signs are to be used in the left members according as the initial projection is in the positive or negative direction.

The constant of the integral (18) is given by the equation

$$(28)\quad C = -\frac{1}{9\tau^4}\left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2\right] + x^2 + y^2 + \frac{2(1-\mu)}{\left[(x+\mu)^2+y^2\right]^{\frac{1}{2}}} + \frac{2\mu}{\left[1-2(x+\mu)+(x+\mu^2+y^2)\right]^{\frac{1}{2}}}.$$

It follows from (27) that the right member of this equation can be developed as a series of the form

$$C = \frac{C_{-2}}{\tau^2} + C_0 + C_2\tau^2 + C_4\tau^4 + \ldots.$$

Since this equation is an identity in τ, it follows that

$$C_{-2} = C_2 = C_4 = C_6 = \ldots = 0, \qquad C = C_0.$$

The expressions which are equated to zero constitute a check on the computation of the coefficients of (27). By direct substitution of (27) in (28), it is found that

$$C = C_0 = -\tfrac{20}{9}c\alpha + \mu(2+\mu) = \pm\frac{2(1-\mu)}{\xi_0} + \mu(2+\mu), \tag{29}$$

where the positive or negative sign is to be used according as ξ_0 is positive or negative.

The force function is sometimes used in the symmetrical form

$$\bar{U} = (1-\mu)\left(r_1^2 + \frac{2}{r_1}\right) + \mu\left(r_2^2 + \frac{2}{r_2}\right) = U + \mu(1-\mu),$$

instead of in the form given in (17). Then the constant C becomes

$$\bar{C} = -\tfrac{20}{9}c\alpha + 3\mu = \pm\frac{2(1-\mu)}{\xi_0} + 3\mu. \tag{30}$$

7. *The Conditions for the Existence of Closed Orbits of Ejection.*

The series (27) converge for all $|\mu| \leqslant \mu_0$ and $|\alpha - \alpha_0| \leqslant \rho$ provided $|\tau| \leqslant R$, where R is a positive constant depending on μ_0, α_0, and ρ. The coefficients of the various powers of τ are polynomials in α and μ, so far as μ occurs explicitly, and they also involve μ implicitly through c. These coefficients are expansible as power series in $\alpha - \alpha_0$ and μ which converge for all finite values of $|\alpha - \alpha_0|$ and for $|\mu| < 1$. Therefore the expressions for $x+\mu$ and y are expansible as power series in $\alpha - \alpha_0 = \beta$ and μ, and if $|\beta| \leqslant \rho$, $|\mu| \leqslant \mu_0$ the series converge for all $|\tau| \leqslant R$. They may be written

$$\begin{cases} x = p_1(\beta, \mu\,;\, \tau), & y = p_3(\beta, \mu\,;\, \tau), \\ \dfrac{dx}{d\tau} = p_2(\beta, \mu\,;\, \tau), & \dfrac{dy}{d\tau} = p_4(\beta, \mu\,;\, \tau), \end{cases} \tag{31}$$

where $p_1, \ldots, p_4$ are power series in β and μ.

Now α_0 will be determined so that when μ is zero the period from ejection to collision shall be $2j\pi$. From (10) it is found that α_0 satisfying this condition is

$$\alpha_0 = -\frac{9^{\frac{2}{3}}}{10}(2j)^{-\frac{2}{3}}. \tag{32}$$

Suppose $T < R$ and $\mu = 0$ and that for $\tau = T$ the coordinates of the infinitesimal body are x_0, x_0', y_0, y_0', where the accents denote derivatives with respect to τ. Suppose now that $0 < \mu < \mu_0$ and let the values of the coordinates at $\tau = T$ be

$$x = x_0+\beta_1, \quad x' = x_0'+\beta_2, \quad y = y_0+\beta_3, \quad y' = y_0'+\beta_4. \tag{33}$$

The conditions that these values of the coordinates shall belong to an orbit of ejection are

$$\begin{cases} x_0+\beta_1 = p_1(\beta, \mu; T), & y_0+\beta_3 = p_3(\beta, \mu; T), \\ x_0'+\beta_2 = p_2(\beta, \mu; T), & y_0'+\beta_4 = p_4(\beta, \mu; T). \end{cases} \tag{34}$$

Since the right members of these equations are expansible as converging power series in β and μ, it follows from the definitions of x_0, x_0', y_0, and y_0' that

$$\beta_1 = q_1(\beta, \mu), \quad \beta_2 = q_2(\beta, \mu), \quad \beta_3 = q_3(\beta, \mu), \quad \beta_4 = q_4(\beta, \mu), \tag{35}$$

where $q_1, \ldots, q_4$ are power series in β and μ, vanishing with β and μ.

If the infinitesimal body crosses the x-axis perpendicularly at any time the orbit is symmetrical with respect to the x-axis. It follows from the definition of x_0, x_0', y_0, and y_0' that, when $\mu = 0$,

$$x_0'(\tfrac{1}{2}P) = 0, \quad y_0(\tfrac{1}{2}P) = 0, \tag{36}$$

where P is the period in τ from ejection to collision. It will be shown that analogous conditions can be satisfied when μ is distinct from zero, and therefore that closed orbits of ejection exist in the restricted problem of three bodies.

Suppose μ is distinct from zero and consider the solution with the initial conditions $x_0+\beta_1$, $x_0'+\beta_2$, $y_0+\beta_3$, $y_0'+\beta_4$. In order to leave the period arbitrary an undetermined parameter δ is introduced by the transformation

$$\tau = \tau_1(1+\delta), \tag{37}$$

where τ_1 is the new independent variable. Now if the orbit for $\mu = 0$ does not pass through the position of μ, the solutions can be expanded as power series in $\beta_1, \ldots, \beta_4, \delta$, and μ; and if Q is arbitrarily chosen in advance, the moduli of $\beta_1, \ldots, \beta_4, \delta$, and μ can be taken so small that the solutions converge for all $T \leqslant \tau_1(1+\delta) \leqslant Q$. The quantity Q will be taken equal to $(1+\delta)\frac{1}{2}P$, where, as before, P is the period from ejection to collision when $\mu = 0$.

In order to complete the discussion in regard to the convergence, it is necessary to show that none of the orbits in question for $\mu = 0$ passes through the position of μ. Suppose $\tau = \tau_0$ is the time at which the infinitesimal body crosses the positive x-axis on which, for $\mu = 0$, the body μ lies at the distance unity from the origin. If the ejection is in the positive direction, τ_0^3 is an even multiple of 2π; and if in the negative direction, an odd multiple of π. That is, it is necessary to show that for none of the values of ξ_0 defined in (16) is equation (4) satisfied by $\xi = 1$ and $t_0 - t_1 = n\pi$, where t_1 is the time of ejection.

It follows from (4) that the larger $|\xi_0|$ is the quicker will the infinitesimal body pass from ejection to the distance unity. Therefore if for the smallest value of ξ_0 belonging to the problem, viz., $\xi_0 = 2$, the infinitesimal body reaches the distance unity for $t - t_1 < \pi$, then it will always be at a distance greater than unity at $t - t_1 = \pi$ and all multiples of π until it reaches the greatest distance ξ_0. Consequently it cannot pass through the point occupied by μ while receding from $1-\mu$; and, since the path referred to rotating axes is symmetrical with respect to the x-axis, it cannot pass through the position of μ on its return to $1-\mu$. In making the computation it is convenient to let $\xi = \xi_0 \rho$. Then, on transferring the origin to t_1, equation (4) becomes

$$t - t_1 = \frac{\sqrt{2}}{8}\left[+\pi - 4\sqrt{\rho - \rho^2} + 2\sin^{-1}(-1+2\rho)\right](\pm\xi_0)^{\frac{3}{2}}. \tag{38}$$

It is found from this equation that if $\xi_0 = 2$, the value of $t - t_1$ for $\xi = 1$ is 0.18π, which is less than π. Therefore none of the orbits in question for $\mu = 0$ passes through the position of μ.

If the infinitesimal body is moving in an orbit of ejection and crosses the x-axis perpendicularly at any time, then it follows from the symmetry of its motion that its orbit is also an orbit of collision. Therefore sufficient conditions for a closed orbit of ejection are

$$\frac{dx}{d\tau_1} = y = 0 \quad \text{at} \quad \tau_1 = \frac{P}{2}.$$

If the initial conditions are $x_0+\beta_1$, $x_0'+\beta_2$, $y_0+\beta_3$, $y_0'+\beta_4$ and the parameter δ has been introduced by (37), these equations become

$$\begin{cases} \dfrac{dx}{d\tau_1} = P_1(\beta_1, \ldots, \beta_4, \delta, \mu; \tfrac{1}{2}P) = 0, \\ y = P_2(\beta_1, \ldots, \beta_4, \delta, \mu; \tfrac{1}{2}P) = 0, \end{cases} \tag{39}$$

where P_1 and P_2 are converging power series in $\beta_1, \ldots, \beta_4, \delta$, and μ. It follows from (36) that P_1 and P_2 vanish for $\beta_1 = \ldots = \beta_4 = \delta = \mu = 0$.

If $\beta_1, \ldots, \beta_4$ are determined by (35), the orbit is one of ejection. Therefore, upon substituting the series for these constants in (39), sufficient conditions for the existence of closed orbits of ejection become

$$(40) \qquad \frac{dx}{d\tau_1} = Q_1(\beta, \delta, \mu\,;\ \tfrac{1}{2}P) = 0, \qquad y = Q_2(\beta, \delta, \mu\,;\ \tfrac{1}{2}P) = 0,$$

where Q_1 and Q_2 are power series in β, δ, and μ which vanish for $\beta = \delta = \mu = 0$. The coordinates can therefore be developed as power series in β, δ, and μ, and the moduli of these parameters can be taken so small that the series converge for $|\tau| < P$, where P is the period from ejection to collision for $\mu = 0$.

8. *Proof of the Existence of Closed Orbits of Ejection.*

The proof of the existence of closed orbits of ejection resolves itself into the demonstration that equations (40) have solutions when μ is distinct from zero. These equations are not satisfied by $\mu = 0$ unless β and δ are both also zero, because, when $\mu = 0$, the problem reduces to that of two bodies in which the period in τ from ejection to greatest distance depends upon β and δ. Therefore equations (40) have one or more solutions for β and δ as power series in μ, vanishing with μ, according as the functional determinant is distinct from zero or is zero for $\beta = \delta = \mu = 0$.

Since the functional determinant involves derivatives only with respect to β and δ, the μ may be put equal to zero before it is formed. Then the determinant in question is

$$(41) \qquad \Delta = \left| \begin{matrix} \dfrac{\partial \dot{x}}{\partial \alpha}, & \dfrac{\partial \dot{x}}{\partial \delta} \\ \dfrac{\partial y}{\partial \alpha}, & \dfrac{\partial y}{\partial \delta} \end{matrix} \right|_{\beta = \alpha - \alpha_0 = \delta = 0},$$

where $\dot{x}$ is the derivative of x with respect to τ_1. Before forming the elements of the first column, δ may be put equal to zero; and, before forming the elements of the second column, $\beta = \alpha - \alpha_0$ may be put equal to zero.

It follows from (15) that, when $\delta = \mu = 0$, $t - t_1 = 2j\pi$, the value of y is zero whatever α may be. Therefore $\partial y/\partial \alpha$ is zero and the determinant becomes simply $\Delta = (\partial \dot{x}/\partial \alpha)(\partial y/\partial \delta)$.

In the case under consideration the value of $t-t_1$ for which Δ is formed is

$$(42) \qquad t-t_1 = \tau^3 = \tau_1^3(1+\delta)^3 = \frac{P^3}{8}(1+\delta)^3 = j\pi(1+\delta)^3.$$

The second factor of Δ will be computed first. Upon putting

$$\alpha-\alpha_0 = \beta = 0,$$

it is found from the second equation of (15) that

$$(43) \qquad \frac{\partial y}{\partial \delta} = \left[-\frac{\partial \xi}{\partial \delta}\sin j\pi(1+\delta)^3 - 3j\pi(1+\delta)^2\,\xi\cos j\pi(1+\delta)^6\right]_{\delta=0}$$
$$= (-1)^{j+1}\,3j\pi\xi_0 \neq 0.$$

Before computing the first factor of Δ, the parameter δ may be put equal to zero. Hence it follows from (42), (15), and (9) that

$$(44) \qquad \frac{\partial \dot{x}}{\partial \alpha} = \frac{\partial \dot{\xi}}{\partial \alpha} = 4\tau^3\left[1-\frac{9}{7}\frac{a}{c}\tau^2+\frac{46}{21}\frac{a^2}{c^2}\tau^4-\frac{18{,}940}{4851}\frac{a^3}{c^3}\tau^6+\ldots\right]_{\substack{\alpha=\alpha_0,\\ \tau=\frac{1}{2}P}}.$$

It was proved in § 2 that the signs in this series alternate and that a is negative for those orbits which lie entirely in the finite part of the plane. Therefore $\partial\dot{x}/\partial\alpha$ is distinct from zero for all the values of α under consideraton.

It follows from this discussion that Δ is distinct from zero for $\beta = \alpha-\alpha_0 = \delta = 0$, $\tau = j\pi$, and consequently that the sufficient conditions for the existence of closed orbits of ejection can be uniquely satisfied for $|\mu|$ sufficiently small. There is a closed orbit of the type in question for ejection in both the positive and the negative direction for all integral values of j upon which the ξ_0, or α_0, of (16) depends.

In the special case in which the finite masses are equal a closed orbit of ejection for $j = 2$, with ejection in the positive direction,* was discovered from numerical experiments by Burrau in two interesting memoirs.† Since in his problem μ had the large value 0.5, it is not to be expected that the results of this analysis would agree very closely with the results of his computations. Hence the comparison will be made only for the constant of the Jacobian integral. Upon taking into account

* Burrau's orbit of ejection was from the body called μ here, but, permuting $1-\mu$ and μ, and changing the positive directions of the axes, the statements are correct.

† "Recherches Numériques concernant des Solutions Périodiques d'un Cas Spécial du Problème des Trois Corps," *Astronomische Nachrichten*, Vol. CXXXV (1894), No. 3230; and *ibid.*, Vol. CXXXVI (1894), No. 3251.

the difference in his units and those employed here, it is found that his Jacobian constant C_B, equation (5) *loc. cit.*, is related to C of (29) by the equation

$$(45) \qquad -2C_B = C = \frac{-20}{9}\,c\alpha + \mu(2+\mu) = \pm\frac{2(1-\mu)}{\xi_0} + \mu(2+\mu).$$

Burrau's computation gave $-2C_B = 2.2528$; and for $\mu = 0.5$, $\xi_0^3 = 2\sqrt{2(1-\mu)} = 2$, it is found that $C = 2.38$. The agreement between this result and that which Burrau found by computation is fully as close as would be expected. It follows from these numbers that a larger value of the constant $-\alpha$, corresponding to a smaller value of ξ_0, belongs to the undisturbed orbit having the period 2π than to that computed by Burrau. In the undisturbed orbit the greatest distance to which the infinitesimal recedes is, by (16), $\xi_0 = 2$; it has this value at $t - t_1 = \pi$, and it is then on the negative half of the x-axis. The greatest distance found by Burrau in his computation was 1.9972, or a little less than that in the undisturbed motion.

If the infinitesimal body is ejected toward or from the body μ with a small value of $|\xi_0|$, it will be disturbed so that on its return it will revolve around $1-\mu$ in the positive direction. This can be seen when the motion is considered in fixed axes, for, under the conditions postulated, the disturbance is positive all the time that the infinitesimal body is going out and returning. If it is ejected farther, it will be accelerated by μ in the negative direction part of the time. While, in general, the body will not collide with $1-\mu$ on its return, it may possibly do so under special conditions. Indeed, Sir George Darwin has discovered one such orbit by numerical experiment* having the period π. The ejection was toward μ, and the body collided with $1-\mu$ going in the same direction. † This orbit is one of a pair which together are the limit of certain periodic orbits, though they are not periodic themselves either physically or mathematically. The constant $\bar{C}$ belonging to Darwin's orbit in the units employed here is $\frac{20}{11} = 1.818$. The values of the masses used by Darwin were $1-\mu = \frac{10}{11}$, $\mu = \frac{1}{11}$. It follows from (16) that $\xi_0 = 2^{\frac{1}{3}}$ for this period, and from (30) that $\bar{C} = 1.716$. In this case the ξ_0 belonging to the undisturbed orbit is slightly larger than that belonging to Darwin's orbit.

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* "On Certain Families of Periodic Orbits," *Monthly Notices of the Royal Astronomical Society*, Vol. LXX (1909), p. 134.

† See further remarks on this orbit at the close of § 12.

The value of ξ_0 is $2^{\frac{1}{3}} = 1.26$; the greatest distance in Darwin's orbit, according to his diagram, is 1.3.

9. *The Conditions at an Arbitrary Point for an Orbit of Ejection.*

Since the motion of the infinitesimal body is regular for all finite values of t and all finite values of the coordinates except those for which it collides with one of the finite masses,* it becomes a matter of interest to determine in any special case whether the trajectory is one of ejection or collision for a finite value of t. It is sufficient, as Painlevé conjectured and as Levi-Civita proved (in case the initial point is near a finite mass),† that the coordinates and velocities shall satisfy one analytic condition in order that the orbit shall collide with one of the finite masses for a finite value of t. This conclusion will be established here in a different way for any point provided μ is sufficiently small.

Suppose μ is zero and consider the problem of defining the initial conditions for an orbit of ejection so that it shall pass through the point in question, and so that the components of velocity at the point shall satisfy as many conditions as possible. The velocity in rotating axes at any distance from the finite mass $1-\mu$ is the resultant of the velocity with respect to fixed axes and that due to the rotation of the axes. The velocity with respect to fixed axes at any finite distance can be made any finite quantity by a suitable determination of the constant α or the equivalent constant ξ_0. Consequently an arbitrary speed, or one of the components of velocity, with respect to rotating axes at any distance can be secured.

Suppose the speed at a given distance has been assigned; then it is possible to determine the initial direction of ejection so that the orbit of the body will pass through any point having the given distance, for it is possible to do it in fixed axes, and the rotation simply changes the direction of ejection by an angle which is proportional to the time required for the body to reach the distance in question. It is clear from this that when $\mu = 0$ the conditions of ejection can be so determined that the infinitesimal body shall pass through any assigned point with any assigned speed. Of the four quantities required to define an orbit the two coordinates, the speed, and direction of motion, three can be taken arbitrarily and the fourth is determined by the condition that the orbit

* Painlevé, *Leçons sur la Théorie Analytique des Equations Différentielles*, p. 583.

† *Acta Mathematica*, Vol. xxx (1906), pp. 305–327.

shall be one of ejection. The determination of the fourth quantity is double because the body has the same speed twice, once when it is receding from $1-\mu$ and once when it is returning toward $1-\mu$.

Suppose that, for $\mu = 0$, an orbit of ejection passes through the point x_T, y_T with the speed

$$v_T = \sqrt{x_T'^2 + y_T'^2} \quad \text{at} \quad (t-t_1)^{\frac{1}{3}} = T.$$

If ξ_T' represents the speed with respect to fixed axes, then, since the component of velocity due to the rotation of the axes equals numerically the distance of the point from the origin, the relation of v_T and ξ_T' is

$$v_T^2 = 1 + \xi_T'^2. \tag{46}$$

Then equation (2) determines ξ_0, the greatest distance to which the body recedes, and (13) the constant α_0. Equation (4) gives the value of T, and the direction of ejection is T degrees in the negative direction from the line joining $1-\mu$ and the point (x_T, y_T). Let the angle of ejection be θ_0.

Now suppose that μ is distinct from zero, but small. Let the initial values of α and θ be $\alpha_0+\beta$ and $\theta_0+\gamma$. Let a new independent variable τ_1 and a parameter δ be introduced by (37). Then the solution can be written in the form

$$\begin{cases} x = p_1(\beta, \gamma, \delta, \mu\,;\ \tau_1), & \dfrac{dx}{d\tau_1} = p_2(\beta, \gamma, \delta, \mu\,;\ \tau_1), \\ y = p_3(\beta, \gamma, \delta, \mu\,;\ \tau_1), & \dfrac{dy}{d\tau_1} = p_4(\beta, \gamma, \delta, \mu\,;\ \tau_1), \end{cases} \tag{47}$$

where $p_1, \ldots, p_4$ are power series in β, γ, δ, and μ. The moduli of these parameters can be taken so small that the series converge for $0 \leqslant \tau_1 \leqslant T$.

The conditions that the body shall pass through the point (x_T, y_T) with the velocity v_T at $\tau_1 = T$ are

$$\begin{cases} p_1(\beta, \gamma, \delta, \mu\,;\ T) - x_T = 0, \\ p_3(\beta, \gamma, \delta, \mu\,;\ T) - y_T = 0, \\ \sqrt{p_2^2 + p_4^2} - v_T = 0. \end{cases} \tag{48}$$

Since these equations are satisfied by $\beta = \gamma = \delta = \mu = 0$, they can be written as power series in β, γ, δ, and μ, vanishing for $\beta = \gamma = \delta = \mu = 0$, of the form

$$P_1(\beta, \gamma, \delta, \mu\,;\ T) = 0, \quad P_2(\beta, \gamma, \delta, \mu\,;\ T) = 0, \quad P_3(\beta, \gamma, \delta, \mu\,;\ T) = 0. \tag{49}$$

Equations (49) are not satisfied by $\mu = 0$ unless also $\beta = \gamma = \delta = 0$.

Therefore they have solutions for β, γ, and δ in terms of μ which vanish for $\mu = 0$. If the determinant of the linear terms in β, γ, and δ is distinct from zero, the solution is unique. In treating the problem it is convenient to use equations derived from (49) rather than these equations themselves. Let ϕ represent the angle between the positive end of the x-axis and the line from the origin to the point (x_T, y_T). Then let Q_1, Q_2, and Q_3 be defined by

$$(50) \quad Q = P_1 \cos\phi + P_2 \sin\phi, \quad Q_2 = -P_1 \sin\phi + P_2 \cos\phi, \quad Q_3 = P_3.$$

This transformation is equivalent to rotating the axes so that (x_T, y_T) lies on the positive half of the new x-axis. The solutions of

$$(51) \quad Q_1(\beta, \gamma, \delta, \mu; T) = 0, \quad Q_2(\beta, \gamma, \delta, \mu; T) = 0, \quad Q_3(\beta, \gamma, \delta, \mu; T) = 0,$$

are identical with those of (49), for the two sets of functions are linearly related with non-vanishing determinant.

The determinant of the terms of the Q_i which are linear in β, γ, and δ is

$$(52) \quad \Delta = \left| \begin{matrix} \frac{\partial Q_1}{\partial \beta}, & \frac{\partial Q_1}{\partial \gamma}, & \frac{\partial Q_1}{\partial \delta} \\ \frac{\partial Q_2}{\partial \beta}, & \frac{\partial Q_2}{\partial \gamma}, & \frac{\partial Q_2}{\partial \delta} \\ \frac{\partial Q_3}{\partial \beta}, & \frac{\partial Q_3}{\partial \gamma}, & \frac{\partial Q_3}{\partial \delta} \end{matrix} \right|_{\beta=\gamma=\delta=\mu=0}.$$

Before forming this determinant μ may be put equal to zero, and before computing the elements of each column, the parameters with respect to which the derivatives are taken in the other columns may be put equal to zero. When $\gamma = \delta = \mu = 0$, the value of Q_2 is zero for all values of β; therefore $\partial Q_2/\partial\beta = 0$. Since, for $\mu = 0$, the distance of the infinitesimal body from the origin is independent of γ and the character of the transformation (50), it follows that the part of Q_1 which is independent of μ is an even function of γ; therefore $\partial Q_1/\partial\gamma = 0$ for $\beta = \delta = \mu = 0$. Also, since, for $\beta = \delta = \mu = 0$, the velocity is independent of γ, it follows that $\partial Q_3/\partial\gamma = 0$. Hence the determinant reduces to

$$(53) \quad \Delta = \frac{\partial Q_2}{\partial \gamma} \left| \begin{matrix} \frac{\partial Q_1}{\partial \beta}, & \frac{\partial Q_1}{\partial \delta} \\ \frac{\partial Q_3}{\partial \beta}, & \frac{\partial Q_3}{\partial \delta} \end{matrix} \right|_{\beta=\gamma=\delta=\mu=0}.$$

When $\mu = 0$ the values of x and y, the coordinates referred to

rotating axes, are

$$x = \xi \cos[\theta_0+\gamma-(t-t_1)], \quad y = \xi \sin[\theta_0+\gamma-(t-t_1)],$$

where ξ has the value given in (9). Therefore the functions P_1 and P_2 become

$$P_1 = \xi(a_0+\beta, \delta; T^3)\cos[\theta_0+\gamma-T^3]-\xi(a_0+0, 0, T^3)\cos(\theta_0+0-T^3),$$

$$P_2 = \xi(a_0+\beta, \delta; T^3)\sin[\theta_0+\gamma-T^3]-\xi_0(a_0+0, 0, T^3)\sin(\theta_0+0-T^3).$$

If $\beta = \delta = 0$, the first terms of the expansions of these expressions as power series in γ are

$$P_1 = -\xi(a_0+0, 0; T^3)\sin(\theta_0-T^3)\gamma+\ldots,$$
$$P_2 = +\xi(a_0+0, 0; T^3)\cos(\theta_0-T^3)\gamma+\ldots.$$

Under the restrictions which have been imposed, $\phi = \theta_0-T^3$ and Q_2 becomes

$$Q_2 = \xi(a_0+0, 0; T^3)\gamma+\ldots. \tag{54}$$

Since $\xi(a_0+0, 0; T^3)$ is distinct from zero, $\partial Q_2/\partial\gamma$ is also distinct from zero.

Now suppose $\gamma = \mu = 0$. Then $\phi = \theta_0-T^3$ and Q_1 becomes

$$Q_1 = \xi(a_0+\beta, \delta; T^3)-\xi(a_0+0, 0; T^3). \tag{55}$$

Therefore $$\partial Q_1/\partial\beta = \partial\xi/\partial\beta, \quad \partial Q_1/\partial\delta = \partial\xi/\partial\delta,$$

the first of which is positive by the properties of ξ which were derived in § 2. The second one of these partial derivatives is positive or negative, according as, for $\mu = \beta = \delta = 0$, the infinitesimal body is receding from, or approaching toward, the origin at $t-t_1 = T^3$. If ξ has its greatest value for $t-t_1 = T^3$, then $\partial Q_1/\partial\delta_1$, which is the derivative of ξ with respect to τ, is zero.

It follows from (46), (48), and (50) that, for $\gamma = \mu = 0$,

$$Q_3 = \sqrt{1+[\xi'(a_0+\beta, \delta; T)]^2}-\sqrt{1+[\xi'(a_0+0, 0; T)]^2}. \tag{56}$$

Therefore the partial derivatives of Q_3, which appear in (53), are

$$\frac{\partial Q_3}{\partial\beta} = \frac{\dfrac{\partial\xi'(a_0+\beta, 0; T)}{\partial\beta}}{\sqrt{1+[\xi'(a_0+0, 0; T)]^2}}, \qquad \frac{\partial Q_3}{\partial\delta} = \frac{\dfrac{\partial\xi'(a_0+0, \delta; T)}{\partial\delta}}{\sqrt{1+[\xi'(a_0+0, 0; T)]^2}}. \tag{57}$$

From equation (9) it is found that

$$\xi' = 2c\tau_1(1+\delta)^2+4a\tau_1^3(1+\delta)^4-\frac{18}{7}\frac{a^2}{c}\tau_1^5(1+\delta)^6+\ldots,$$

where the signs of the coefficients alternate beyond the first term. Therefore the partial derivatives in question are

$$(58)\qquad \begin{cases} \dfrac{\partial \xi'}{\partial \beta} = +4\tau_1^3 - \dfrac{36}{7}\dfrac{\alpha_0}{c}\tau_1^5 + \ldots, \\ \dfrac{\partial \xi'}{\partial \delta} = 4c\tau_1 + 16\alpha_0\tau_1^3 - \dfrac{108}{7}\dfrac{\alpha_0^2}{c}\tau_1^5 + \ldots. \end{cases}$$

Since α_0 is negative, the first of these expressions is positive; since the velocity decreases or increases with increasing time according as the infinitesimal body is receding from or approaching toward the origin, the second of these expressions is negative if the body is receding from, and positive if it is approaching toward, the origin. If ξ' is zero for $t-t_1 = T^3$, ξ is an even function of δ because the motion is symmetrical with respect to T^3 as initial time. In this case the second of (58) is zero. Therefore, if the infinitesimal body for $\mu = \beta = \delta = 0$ is not at its greatest distance at $t-t_1 = T^3$, the form of Δ is

$$(59)\qquad \Delta = \frac{\partial Q_2}{\partial \gamma}\begin{vmatrix} + & \pm \\ + & \mp \end{vmatrix},$$

where the upper sign is to be used if it is receding from, and the lower if it is approaching toward, the origin. Since $\partial Q_2/\partial\gamma$ is distinct from zero, Δ is distinct from zero. Therefore in this case equations (48) are uniquely solvable for β, γ, and δ as power series in μ, vanishing with μ. This means that, if an arbitrary point in the xy-plane, and a velocity greater than that of this point with respect to fixed axes, be selected, then there exist two orbits of ejection passing through this point such that for μ sufficiently small the infinitesimal body will pass the point with the given velocity. The direction with which the body passes the point depends upon the initial conditions, of which it is a regular analytic function. This is Painlevé's theorem for plane motion in the restricted problem of three bodies with one of the finite masses sufficiently small.

If the velocity chosen equals that of the arbitrary point with respect to fixed axes, so that for $\mu = 0$ the point in question is at the greatest distance to which the infinitesimal body recedes, then the determinant Δ is zero and the solution is multiple. The reason is that the two solutions which were distinct in the other case have united, and the solution has become double.

10. *Closed Orbits of Ejection for Large Values of μ.*

It was proved in § 8 that closed orbits of ejection exist provided $|\mu|$ is sufficiently small. The question of their existence for large values of μ will now be considered.

If $\tau = \tau_1(1+\delta)$ the solutions of (23) may be written

$$x = f_1(a_0+\beta, \delta, \mu; \tau_1), \quad y = f_2(a_3+\beta, \delta, \mu; \tau_1), \tag{60}$$

and the conditions for a closed orbit of ejection with the period 2π in τ_1 are

$$\frac{dx(\pi)}{d\tau_1} = f_1'(a_0+\beta, \delta, \mu; \pi) = 0, \qquad y(\pi) = f_2(a_0+\beta, \delta, \mu; \pi) = 0. \tag{61}$$

It has been shown that for $|\mu|$ sufficiently small equations (61) can be solved for β and δ as converging power series in μ, vanishing with μ, and that when these results are substituted in (60) the latter become power series in μ which converge for $0 \leqslant \tau \leqslant \pi$ provided $|\mu|$ is sufficiently small.

Suppose the series for the closed orbit converge for $\mu = \mu_1$, and that the values of $a_0+\beta$ and δ for this value of μ are a_1 and δ_1. The solutions of (23) are expansible as power series in τ for values of a, δ, and μ in the vicinity of a_1, δ_1, and μ_1. If $|a-a_1| < r_1$, $|\delta-\delta_1| < r_2$, $|\mu-\mu_1| < r_3$, the series converge if $\tau_1 < T$, where T is any quantity less than the period from ejection to collision. The result will have the form of (27) where τ is replaced by $\tau_1(1+\delta)$. Each term of (27) can be expanded as a power series in $a-a_1$, $\delta-\delta_1$, $\mu-\mu_1$, which will converge provided $|\mu-\mu_1| < 1-\mu_1$. The solution may be written in the form

$$x = p_1(a-a_1, \delta-\delta_1, \mu-\mu_1; \tau_1), \quad y = p_2(a-a_1, \delta-\delta_1, \mu-\mu_1; \tau_1), \tag{62}$$

where p_1 and p_2 are power series in $a-a_1$, $\delta-\delta_1$, and $\mu-\mu_1$. Suppose the values of x, y and their derivatives at

$$\tau_1 = T \text{ for } a-a_1 = \delta-\delta_1 = \mu-\mu_1 = 0$$

are x_T, y_T, x_T', and y_T'. Let their values for an arbitrary set of values of $a-a_1$, $\delta-\delta_1$, and $\mu-\mu_1$, satisfying the inequalities which ensure convergency, be $x_T+\beta_1$, $y_T+\beta_2$, $x_T'+\beta_3$, and $y_T'+\beta_4$. Then $\beta_1, \ldots, \beta_4$ are expansible as power series in $a-a_1$, $\delta-\delta_1$, $\mu-\mu_1$ of the form

$$\beta_i = q_i(a-a_1, \delta-\delta_1, \mu-\mu_1; T) \quad (i = 1, \ldots, 4), \tag{63}$$

where the q_i vanish for $a-a_1 = \delta-\delta_1 = \mu-\mu_1 = 0$.

Now consider a solution with the initial conditions $x_T+\beta_1$, $y_T+\beta_2$, $x'_T+\beta_3$, $y'_T+\beta_4$. Suppose for $\beta_1=\beta_2=\beta_3=\beta_4=0$ the infinitesimal body does not pass through the position of μ for $T \leqslant \tau_1 \leqslant \pi$. Therefore the solutions can be expanded as power series in $\beta_1, \ldots, \beta_4$, which will converge for $T \leqslant \tau_1(1+\delta_1) \leqslant \pi$, provided the moduli of $\beta_1, \ldots, \beta_4$ are sufficiently small. At $\tau_1=\pi$ the expressions for the coordinates become, making use of (63),

$$(64)\qquad \begin{cases} x = P_1(\beta_1, \ldots, \beta_4;\ \pi) = Q_1(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1), \\ y = P_2(\beta_1, \ldots, \beta_4;\ \pi) = Q_2(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1), \\ x' = P_3(\beta_1, \ldots, \beta_4;\ \pi) = Q_3(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1), \\ y' = P_4(\beta_1, \ldots, \beta_4;\ \pi) = Q_4(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1). \end{cases}$$

The conditions that the orbit shall be closed with the period $2\pi(1+\delta)$ are

$$(65)\qquad Q_2(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1)=0, \qquad Q_3(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1)=0.$$

It has been seen that in general the solution of these equations for $\alpha-\alpha_1$ and $\delta-\delta_1$ in terms of $\mu-\mu_1$, vanishing with $\mu-\mu_1$, is unique. This is always true unless the solution becomes multiple. If the multiplicity is even there are two real solutions when $\mu-\mu_1$ has one sign and none when it has the other; if the multiplicity is odd there is one real solution for both positive and negative values of $\mu-\mu_1$. There are three solutions altogether for $|\mu|$ sufficiently small because ξ_0, defined in (16), has three determinations for which the conditions of a closed orbit of ejection can be satisfied, but only one of them is real. Consequently the real solution cannot disappear by uniting with one of the others unless they first unite and become real. Then, if two of the real solutions should unite and become complex, there would be one real one left. That is, there is one real closed orbit of ejection from $1-\mu$ for all values of μ from 0 to 1, excluding the value unity. The argument has been made for the period 2π, but it is entirely similar for any multiple of 2π.

It has been tacitly assumed in the argument that none of the orbits of ejection under consideration passes through the position of μ for any value of μ; for it was only under this condition that the convergence of the series was assured. It has been proved that the closed orbits of ejection do not pass through μ for μ sufficiently small. Since the coordinates in these orbits are regular analytic functions of μ, it follows that if any one of them passes through the position of μ for any value of μ, then for values near this one it will pass near μ. The motion of the infinitesimal

body in the vicinity of a finite body when referred to rotating axes is always in the retrograde direction, and the orbits are always symmetrical with respect to the x-axis.

Consider the motion with respect to μ in a closed orbit of ejection from $1-\mu$. Whether the ejection is toward or from μ, the motion with respect to μ in those parts of the orbit which are near to it is direct instead of retrograde. Therefore the orbit cannot pass near μ without first developing folds so that a line from μ in certain directions will intersect it three times. It is extremely improbable, though not absolutely certain, that this sort of development can take place. It is probable that the real orbits of ejection which exist for μ sufficiently small continue in the analytic sense for all values of μ from 0 to unity, and that they do not pass near μ.

For $\mu = 0$ the orbits in which the ejection is toward μ are symmetrically opposite to those in which the ejection is away from μ, and these conditions are approximately realized when μ is small. When μ increases, the loops which surround $1-\mu$ diminish in size and preserve their approximate forms, while those which surround μ approach circles whose radii are $j^{\frac{2}{3}}$, where $2j\pi$ is the period, and whose centres are at μ. For $\mu = 1-\epsilon$, where ϵ is very small, the orbits have, for $j = 1$ and $j = 2$, the forms shown in Fig. 2. There are, of course, closed orbits of ejection from both of the finite bodies.

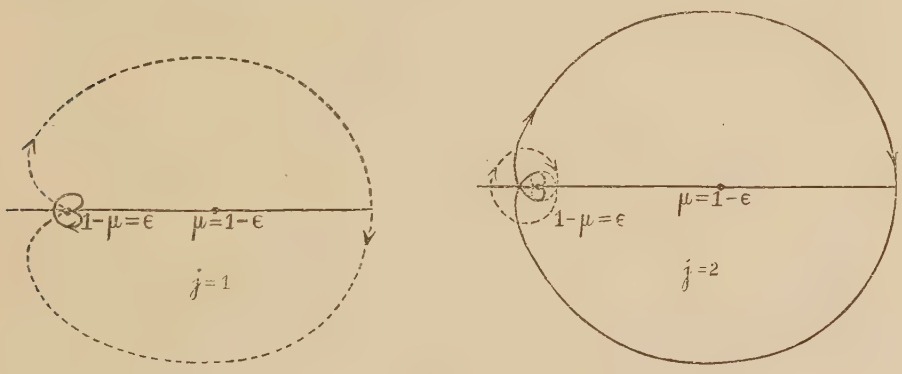

FIG. 2.

11. *Periodic Orbits Related to Closed Orbits of Ejection.*

There are periodic orbits passing near one of the finite bodies of which the closed orbits of ejection are the limits. Suppose $\mu = 0$ and consider the motion of the infinitesimal body with respect to the finite body $1-\mu$.

Let the infinitesimal body cross the x-axis perpendicularly at $t = 0$ near the body $1-\mu$, and let the initial velocity be determined so that the period is 2π or a multiple of 2π. Then the motion with respect to the rotating axes is periodic, the orbit is symmetrical with respect to the x-axis and crosses it perpendicularly at the half period. The limits of these orbits as the nearest approach to $1-\mu$ becomes zero are the closed orbits of ejection for $\mu = 0$.

The orbits under consideration exist for initial motion near the finite body in both the positive and the retrograde direction; but in both cases the motion in the remote parts of the orbits when referred to rotating axes is in the retrograde direction. Therefore only those orbits in which the motion in the vicinity of the finite body is direct have loops. The character of the two classes of orbits for periods 2π and 4π are shown in Fig. 3. There are, of course, orbits of a similar character which are symmetrically opposite with respect to the y-axis.

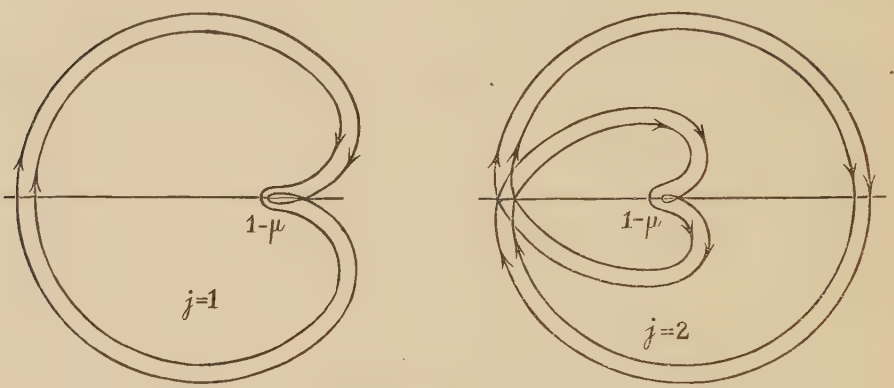

FIG. 3.

Suppose the initial conditions for one of the periodic orbits in question, when $\mu = 0$, are $x(0) = a$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = b$, and represent the coordinates for this solution by x_0, x_0', y_0, and y_0'. The distance a is small and the orbit does not pass through the point (1, 0) occupied by μ.

Now suppose μ is distinct from zero and that the initial conditions are $x(0) = a+\alpha$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = b+\beta$. Let the variable τ be introduced by $t = \tau(1+\delta)$, where δ is an undetermined parameter. Then the solutions of the differential equations of motion, which are regular functions of the coordinates and δ in the vicinity of $x = x_0$, $x' = x_0'$, $y = y_0$, $y' = y_0'$, $\delta = 0$ for $0 \leqslant \tau \leqslant j\pi$ (j an integer), can be written

in the form

$$(66)\qquad \begin{cases} x = x_0+\xi = x_0(\tau)+p_1(\alpha, \beta, \delta, \mu\,;\ \tau), \\ x' = x_0'+\xi' = x_0'(\tau)+p_2(\alpha, \beta, \delta, \mu\,;\ \tau), \\ y = y_0+\eta = y_0(\tau)+p_3(\alpha, \beta, \delta, \mu\,;\ \tau), \\ y' = y_0'+\eta' = y_0'(\tau)+p_4(\alpha, \beta, \delta, \mu\,;\ \tau), \end{cases}$$

where $p_1, \ldots, p_4$ are power series in α, β, and δ, vanishing with α, β, and δ. Moreover, the moduli of α, β, and δ can be taken so small that the series converge for $0 \leqslant \tau \leqslant j\pi$, where j is an integer.

Sufficient conditions that the orbit for μ distinct from zero shall be periodic with the period $2j\pi$ are

$$(67)\qquad p_2(\alpha, \beta, \delta, \mu\,;\ j\pi) = 0, \qquad p_3(\alpha, \beta, \delta, \mu\,;\ j\pi) = 0.$$

The problem is to show that α, β, and δ can be determined so that these equations shall be satisfied for an arbitrary μ sufficiently small. In the problem for $\mu = 0$ either a or b can be taken arbitrarily when the other is determined in terms of j except for sign; one sign belongs to the direct and the other to the retrograde orbit. It will be supposed that a is the arbitrary. Therefore it may be supposed that it absorbs the undetermined α, and leaves only two parameters in (67) besides the arbitrary μ.

Equations (67) can be solved uniquely for β and δ as power series in μ, vanishing with μ, provided

$$(68)\qquad \Delta = \left| \begin{matrix} \dfrac{\partial p_2}{\partial \beta}, & \dfrac{\partial p_2}{\partial \delta} \\ \dfrac{\partial p_3}{\partial \beta}, & \dfrac{\partial p_3}{\partial \delta} \end{matrix} \right|_{\beta=\delta=\mu=0}$$

is distinct from zero. Before this determinant is formed μ can be put equal to zero, and therefore Δ depends only upon the two-body problem. Before the second column is formed β can be put equal to zero. When $\mu = \beta = 0$ the period is $2j\pi$ in t; hence, at the half period the infinitesimal body is on the x-axis, and the value of x is an even function of $t-j\pi$. Now the parameter δ serves only to vary the period in t (keeping it fixed in τ), and is therefore equivalent to varying t from the half-period. When t is near $j\pi$, δ can be determined so that

$$t-j\pi = j\pi(1+\delta)-j\pi$$

consistently with the definition of τ. Therefore $p_2(0, 0, \delta, \mu\,;\ j\pi)$ is an even function of δ, and consequently $\partial p_2/\partial\delta = 0$ for $\beta = \delta = \mu = 0$.

Hence the determinant δ becomes

$$\Delta = \frac{\partial p_2}{\partial \beta} \frac{\partial p_3}{\partial \delta}. \tag{69}$$

The second factor is distinct from zero because it is the derivative of y with respect to t, except for a non-vanishing constant factor, at $\tau = j\pi$, and this derivative is distinct from zero.

In considering the first factor of (69) the parameter δ can be put equal to zero. If ξ and η represent the coordinates referred to fixed axes, the expression for x becomes

$$x = \xi \cos t + \eta \sin t.$$

Therefore the value of x' at $t = j\pi$ is

$$x' = (-1)^j [\xi' + \eta].$$

The problem is reduced to finding whether

$$(-1)^j \frac{\partial x'}{\partial \beta} = \frac{\partial \xi'}{\partial \beta} + \frac{\partial \eta}{\partial \beta} \tag{70}$$

is zero or not.

The expressions for ξ, ξ', and η are

$$\xi = \bar{a}[\cos E - e], \quad \xi' = -\bar{a} \sin E \frac{dE}{dt} = -\frac{\omega \bar{a} \sin E}{1 - e \cos E},$$

$$\eta = \bar{a}\sqrt{1-e^2} \sin E,$$

where E is the eccentric anomaly, $\bar{a}$ is the major semi-axis of the orbit, and ω is the mean angular motion in the orbit. Since $\sin E = 0$, $\cos E = (-1)^j$ for $t = j\pi$ and $\beta = 0$, it follows that

$$\frac{\partial \xi'}{\partial \beta} = -\frac{\omega \bar{a}}{1+e} \frac{\partial E}{\partial \beta}, \quad \frac{\partial \eta}{\partial \beta} = \bar{a}\sqrt{1-e^2} \frac{\partial E}{\partial \beta}.$$

Therefore (70) is not zero unless $\partial E/\partial \beta$ is zero. But it is found from Kepler's equation that

$$\frac{\partial E}{\partial \beta} = \frac{j\pi}{1+e} \frac{\partial \omega}{\partial \beta}.$$

It follows from the properties of the two-body problem at $t = 0$ that

$$\bar{a}(1-e) = a, \quad \bar{a} = \omega^{-\frac{2}{3}}, \quad (b+\beta)^2 = \frac{1}{\bar{a}} \frac{1+e}{1-e}. \tag{71}$$

From these equations it is found that

$$2 = a[\omega^{\frac{2}{3}} + (b+\beta)^2], \quad \frac{\partial \omega}{\partial \beta} = -3b\omega^{\frac{1}{3}} \neq 0.$$

Therefore neither factor of the right member is zero, and consequently the solution of (67) for β and δ as power series in μ, vanishing with μ, is unique. When the results of the solution of (67) are substituted in equations (66) the expressions for x, x', y, and y' become power series in μ alone (a having been taken equal to zero), and they are periodic with the period $2j\pi$.

When $\mu = 0$ the limits of the periodic orbits which are being considered as a approaches zero are the closed orbits of ejection. There are two families, depending upon the sign of b, which have the same limit. Now when μ is distinct from zero the expressions for the coordinates are expansible as power series in μ, the parts independent of μ are the periodic orbits for $\mu = 0$, and the series converge for $|\mu|$ sufficiently small, while t runs through at least half a period. Therefore the coordinates for any value of t are continuous functions of μ. Since the solutions were developed as power series in a, they are continuous functions of a for any t and μ if a is distinct from zero. But in the variables of Levi-Civita it is not necessary to make any restrictions on the initial conditions. The coordinates are in all cases uniformly continuous functions of the initial conditions for all μ, and, therefore, the limits of the periodic orbits under discussion as a becomes zero are the closed orbits of ejection, and this holds not only for μ equal to zero, but also for all μ sufficiently small.

12. *Periodic Orbits having many Near Approaches.*

The orbits considered in the preceding article are characterized by the fact that, at least for small values of μ, after the infinitesimal body leaves the point nearest $1-\mu$ it continually recedes until the mid-period, which is a multiple of π, and then it returns symmetrically. Orbits will now be considered in which the infinitesimal body recedes from and returns toward μ many times before they re-enter.

Suppose μ is zero, and that the infinitesimal body is started near $1-\mu$ on and perpendicularly to the line joining $1-\mu$ and μ; and suppose the initial conditions are such that its period is commensurable with 2π without being a multiple of 2π. Let the period be

$$P = 2\pi \frac{p}{q}, \tag{72}$$

where p and q are relatively prime integers. Then the motion with respect to the rotating axes is periodic with the period

$$T = Pq = 2\pi p. \tag{73}$$

In this period the infinitesimal body runs through q of its periods with respect to fixed axes, and the movable axes make p rotations.

Now suppose that μ is distinct from zero, and that the initial conditions are given slight variations, but of such a character that the infinitesimal body is started at right angles to the line joining the finite bodies. A new independent variable τ and a parameter δ are introduced by the relation $t = \tau(1+\delta)$. The solutions can be developed as power series in μ, δ, and the increments to the initial conditions, and the moduli of these quantities can be taken so small that these series converge for $0 \leqslant \tau \leqslant \frac{1}{2}T$. Then the conditions that the solution shall be periodic are that the orbit shall cross the x-axis perpendicularly at $\tau = \frac{1}{2}T$. These conditions have the form (67) and all of the properties of (67) which were used in proving the existence and character of their solution. Therefore the periodic orbits which are in question exist.

Now suppose that the initial distance from $1-\mu$, which was arbitrary, is made to approach zero as a limit. During this approach to zero the distances to the other near apses vary, but there is no apparent reason why all these apsidal distances should vanish at the same time. In fact, from the lack of symmetry, it is doubtful whether any two of them are simultaneously zero.

The simplest orbits of the type under consideration are those for which $p = 1$, $q = 2$. Their general form for retrograde motion in the vicinity of the finite body $1-\mu$ is shown in Fig. 4. If the distance from $1-\mu$ to

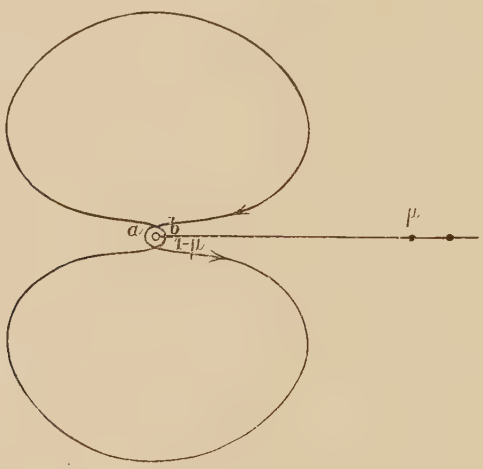

FIG. 4.

a becomes zero, the orbit of ejection discovered by Sir George Darwin is obtained. The question whether the distance from $1-\mu$ to b becomes

zero at the same time is one that is hard to answer. Certainly it can not be answered affirmatively with complete rigour by numerical experiments, though the existence of certain classes of periodic orbits can be proved in this way. If, for perpendicular ejection from a given point on the x-axis with a certain speed, the next crossing of the x-axis is at an angle which is greater than $\frac{1}{2}\pi$; and if, for a perpendicular ejection from the same point with a different speed, the next crossing is at an angle less than $\frac{1}{2}\pi$, then, from the analytic continuity, it can be inferred that there is an intermediate speed at which the crossing will be perpendicular. But in the present case the conditions are different, and all that can be said is that when the distance from $1-\mu$ to a vanishes, that from $1-\mu$ to b is small, and that the approach to $1-\mu$ is almost exactly along the x-axis. This is, of course, to be expected from the nature of these orbits.

CURVES OF CONTACT OF ANY ORDER ON ALGEBRAIC SURFACES

By HILDA P. HUDSON.

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I. *Contact along a Simple Curve.*

A recent paper* referred incidentally to contact of cubic surfaces along a straight line; the present investigation generalizes these ideas and determines the postulation and equivalence† of contact of any order of a family of algebraic surfaces along a curve, subject only to certain restrictions as to the nature of the singularities.

Let there be given a curve C_m of order m and also a surface f_κ of degree κ passing through C, and let us find the postulation, or number of independent conditions imposed upon a surface f_λ which has i-contact (contact of order $i-1$) with f_κ along C; and the equivalence, or number absorbed of the $\lambda\mu\nu$ intersections of three general surfaces f_λ, f_μ, f_ν, each of which has i-contact with f_κ along C.

Let C have no singular points except d double points with distinct tangents; let r be its rank, and† $\rho = r+2d+2m$.

Let‡ $(P_i)_\lambda = \frac{1}{2}i(i+1)(\lambda+2)m - \frac{1}{12}i(i+1)(2i+1)\rho$

$= $ postulation of C as an i-fold curve on f_λ,

$E_i = i^2(\lambda+\mu+\nu)m - i^3\rho$

$= $ equivalence of C as a common i-fold curve on f_λ, f_μ, f_ν.

It will appear that, even if f_κ had no singular points, f_λ would have at points of C a certain number of nodes of the type B_i (each of which reduces the class of the surface by i). It is therefore natural to allow f_κ to have D_l nodes of the type B_l ($l = 2, 3, \ldots, i$; B_2's are conic nodes, the

* *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 18, 1911. See also Loria, *Atti R. Acc. Sci., Torino*, tom. XXVI, p. 294, footnote, 1891.

† Cayley, *Coll. Math. Papers*, Vol. VII, p. 189, 1869 (where ρ is used for $r+2d$).

‡ Noether, *An. di Mat.*, Ser. 2, tom. V, p. 163, 1871.

other B's are binodes). Then f_λ has the same nodes except the B_i's, which are in general different both in number and position from those of f_κ. We assume that no surface has on C any more complicated singular point, thus excluding unodes and triple points, and also that at none of the nodes does C have a double point, or touch the edge of a binode.

In the case of simple contact, the tangent planes to f_λ, f_κ must coincide at all points of C at which both are determinate. If the point is a node on f_κ and not on f_λ, then the tangent plane to f_λ coincides with the limit to the tangent planes to f_κ at adjacent points of C; there is no necessity for f_λ to have a node, and all singularities are variable. But in the case of osculation, not only do the tangent planes coincide, but also the sections of f_λ, f_κ by any plane have the same curvature at points of C. Now as we approach a conic node of f_κ, this curvature becomes infinite for f_κ and therefore for f_λ, which must also have a conic node; the singularity is fixed by the condition of osculation. But for a binode this does not hold, since the curvature does not become infinite, provided the edge does not touch C. In the general case of i-contact, all variable singularities are B_i's, and all B_i's are variable.

For, let the origin O be a simple point on f_λ and a node on f_κ; if it were a conic node, we have seen that there could not be more than simple contact; it is therefore a binode, and one sheet of f_κ has $(i-1)$-contact with f_λ.

Let $$f_\lambda = \phi_1+\phi_2+\ldots+\phi_l+\ldots+\phi_\lambda,$$

where ϕ_l is a homogeneous function of xyz of degree l, and let

$$F_l = \phi_1+\phi_2+\ldots+\phi_l.$$

Then f_κ must have the form

$$f_\kappa = F_{i-2}M+\psi_i+\ldots+\psi_\kappa,$$

where M is a surface having $(i-2)$-contact at O with the second sheet of f_κ. From this form it follows that O is a B_i on f_κ.

Conversely, if O is a B_i, then f_κ has this form, and we can prove that there exists a surface f', having i-contact with f along C, but not singular at O, and independent of f in the sense of not being merely the sum of a multiple of f and terms which, equated to zero, give a surface having C as an i-fold curve, and which will be called i-fold terms. Stated algebraically, there exist multipliers M, M', such that

$$Mf'-M'f \equiv i\text{-fold terms.} \qquad \text{(I)}$$

Now, if M is arbitrary, there are no solutions of (I) other than

$$\left.\begin{aligned} f' &= af + i\text{-fold terms} \\ M' &= aM + (i-1)\text{-fold terms} \end{aligned}\right\}, \qquad \text{(II)}$$

which are not useful for our purpose, as f' is not independent of f.

Now (I) imposes a certain number of independent conditions, which, if f, M are general, fall upon the coefficients of f', M' and reduce them to the form (II). But, if f, M are specially related, a certain number of the conditions are satisfied by reason of these relations, and do not fall upon f', M'. In this case, certain arbitrary constants are left in f', besides those of (II), and there exist useful solutions of (I), whose number is that of the conditions transferred from f', M' to f, M.

Now the conditions imposed by (I) are concerned only with the behaviour of $Mf' - M'f$ at points of C; therefore this transference can only take place by reason of the behaviour of M at its points of intersection with C. Take one of these as origin; we have

$$M'f = Mf' + (x, y, z)^i + \dots .$$

Since M passes through O, and f', which contains C, also passes through O, the right hand side has a B_i at O, or a more complicated singularity if f' is singular. Therefore $M'f$ has a sheet through O which has $(i-1)$-contact with M, and if f is general, this sheet is M', and therefore $\frac{1}{2}i(i-1)$ of the conditions imposed by (I) express that M' has $(i-1)$-contact with M.

If f is singular, even if it has a sheet having l-contact with M at O, where $l < i-1$, these conditions still fall on M', and it also follows that f' has a singular point at O of the same nature as that of f. But if f has a B_i, one sheet having $(i-1)$-contact with M, then all these conditions fall away; M' does not even pass through O, and f' has a simple point there; and there exist exactly $\frac{1}{2}i(i-1)$ independent useful terms in f'.

Any multiple of f' has the same properties as f', so only $\frac{1}{2}i(i-1)$ are independent, *i.e.*, Mf' ceases to be independent when M satisfies a certain set of $\frac{1}{2}i(i-1)$ conditions, which are those of $(i-1)$-contact at O with the second sheet of f.

In the same way, corresponding to each B_l of f_κ there exists a surface f' having l-contact with f along C, and having a simple point instead of the B_l, and such that, if M is any surface having $(l-1)$-contact with the second sheet of f, then Mf' is the sum of a multiple of f and l-fold terms. With slight modifications all this applies to the case $l = 2$, when the point is a conic node. Then in the intersection of f_κ with $f'^a\phi$, where ϕ is a

β-fold term, C counts $al+\beta$ times, and if this $\geqslant i$, the term when added to f_λ does not disturb its i-contact with f_κ.

Therefore, in the equation of the general surface f_λ we admit the following terms:—

(i) All i-fold and higher terms, in number $\binom{\lambda+3}{3}-(P_i)_\lambda$.

(ii) Multiples of f_κ; in order not to repeat terms of (i) we exclude from the multiplier all $(i-1)$-fold and higher terms. The number remaining is $(P_{i-1})_{\lambda-\kappa}$.

(iii) Terms such as $f'^\alpha\phi$, where f' is the surface having l-contact with f_κ corresponding to any B_l of f_κ, and ϕ is a β-fold term where

$$al+\beta \geqslant i \quad (l = 2, 3, \ldots, i).$$

It remains to consider the number of independent terms in (iii). Let O be a B_l on f_κ, and let xy be the biplanes; let $x = z = 0$ be the tangent to C at O, and let C be given as the common intersection of as many surfaces $g, g', \ldots$ as may be necessary. We may assume

$$\begin{aligned} g &= x+(x, y, z)^2+\ldots, \\ g' &= z+(x, y, z)^2+\ldots, \\ g'', \ldots &= \quad (x, y, z)^l+\ldots, \ldots, \\ f' &= g+(g, g', \ldots)(x, y, z)^1+\ldots, \\ M &= y+(x, y, z)^2+\ldots; \end{aligned}$$

and therefore

$$f'y = \text{multiple of } f_\kappa+l\text{-fold terms}+f'\{(x, y, z)^2+\ldots\},$$

also $f'(x, y, z)^l = \text{multiple of } f_\kappa+l\text{-fold terms}$.

Now $f'^\alpha\phi$ is the sum of terms such as $f'^\alpha(g, g', \ldots)^\beta(x, y, z)^\gamma$. By means of the above equations we can eliminate $x, z, g'', \ldots, g, y$ from all the terms with $\gamma = 1, 2, \ldots, l-1$, and omit all the terms with $\gamma \geqslant l$, or with $\alpha+\beta \geqslant i$, replacing them by sums of terms counted under (i), (ii), or under other values of α, β.

Let $i = al+b$, $0 \leqslant b < l$. Then the only terms to admit are

$$f'^\alpha g'^\beta; \qquad \alpha = 1, 2, \ldots, a, \quad \beta = i-al, \ldots, i-\alpha-1\ i;$$

and $\qquad f'^{\alpha+1}(g, g')^\beta; \qquad \beta = 0, 1, \ldots, i-\alpha-2.$

Therefore the number of independent terms formed corresponding to each B_l of f_κ is

$$\sum_{\alpha=1}^{a} a(l-1)+\sum_{\beta=0}^{i-a-2}(\beta+1)=\tfrac{1}{2}\{i(i-a)-b(a+1)\}.$$

We verify that this is 0 at a simple point, where $l=1,\ a=i,\ b=0$; and is $\frac{1}{2}i(i-1)$ at a variable singularity, where $l=i,\ a=1,\ b=0$.

The postulation on f_λ of i-contact with f_κ is therefore

$$P=\binom{\lambda+3}{3}-\left[\binom{\lambda+3}{3}-(P_i)_\lambda+(P_{i-1})_{\lambda-\kappa}+\tfrac{1}{2}\sum_{l=2}^{i}\{i(i-a)-b(a+1)\}D_l\right]$$

$$=\{i(\lambda+2)+\tfrac{1}{2}i(i-1)\kappa\}m-\tfrac{1}{2}i^2\rho-\tfrac{1}{2}\Sigma\{i(i-a)-b(a+1)\}D_l$$

$$=i(P_1)_\lambda-\tfrac{1}{2}i(i-1)(\rho-\kappa m)-\tfrac{1}{2}\Sigma\{i(i-a)-b(a+1)\}D_l,$$

the sum extending to all nodes of f_κ.

For example, in the paper referred to on p. 398, there are found six cases of osculation of cubic surfaces along a straight line, for which

$$\begin{aligned} D_2 &= 0 \quad 1 \quad 2 \quad 0 \quad 1 \quad 0, \\ D_3 &= 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 2, \\ P &= 15 \quad 13 \quad 11 \quad 12 \quad 10 \quad 9. \end{aligned}$$

Here $i=3,\ \kappa=\lambda=3,\ m=1,\ \rho=2$, and though the expression for $(P_{i-1})_{\lambda-\kappa}$ is not strictly applicable, yet it gives the right value 1; these results verify the formula, which becomes

$$P=15-2D_2-3D_3.$$

In order to find the equivalence, we need to know the behaviour, at the nodes, of the residual intersection $C'_{\lambda\mu-im}$ of f_λ, f_μ.

(1) At a B_i of f_λ, we suppose that f_μ is not singular, and assume

$$f_\mu=F_{i-2}+\psi_{i-1}+\ldots,$$

$$f_\lambda=F_{i-2}M+\phi_i+\ldots,$$

where

$$M=y+(x,\ y,\ z)^2+\ldots.$$

The tangents to the total intersection $iC+C'$ are given by

$$x=0,\quad y\psi_{i-1}=\phi_i,$$

and there are i branches, viz., C counted i times, and no branch of C'.

Any other curve C'' lying on the second sheet of f_λ meets f_μ in one point only at O.

(2) At a common B_l $(i > l > 2)$ of f_λ, f_μ, we assume

$$f_\lambda = F_{l-2}M + \phi_l + \ldots,$$

and f_μ differs from a multiple of f_λ by a set of terms of which the important parts are

$$\phi_i,\ F\phi_{i-l},\ F^2\phi_{i-2l},\ \ldots,\ F^a\phi_b,\ F^{a+1},\ \ldots.$$

There are two sets of branches of the total intersection:

(i) On the first, F is small of order l, having one of a different values given by

$$\phi_i + F\phi_{i-l} + \ldots + F^a\phi_b = 0,$$

and the important term in f_λ is ϕ_l; this leads to C counted al times.

(ii) If $b > 0$, there is a second set of branches on which F is small of order b, given by

$$F^a\phi_b + F^{a+1} = 0,$$

and the important term in f_λ is $-\phi_b M$; this leads to C counted b times, and a single branch of C', which touches the edge of the binode (unless $b = l-1$), and meets the third surface f_ν in $b(a+1)$ points coinciding at O. Here C'' meets f_μ in $a+1$ points at O.

But if $b = 0$, this second set is absent; there is no branch of C', and C'' meets f_μ in a points only.

(3) At a B_2, the term FM is absent, and is replaced by ϕ_2. A similar argument leads to the same result, putting $l = 2$.

In every case C' meets f_ν in $b(a+1)$ points coinciding at a B_l. Let it also meet C in I simple points, at each of which it meets f_ν in i points. Then the number of free intersections (*i.e.*, not lying on C) of f_λ, f_μ, f_ν, or of C', f_ν is

$$\nu(\lambda\mu - im) - iI - \sum_{l=2}^{i} b(a+1)D_l.$$

The rest of the $\lambda\mu\nu$ intersections are absorbed by C, and the equivalence is

$$E = i\nu m + iI + \Sigma b(a+1)D_l.$$

To find I, consider* an auxiliary non-singular surface f_ϖ, passing through C, but not touching f_λ along it. This meets C' in $I+\Sigma_1 D_l$ points on C (where Σ_1 extends to all nodes of f_λ for which $b>0$, and Σ_2 to those for which $b=0$), and in

$$\varpi(\lambda\mu-im)-I-\Sigma_1 D_l$$

other points, which are the free intersections of $f_\lambda, f_\mu, f_\varpi$.

But the residual intersection $C'_{\lambda\varpi-m}$ of f_λ, f_ϖ passes once through each of the nodes of f_λ and meets f_μ in $\Sigma_1(a+1)D_l+\Sigma_2 aD_l$ points at these nodes; let it also meet C in I' simple points, at each of which it meets f_μ in i points; then C'' meets f_μ in

$$\mu(\lambda\varpi-m)-iI'-\Sigma aD_l-\Sigma_1 D_l$$

other points, which are again the free intersections of $f_\lambda, f_\mu, f_\varpi$. Equating these values

$$I=(\mu-i\varpi)m+iI'+\Sigma aD_l.$$

To find I', consider† the locus $f_{\lambda+\varpi-2}$ of points whose polar planes with regard to f_λ, f_ϖ meet in a straight line which meets an arbitrary straight line; this locus meets C in $(\lambda+\varpi-2)m$ points consisting of:
the r points of C at which the tangent line meets the arbitrary straight line;
the d double points of C
the I' intersections of C, C'' } which are points of contact of f_λ, f_ϖ;
and the ΣD_l nodes of f_λ.

Therefore $I'=(\lambda+\varpi)m-\rho-\Sigma D_l$

$$I=(i\lambda+\mu)m-i\rho-\Sigma(i-a)D_l,$$

$$E=i(i\lambda+\mu+\nu)m-i^2\rho-\Sigma\{i(i-a)-b(a+1)\}D_l$$

$$=iE_1-i(i-1)(\rho-\lambda m)-\Sigma,$$

the sum extending to all nodes of f_λ.

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* Cayley, *Coll. Math. Papers*, Vol. v, p. 350, 1869.
† Salmon, *Geometry of Three Dimensions*, 5th ed., § 355, p. 371, 1912.

Now E must be symmetrical in λ, μ, ν; but if $l < i$, D_l is a constant; let D_λ be the value of D_i for f_λ.

Therefore $\lambda m - D_\lambda$ is independent of λ, and equal to $\kappa m - D_\kappa$. Hence even if f_κ has no nodes, f_λ will have B_i's if $\lambda > \kappa$.

The postulation and equivalence can be expressed in the form

$$P = i(P_1)_\lambda - N, \qquad E = iE_1 - 2N,$$

where $$N = \tfrac{1}{2}i(i-1)(\rho - \lambda m + D_\lambda) + \tfrac{1}{2}\sum_{l=2}^{i-1}\{i(i-a) - b(a+1)\}D_l,$$

the sum extending to all the fixed singularities. Hence, if, and only if, $N \geqslant 0$, the curve of i-contact can be regarded as the limit of i distinct adjacent curves of intersection, and these meet one another in N points.

The foregoing also applies when f_κ has, in addition, a certain number of nodes at double points of C, provided the two branches do not lie on the same sheet at a binode. Neither C' nor C'' passes through such a point, which only affects the results by entering into ρ.

For example, let C_m be the twisted cubic given by

$$g = y - x^2, \qquad g' = z - xy, \qquad g'' = xz - y^2,$$

and let $$f_\kappa = xg + zg' + z^2 g'',$$

with a conic node at the origin. We have

$$f' = g + yg' + yzg'',$$

touching f_κ along C.

In the family f_λ osculating f_κ along C the admissible terms are:

$$f_\kappa f_{\lambda-4}, \quad f'(g, g', g'')1, x, y, z)^{\lambda-6}, \quad (g, g', g'')^3 1, x, y, z)^{\lambda-6};$$

where, to avoid repetitions, we exclude 2-fold terms from the multiplier of f_κ, and all terms but (g, g') from the multiplier of f'.

The postulation is $$9\lambda + 7 = 3P_1 + 4,$$

which agrees with the formula, putting

$$m = 3, \quad \rho = 10, \quad i = 3, \quad \kappa = 4, \quad D_\kappa = 0, \quad D_2 = 1.$$

Consider the intersection of the two members of the family f_λ, viz.,

$$f_\kappa f_{\lambda-4} \quad \text{and} \quad f_\kappa \phi_{\lambda-4} + f'(g, g').$$

We find that the residual C'_{λ^2-9} meets C in the node of f_κ and in $6\lambda - 8$ other points. It meets a third surface of the family in two points at the

node, three points at each of its other intersections with C and in $\lambda^3-27\lambda+22$ free points.

The equivalence is $27\lambda-22 = 3E_1+8$, which agrees with the formula.

II. *Contact along a Multiple Curve.*

Now let C be a j-fold line on $f_\kappa, f_\lambda, \ldots$, and let each sheet of f_λ have i-contact with the corresponding sheet of f_κ along C, which counts jk times in the intersection, where $k = j+i-1$. Then k-fold terms can be added to f_λ without disturbing the contact, and terms must now be called independent which are not the sum of a multiple of f_κ and k-fold terms.

The only higher singularities of f_κ on C that we consider are $(j+1)$-fold points. As before, we can prove that the variable singularities are $(j+1)$-planar points, at which j of the tangent planes are continuous with the tangent planes at adjacent points of C, and the last sheet of f_κ has $(i-1)$-contact with a non-singular surface M; and there exists an independent surface f' having i-contact along C with all the sheets of f_κ, and such that $M'f'$ ceases to be independent if M' has $(i-1)$-contact with M. Let f_κ have D_κ such points, and also D_l points of a similar nature with l instead of i ($l = 2, 3, \ldots, i-1$); the D_2 points are conical. Let all these singular points fall at simple points of C; then a double point of C is a j-fold point of f_κ at which all the j tangent planes along each branch coincide with the plane of the two tangents to C. The residual does not pass through such a point.

In the equation of the general surface f_λ we admit the following terms:—

(i) All k-fold and higher terms, in number $\binom{\lambda+3}{3} - (P_k)_\lambda$.

(ii) Multiples of f_κ; the number of independent terms is $(P_{i-1})_{\lambda-\kappa}$.

(iii) Terms such as $f'^{\alpha}(g, g', \ldots)^{\beta}(x, y, z)^{\gamma}$,

where $\alpha(j+l-1)+\beta \geqslant k > \alpha j+\beta$.

As before, we need only consider $\gamma = 0$, and we exclude $g'', \ldots$; since $f' = (g, g', \ldots)^j$ we can also exclude g^j and higher powers of g. Now let

$$k = \alpha(j+l-1)+b, \quad 0 \leqslant b < j+l-1.$$

(1) Then, if $b \geqslant j$, the terms to admit are

$$f'^{\alpha} g'^{\beta-j+1}(g, g')^{j-1}, \quad \alpha = 1, 2, \ldots, a, \quad \beta = k-\alpha(j+l-1) \ldots k-aj-1,$$

and $f'^{a+1}(g, g')^{\beta}, \qquad \beta = 0, 1, \ldots, k-(a+1)j-1,$

the last value of β being positive when $b \geqslant j$, and the number of independent terms formed corresponding to each $(j+1)$-fold point of f_κ is

$$\sum_{\alpha=1}^{a} \alpha(l-1)j + \sum_{\beta=0}^{k-(a+1)j-1} (\beta+1).$$

(2) But, if $b < j$, and $k > (a+1)j$, then in the multiplier of f'^{α} we admit $(g, g')^{\beta}$ instead of $g'^{\beta-j+1}(g, g')^{j-1}$ ($\beta = b, b+1, \ldots, j-1$), and the number must be diminished by $\frac{1}{2}(j-b)(j-b-1)$.

If $k \leqslant a(j+1)$, this series of values of β stops at $\beta = k-aj-1$ instead of $j-1$, but we must also omit f'^{a+1} entirely, giving the same reduction.

Let Σ extend to all the $(j+1)$-fold points of f_κ, and Σ_1 to those for which $b < j$; then the postulation is

$$\begin{aligned} P = {} & (P_k)_\lambda - (P_{i-1})_{\lambda-\kappa} \\ & - \Sigma\left[\tfrac{1}{2}j(a+1)(k-aj-b) + \tfrac{1}{2}\{k-(a+1)j\}\{k-(a+1)j+1\}\right] \\ & + \Sigma_1 \tfrac{1}{2}(j-b)(j-b-1). \end{aligned}$$

To determine the behaviour of the residual, let the origin be one of the D_l points; we assume

$$f_\lambda = FM + \phi_{j+l-1} + \ldots,$$

$$f_\mu = \text{multiple of } f_\lambda + \phi_k + F\phi_{k-(j+l-1)} + \ldots + F^a\phi_b + F^{a+1} + \ldots,$$

where the lowest terms in F are of degree j.

(1) If $b \geqslant j$, the term of lowest degree in f_μ is F^{a+1}.

(i) On one set of branches, F is small of order $j+l-1$, having one of a different values, and the important term in f_λ is ϕ_{j+l-1}; this leads to C counted $aj(j+l-1)$ times.

(ii) On the second set, F is small of order b, and the important term in f_λ is $-\phi_b M$; this leads to C counted bj times, and j branches of C', which touch $F = M = 0$ (unless $b = j+l-2$), and each meets f_ν in $b(a+1)$ points at O. Any other curve C'' on the last sheet of f_λ meets f_μ in $j(a+1)$ points at O.

(2) If $b < j$, the term of lowest degree in f_μ is $F^a\phi_b$; there are b branches of C', which touch $\phi_b = M = 0$, and each meets f_ν in $aj+b$ points at O. Here C'' meets f_μ in $aj+b$ points at O.

Let C' also meet C in I ordinary points, at each of which it meets f_ν in k points. Then the number of free intersections of f_λ, f_μ, f_ν is

$$\nu(\lambda\mu - jkm) - kI - \Sigma jb(a+1) + \Sigma_1 b(j-b).$$

To find I, consider the auxiliary surface f_ϖ, passing once only through C. The number of free intersections of C', f_ϖ is

$$\varpi(\lambda\mu - jkm) - I - \Sigma j + \Sigma_1(j-b),$$

and this is equal to the number of free intersections of f_μ and the residual intersection C'' of f_λ, f_ϖ, viz.,

$$\mu(\lambda\varpi - jm) - kI' - \Sigma j(a+1) + \Sigma_1(j-b),$$

where I' is the number of intersections of C'', C at ordinary points of f_λ. We find

$$I' = (\lambda + j\varpi)m - j\rho - \Sigma 1,$$

whence

$$I = (j\mu + k\lambda)m - jk\rho - \Sigma(k - aj),$$

and the equivalence is

$$E = jk(\lambda+\mu+\nu)m - k(j-k)\lambda m - jk^2\rho - \Sigma\{k(k-aj) - jb(a+1)\} - \Sigma_1 b(j-b),$$

the sums extending to the $(j+1)$-fold points of f_λ.

As before, $\lambda m - D_\lambda$ is independent of λ, and the postulation and equivalence can be expressed in the forms

$$P = (P_j)_\lambda + j(i-1)(P_1)_\lambda - jN_1 - N_2, \quad E = E_j + j(i-1)E_1 - (3j-1)N_1 - 2N_2,$$

where

$$N_1 = -(i-1)\lambda m + j(i-1)\rho + \Sigma\{k - (a+1)j\} + \Sigma_1(j-b),$$

$$N_2 = -\tfrac{1}{2}(i-1)(i-2j)\lambda m + \tfrac{1}{2}j(i-1)(i-j-1)\rho + \Sigma[\tfrac{1}{2}(a+1)j(2j-i-b) + \tfrac{1}{2}k(i-j)] - \Sigma_1 \tfrac{1}{2}(j-b)(3j-b-1),$$

which would be the formulæ for a system consisting of a j-fold curve and $j(i-1)$ distinct simple curves having N_1 intersections with the j-fold curve and N_2 with one another.

For example, consider a quartic family containing a double straight line and four simple straight lines all passing through a common triple point of the surfaces at the origin:

$$f_\lambda = \phi_3+\phi_4, \quad f_\mu = \psi_3+\psi_4, \quad f_\nu = \chi_3+\chi_4,$$

where $\phi_3 \ldots \chi_4$ can be interpreted as plane curves having in common a double point and four simple points; therefore $P = 24$.

Now the residual C'_8 projects into the plane septic $\phi_3\psi_4-\phi_4\psi_3$ (shewing that C' has one branch through the origin), which has a fixed quadruple point, four fixed double points, and also passes through the remaining four intersections of ϕ_3, ϕ_4. It therefore meets the corresponding curve formed from f_λ, f_ν, viz., $\phi_3\chi_4-\phi_4\chi_3$, in thirteen variable points which are the projections of the free intersections of f_λ, f_μ, f_ν; therefore $E = 51$.

Now let f_κ be the simplest surface containing the system, viz., a cubic cone, and let two of the simple lines approach the double line along each sheet of f_κ. In the limit, we have

$$m = 1, \quad \rho = 2, \quad j = 2, \quad i = 3, \quad \kappa = 3, \quad D_\kappa = 0, \quad D_2 = 1,$$

$$\lambda = \mu = \nu = 4,$$

and the formulæ are verified. Note that $N_1 = N_2 = 3$.

It is possible to carry the theory farther in many directions, *e.g.*, so as to take account of higher singularities of the curve and surfaces, or of the cases in which the curve is a cuspidal edge, curve of self-contact, &c., on f_κ. For a multiple curve we may also investigate the conditions that the sheets of f_λ fall into several sets, each set having contact of a given order with a given surface.

As a simple example, let C be a straight line of multiplicity j, α, β, γ on $f_\kappa, f_\lambda, f_\mu, f_\nu$ respectively, and let j sheets of each surface touch the corresponding sheets of f_κ. If f_κ has no $(j+1)$-fold points, we find

$$P = (P_\alpha)_\lambda + j(P_1)_\lambda - \{\alpha N_1 + N_2\},$$

$$E = E_{\alpha\beta\gamma} + jE_1 - \{(\alpha+\beta+\gamma-1)N_1 + 2N_2\},$$

where $$N_1 = 2j-\kappa, \quad N_2 = (\kappa-j)(j-1),$$

as for a common straight line, of multiplicity α, β, γ on f_λ, f_μ, f_ν respectively, and j distinct simple lines having N_1 intersections with the multiple line and N_2 with one another. This system is only possible provided

$$0 \leqslant N_1 \leqslant j, \quad 0 \leqslant N_2 \leqslant \tfrac{1}{2}j(j-1),$$

which are equivalent to $\kappa \leqslant \frac{3}{2}j$, unless $j = 1$, when $\kappa \leqslant 2$.

The subject is full of curious paradoxes. For example, in the case of simple contact, let f_κ be a general cubic surface, and C a simple straight line; the correct formulæ are obtained by supposing f_λ to contain a pair of straight lines intersecting in -1 points. Again, let f_κ consist of a pair of quadrics with a common generator, and let C be this nodal straight line; then f_λ can be similarly regarded as containing a nodal line and two distinct simple lines, the latter not meeting the nodal line, but meeting one another in two points. The explanation in the first example is that f_κ does not contain any straight line adjacent to C, for its twenty-six other straight lines are all at a finite distance. In the second example, the two points of contact of the two quadrics which lie on C are points of self-contact (tacnodes) in the case which arises as a limit, and only uniplanar points (unodes) in the case to which the formulæ apply.

THE LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT BY THE TREASURER FOR THE YEAR 1911–12 (NOV. 9TH, 1911, TO NOV. 14TH, 1912).

GENERAL CASH ACCOUNT OF THE SOCIETY.

Cr.

	£.	s.	d.
Balance from 1910–11 :—			
In General Fund ... ... £ 38 12 3			
In Life Compositions Fund 141 15 0			
	180	7	3
Interest on Capital—			
Lieut.-Col. Campbell's Fund 14 10 0			
Life Compositions Fund ... 66 14 0			
Invested Surplus Fund ... 19 15 8			
	100	19	8
Interest on Lord Rayleigh's Fund	47	18	3
22 Entrance Fees	23	2	0
8 Life Compositions (7 new scale, 1 old) ...	120	15	0
Income Tax returned (3 years)	18	0	0
112 Annual Subscriptions—			
3 for 1908–09 ... ... 3 3 0			
8 for 1909–10 ... ... 8 8 0			
15 for 1910–11 ... ... 15 15 0			
83 for 1911–12 ... ... 87 3 0			
3 in advance 3 3 0			
	117	12	0
Sale of *Proceedings*, &c.	165	13	6
	£774	7	8

Dr.

	£.	s.	d.
Printing and Circulating *Proceedings*, &c. ...	382	16	9
Purchase of Journals, &c., for Library	7	17	9
Postage and Sundries	5	12	7
Rent	50	0	0
Attendance	8	8	0
Providing Tea at Meetings	1	11	6
Cost of £106. 10*s.* 11*d.* New South Wales 3½ per Cent. Stock of 1918	105	0	0
Balance at Bank—			
In General Fund... ... £55 11 1			
In Life Compositions Fund 157 10 0			
	213	1	1
	£774	7	8

TRUST FUND ACCOUNTS.

I. Lord Rayleigh's Fund.

	£.	s.	d.
Annuity (less transfer to Sinking Fund)... ...	34	0	11
Dividends on Stock	6	13	9
Income Tax returned (3 years)	7	3	7
	£47	18	3

	£.	s.	d.
Transferred to the General Account of the Society, as above	47	18	3
	£47	18	3

II. De Morgan Medal Fund.

	£.	s.	d.
Balance from previous account	0	11	4
Interest during the year 1911–12	2	16	6
Income Tax returned (3 years)	0	9	11
	£3	17	9

	£.	s.	d.
Balance at Bank	3	17	9
	£3	17	9

STATEMENT OF CAPITAL, November, 1912.

	Sum Invested. £.	s.	d.	£.	s.	d.	Description of Investment.
1. Lieut.-Col. Campbell's Fund	499	7	5	385	0	0	Guaranteed 4 per Cent. London & North Western Railway Consolidated Stock.
2. Life Compositions Fund, from 1910–11 ...	2080	15	0	2076	19	3	New South Wales 3½ per Cent. Inscribed Stock of 1918.
Amount received during 1911–12 ...	120	15	0	157	10	0	In Bank awaiting investment.
3. Invested Surplus Fund	700	0	0	700	7	1	India 3 per Cent. Stock.
Trust Funds.							
1. Lord Rayleigh's Fund	1000	0	0				Terminable Annuity of £50. 8*s.* 8*d.* (Class B, deduction for Sinking Fund) Great Indian Peninsula Railway.
				200	0	0	Guaranteed 3 per Cent. Stock of same Railway Co.
2. De Morgan Medal Fund	102	5	9	60	0	0	Great Western Railway 5 per Cent. Preference Stock.

Signed on behalf of the Council by A. E. H. LOVE,
President.

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

3 *December*, 1912.

J. G. LEATHEM.

CONTRIBUTIONS TO THE ARITHMETIC THEORY OF SERIES

By G. H. HARDY *and* J. E. LITTLEWOOD.*

[Received March 24th, 1912.—Read April 11th, 1912.]

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I.	§§ 1–4.	Introduction and Summary of Results.
II.	5–15.	Theorems concerning the Successive Derivatives of a Continuous Function of a Real Variable.
III.	16–27.	Applications to the Theory of Dirichlet's Series.
IV.	28–43.	Additional Applications. Theorems connected with the General Abel-Tauber Theorem. Theorems whose Conditions involve the Differences of the Terms of a Series.
V.	44–58.	Miscellaneous Theorems.
	44–50.	Negative Indices and the Problem of Multiplication.
	51–54.	Tauber's Theorem for Double Series.
	55–58.	Various Extensions of the Ordinary Form of Tauber's Theorem.

I.

Introduction and Summary of Results.

1. This paper represents a continuation of researches published in three recent papers in these *Proceedings*.† It resembles them in that the general character of the theorems which it contains is "Tauberian": they are theorems of the type whose first example was the beautiful converse of Abel's theorem originally proved by Tauber.‡

Section II contains a systematic development of an idea already to be

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* Some of the results of this paper were communicated to the Society at the meeting of January 11th, 1912, and an Abstract of them published under the title "A New Condition for the Truth of the Converse of Abel's Theorem relating to Power Series." These results are now included in Section IV of the paper.

† Hardy, "Theorems relating to the Summability and Convergence of Slowly Oscillating Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320.

Littlewood, "The Converse of Abel's Theorem," *ibid.*, Ser. 2, Vol. 9, pp. 434–448.

Hardy and Littlewood, "The Relations between Borel's and Cesàro's Methods of Summation," *ibid.*, Ser. 2, Vol. 11, pp. 1–16.

‡ A general explanation of the nature of a "Tauberian" theorem is given in § 1 of the last of the three papers just quoted.

found in the second of the three papers referred to.* It was there proved that, if $f(x)$ is a function of x, with first and second derivatives continuous for $x > x_0$, and if

$$f(x) = s + o(1), \qquad f''(x) = O(1),$$

as $x \to \infty$, then $f'(x) = o(1)$; and thus that, if $f(x)$ has derivatives of all orders, which remain limited as $x \to \infty$, then $f(x)$ cannot tend to a limit unless all its derivatives tend to zero. Here we discuss a number of theorems of this character more thoroughly. In Theorem 1 we show that if

$$f(x) = O(\phi), \qquad f''(x) = O(\psi),$$

where ϕ and ψ are increasing functions, then

$$f'(x) = O\{\sqrt{(\phi\psi)}\};$$

with corresponding results when one or other of the O's is replaced by o. From this theorem we deduce others involving derivatives of higher orders. All these theorems seem to us of considerable interest in themselves, apart from any applications to the theory of series.

So far we have considered increasing functions ϕ, ψ. It is important, for subsequent applications, to obtain theorems of a similar character in which ϕ and ψ are replaced by decreasing functions. These theorems, of which Theorem 5 is the first and most fundamental, are a little more delicate and require some additional restrictions. We have found it most convenient to continue to suppose that $\phi, \psi, \ldots$ are increasing functions, but to consider, instead of relations of the type $f^{(r)} = O(\psi)$, relations of the type $x^r f^{(r)} = O(\psi)$.

2. In Section III we proceed to apply the results of Section II to the theory of ordinary Dirichlet's series—series, that is to say, of the type $\Sigma a_n n^{-s}$. There are two theorems in this section more important than the rest. The first is Theorem 18, which asserts that, if Σa_n is summable (Cr), and $a_n = O(n^a)$, where $a > -1$, then $\Sigma a_n n^{-s}$ is summable (Ck), where $0 \leqslant k \leqslant r$, for

$$s = (a+1)(r-k)/(r+1).$$

When $a = -1$, this theorem reduces to the principal theorem of the first of the three papers quoted at the beginning of § 1. This last theorem we shall describe as the *general Cesàro-Tauber theorem*—the theorem of Tauberian type which is concerned with Cesàro's method of summation

* Littlewood, *l.c.*, p. 437.

and whose hypotheses involve an O. It will be convenient to extend this system of nomenclature, calling theorems *general* if they involve an O and *special* if they involve an o. Thus Tauber's theorem is the special Abel-Tauber theorem, its extension the general Abel-Tauber theorem, Theorem 1 of our paper on Borel's method of summation is the special Borel-Tauber theorem, and Fatou's well known theorem may be described as the Euler-Tauber theorem.*

The second principal theorem of Section II is Theorem 19. In this, we suppose, as in Theorem 18, that Σa_n is summable (Cr), but instead of assuming a hypothesis of the type $a_n = O(n^\alpha)$, we suppose that $\Sigma a_n n^{-s}$ is known to be summable $(C, r-k)$, where $0 \leqslant k \leqslant r$, for $s = \beta > 0$. We then prove that the series is summable $(C, r-l)$, for $0 \leqslant l \leqslant k$, when $s = l\beta/k$. An interesting corollary of the special case of Theorem 19 in which $k = 2$, $l = 1$ is Bohr's theorem that the breadth of the strip between the r-th and $(r+1)$-th lines of summability is a decreasing function of r.

3. Theorem 26, the chief of the early theorems of Section IV, is an extension of the general Abel-Tauber theorem, and related to it much in the same way as some of the preceding theorems are related to the general Cesàro-Tauber theorem. But, in this section, we are, for the most part, concerned with theorems of a different character. These theorems now take a subordinate position in the paper, but it is from them that our investigations started. Their distinguishing characteristic is that their conditions involve the differences of successive terms of the series. Thus Theorem 27, the principal result of this section, states that a sufficient condition for the truth of the converse of Abel's theorem is that

$$\Delta a_n = a_n - a_{n+1} = O(a_n/n).$$

The proof, which is intricate and difficult, shows first that this condition, together with the existence of Abel's limit, involves

$$a_n = o(1/n);$$

the main result then follows from Tauber's theorem. In view of the difficulty of the proof, we have thought it worth while to discuss separately the corresponding theorem for Cesaro summability of the first order, the

* It is reasonably near the truth to represent Euler as defining the sum of Σa_n as the value of $f(x) = \Sigma a_n x^n$ for $x = 1$, when the function is regular there; cf. Bromwich, *Infinite Series*, pp. 265 *et seq.* Fatou's theorem is essentially a theorem for o: it becomes false if o is replaced by O. The general Borel-Tauber theorem, on the other hand, is no doubt true, but as yet unproved.

proof of which is fairly simple. We have also indicated an alternative form of condition, suggested by a theorem of Landau. We also show that, when $\Delta a_n = o(a_n/n)$, the series *cannot* converge, whence it follows that, for such a series, Abel's limit cannot exist. Further, $|na_n| \to \infty$, and $s_n \sim na_n$.

It is to be observed that these theorems become trivial when a_n is real, as then a_n cannot change its sign. They have interesting applications to real series, the real parts of complex series which satisfy their conditions, such as

$$\Sigma n^{-s} \cos(a \log n);$$

but these real series do not themselves satisfy the conditions.

We also establish the analogous theorem (Theorem 31) relating to Borel's method.

These theorems seem at first sight curious, but artificial. We are, however, inclined to think that they are in reality of some interest. This appears more clearly if we try to frame a theorem of this character but connected with Fatou's theorem. The theorem would evidently run: *if* $\Delta a_n = o(a_n)$, *and* $f(x) = \Sigma a_n x^n$ *is regular for* $x = 1$, *then* Σa_n *is convergent.* This is true; but for the reason that, if $\Delta a_n = o(a_n)$, $f(x)$ *cannot* be regular for $x = 1$.* For $\Delta a_n = o(a_n)$ is equivalent to $a_{n+1}/a_n \to 1$, and a well known theorem first rigorously established by Fabry,† tells us that $x = 1$ is certainly a singular point of $f(x)$. That the theorem thus suggested, though it proves in fact to be trivial, should appear as a link between two such important theorems, leads us to think that the artificiality of some of the theorems of this section is more apparent than real, and that conditions of the types here considered do correspond to important classifications of "slowly oscillating" series.

We conclude this section by pointing out an easy generalisation of Fatou's theorem, in which the condition $a_n = o(1)$ is replaced by $\Delta a_n = o(1)$ or $\Delta^r a_n = o(1)$, and by showing that the general Abel-Tauber theorem is a simple corollary of the theorem that $\Delta a_n = O(1/n^2)$, together with the existence of Abel's limit, implies the convergence of Σa_n. This theorem is an obvious consequence of the Abel-Tauber theorem, but it is rather surprising that the latter should be deducible from it in an elementary manner.

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* The theorem is true because a false proposition implies *all* propositions, and therefore any theorem with contradictory hypotheses is true (Whitehead and Russell, *Principia Mathematica*, Prop. 2·21).

† Fabry, *Annales de l'École Normale*, Ser. 3, Vol. 13; Hadamard, *La Série de Taylor*, pp. 23 *et seq.*

4. The contents of Section V are of a more miscellaneous character, and do not depend, as does nearly all the rest of the paper, on the general theorems of Section II. The chief theorem of §§ 44–50 is Theorem 37, which states that a convergent series for which $a_n = O(1/n)$ is summable $(C, -1+\delta)$ for all positive values of δ. This result is interesting in several ways. In the first place it shows that, for such series, *all** methods of Cesàro summation are equivalent, and so supplies an interesting completion of the Cesàro-Tauber and Abel-Tauber theorems. It also enables us to generalise and to connect with these theorems the "multiplicative" property of series for which $a_n = O(1/n)$.

In §§ 51–54 we discuss the analogue of Tauber's theorem for double series. We have had in our possession for several years a special form of this theorem (Theorem 42), in which the fundamental condition is taken to be

$$(m+n)^2 a_{m,n} \to 0.$$

The more general conditions employed in Theorem 40 were suggested† to us by Prof. W. H. Young, who had used them to establish the corresponding "double Cesàro-Tauber" theorem. Prof. Young's very opportune suggestion has led us to a further generalisation and enabled us for the first time to get a clear view of the analogy between the one- and two-dimensional cases. The paper concludes (§§ 55–58) by a discussion of certain further extensions of the ordinary form of Tauber's theorem. The main theorem here is Theorem 49, the purpose of which is to show the striking contrast which arises between Abel's theorem and its converse when paths which touch the circle of convergence are considered.

In spite of the length of the paper and the variety of the topics with which it deals, it leaves open many obvious questions for future discussion. The chief of these are connected with the general theorems of Section II. In the first place, there are clearly theorems of a similar character to be proved involving Riemann-Liouville derivatives of non-integral order, which should enable the results of Section III to be extended to non-integral orders of summability. There must also be theorems involving an arbitrary increasing function $\lambda(x)$, which should enable us to deal with the Rieszian methods of summation appropriate to general Dirichlet's

* Methods of non-integral order less than -1 can be defined, but, as Chapman (*Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 376) has shown, do not give results satisfactory from the point of view of a general theory. No practicable definition of negative integral orders of summability has been given.

† At the meeting of January 11th, 1912. See also Young, "On Multiple Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 167.

series of the type

$$\Sigma a_n e^{-\lambda_n s}.$$

Finally, there is the question of substituting, for conditions of the type $f = O(\phi)$, right-handed or left-handed conditions of the type

$$f = O_R(\phi), \qquad f = O_L(\phi)$$

i.e.,

$$f < K\phi, \qquad f > -K\phi.$$

Landau* has extended the general Cesàro-Tauber theorem to the case in which

$$a_n = O_L(1/n);$$

and it is fairly obvious that many of the theorems of this paper are susceptible of a similar generalisation.†

II.

Theorems concerning the Successive Derivatives of a Continuous Function of a Real Variable.

5. It may facilitate the understanding of the proofs of the theorems of this section if we begin by considering the simplest case in a geometrical and not in an analytical manner. In this case the single idea which underlies all the theorems becomes obvious intuitively. We wish to establish in this rough way the truth of the following theorem :—

If $f''(x)$ is continuous, and $f \to s$, $f'' = O(1)$, as $x \to \infty$, then $f' \to 0$.

Consider (Fig. 1) the curve

$$y = f'(x).$$

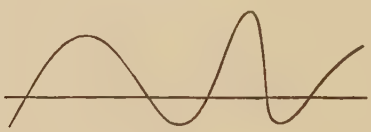

FIG. 1.

If $f'(x)$ does not tend to zero the curve has peaks whose height does not tend to zero. But since $\int^x y\,dx$ tends to a limit as $x \to \infty$, these peaks

* *Prac Matematyczno-Fizycznych*, Vol. 21, p. 97.

† See the additional note at the end of the paper.

must each have an area which tends to zero. They must therefore tend to become (in some places) indefinitely steep, and this is incompatible with $f''(x) = O(1)$.

The theorems which follow contain a rigorous development of this idea.

6. THEOREM 1.—*If $\phi(x)$ and $\psi(x)$ are positive increasing* functions of x, and $f''(x)$ is continuous,† then*

(*a*) *if $f = O(\phi)$ and $f'' = O(\psi)$, then $f' = O\{\sqrt{(\phi\psi)}\}$;*

(*b*) *if $f = o(\phi)$, or if $\phi = 1$ and $f \to s$, and $f'' = O(\psi)$, then $f' = o\{\sqrt{(\phi\psi)}\}$;*

(*c*) *if $f = O(\phi)$ and $f'' = o(\psi)$, then $f' = o\{\sqrt{(\phi\psi)}\}$.*

We shall suppose throughout the proofs in the whole section that $f(x)$ is real. The theorems, however, may be extended at once to the case when $f(x)$ is complex. It is legitimate and convenient to suppose that $f(x)$ is defined for $x \geqslant 0$.

We shall content ourselves by giving the proof of (*c*), which requires a little more consideration than (*a*) or (*b*). The modifications necessary in the other cases will easily be supplied by the reader. It should be observed that the second case in (*b*) reduces to a special case of the first when we consider the function $f - s = o(1)$ instead of f.

We have $|f(x)| < A\phi(x)$, where A is a constant, for all values of x, and $|f''(x)| < \epsilon\psi(x)$, where ϵ is an arbitrary positive number, for every x to the right of some point C depending on ϵ.

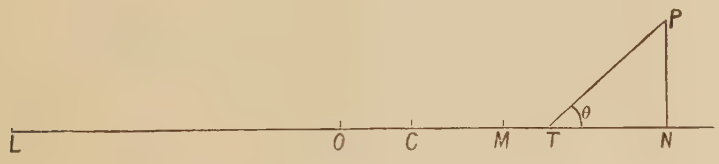

FIG. 2 (i). *Case* (i).

Consider the curve $y = f'(x)$. Let P or (x_0, y_0) be a point of it to the

* The word "increasing" is used throughout the paper to mean "not decreasing."

† The hypothesis of continuity is much more stringent than is necessary for the truth of the theorem. We adopt it partly because we are not aiming at generality of this particular kind, and partly because the more or less geometrical line of argument which the hypothesis makes legitimate seems simpler to follow than a purely analytical one.

right of C, N the foot of its ordinate, M the mid-point of ON, L the point to the left of O such that $OL = ON$. We suppose that y_0 is, say, positive. Let T or $(x_1, 0)$ be the point to the left of N, such that

$$\tan\theta = \tan PTN = \epsilon\psi(x_0).$$

Then between C and N we have

$$\frac{dy}{dx} < \epsilon\psi(x) < \epsilon\psi(x_0) = \tan\theta,$$

and hence the curve lies above the straight line PT between C and N.

We shall distinguish three cases:

(i) T lies between M and N;

(ii) T lies between M and L;

(iii) T lies to the left of L.

Case (i).—M, and therefore T, is to the right of C when x_0 is large, and so

$$2A\phi(x_0) \geqslant A\{\phi(x_0)+\phi(x_1)\}$$

$$\geqslant |f(x_0)-f(x_1)| = \left|\int_{x_1}^{x_0} y\,dx\right|$$

$$> \text{area } PTN = \tfrac{1}{2}y_0^2\cot\theta.$$

Thus

$$y_0^2 < 4A\epsilon\phi(x_0)\psi(x_0). \tag{1}$$

Case (ii).—The curve lies above PM between M and N. Hence, arguing as above, we obtain

$$2A\phi(x_0) > \text{area } PMN = \tfrac{1}{4}\text{area } PLN > \tfrac{1}{4}\text{area } PTN,$$

and so

$$y_0^2 < 16A\epsilon\phi(x_0)\psi(x_0). \tag{2}$$

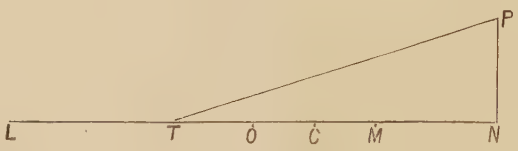

FIG. 2 (ii). *Case* (ii).

Case (iii).—If T lies to the left of L, and so of course to the left of O, it is evident that f' is positive and f monotonic for $0 < x \leqslant x_0$.

Let PT meet the ordinate at C in B. Then

$$PN/BC = TN/TC < LN/LC < LN/LO = 2.$$

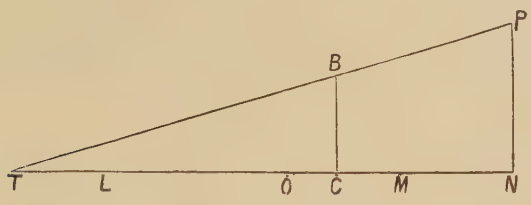

FIG. 2 (iii). *Case* (iii).

Hence y_0 is less than twice the ordinate of the curve at C, a number which depends only on ϵ. If either of the increasing functions ϕ and ψ tends to infinity, their product tends to infinity; and so evidently

$$y_0^2 < \epsilon\phi(x_0)\psi(x_0), \tag{3}$$

for sufficiently large values of x_0. If, on the other hand, both ϕ and ψ remain finite, we have

$$2A\phi(x_0) > \text{area } PBCN > \text{area } PMN = \tfrac{1}{4}x_0 y_0.$$

Hence $y_0^2 < K/x_0^2$, and so (3) again holds for sufficiently large values of x_0.

It follows that, for *all* sufficiently large values of x_0, one or other of (1), (2), (3) must hold. Hence

$$y = o\{\sqrt{(\phi\psi)}\},$$

and the theorem is established.

7. It should be observed that the theorem ceases to be always true if the condition that ϕ and ψ are increasing functions is removed. Thus if

$$f = x\log x - x, \quad f' = \log x, \quad f'' = 1/x,$$

and

$$\phi = x\log x, \quad \psi = 1/x,$$

we have

$$f = O(\phi), \quad f'' = O(\psi),$$

but it is not true that

$$f' = O\{\sqrt{(\phi\psi)}\}.$$

It is instructive to consider Theorem 1 in the light of known results in the theory of "logarithmico-exponential" functions (L-functions).* If

* Hardy, "Orders of Infinity," *Camb. Tracts*, No. 12, pp. 39 *et seq.*; *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 54 *et seq.*

f is an L-function it is *generally* true that

$$f' \asymp \sqrt{(\pm ff'')}$$

(the sign being chosen so as to make the contents of the bracket positive). There are exceptional cases when the increase or decrease of f or f' is less than that of any power of x: thus the result ceases to be true if f is any one of the functions

$$\log x, \quad 1/(\log x), \quad x \log x, \quad x/(\log x).$$

Similar results can be obtained when $f = \lambda \cos \mu$ or $\lambda \sin \mu$, where λ and μ are L-functions.

8. THEOREM 2.—*If* $\phi(x)$ *and* $\psi(x)$ *are increasing functions of* x *and* $\psi \to \infty$; *and if* $f = O(\phi)$, *and* $f^{(r)} = O(\phi\psi)$ *for* $r \geqslant 1$;* *then* $f^{(r)} = O(\phi\psi^{\epsilon})$ *for any arbitrarily small* ϵ.

By Theorem 1 we have

$$f' = O(\phi\psi^{1/2}). \tag{1}$$

From (1) and $f''' = O(\phi\psi)$, we have similarly

$$f'' = O(\phi\psi^{3/4}). \tag{2}$$

Similarly,

$$f''' = O(\phi\psi^{7/8}), \tag{3}$$

and so on.

If now we apply Theorem 1 successively to $f, f', \ldots$, using (2), (3), ..., we obtain a new set of equations similar to (1), (2), (3), in which the indices are reduced. This process may evidently be repeated indefinitely: suppose that after n repetitions we obtain

$$f' = O\{\phi\psi^{\alpha_n^{(1)}}\},$$
$$f'' = O\{\phi\psi^{\alpha_n^{(2)}}\},$$
$$\ldots \quad \ldots \quad \ldots.$$

The law of formation of the indices is

$$\alpha_n^{(r)} = \tfrac{1}{2}\{\alpha_{n-1}^{(r-1)} + \alpha_{n-1}^{(r+1)}\}, \quad \alpha_0^{(r)} = 1. \tag{4}$$

* The O's and o's occurring in the theorems of this section are not, in general, uniform with respect to r, that is, the K in the inequality $|f^{(r)}| < K\chi$, the existence of which is asserted by the equation $f^{(r)} = O(\chi)$, is not necessarily independent of r.

Since $f = O(\phi)$, we may suppose that $a_n^{(0)} = 0$. It is then clear from a little consideration of (4) that if we introduce the convention that $a_n^{(r)}$ is zero when $r \leqslant 0$, we have, for all values of r,

$$a_n^{(r)} \leqslant \tfrac{1}{2}\{a_{n-1}^{(r-1)} + a_{n-1}^{(r+1)}\}. \tag{5}$$

We shall now show that

$$a_n^{(r)} \leqslant a_{n-1}^{(r)}. \tag{6}$$

This is certainly the case if

$$a_{n-1}^{(r)} \geqslant \tfrac{1}{2}\{a_{n-1}^{(r-1)} + a_{n-1}^{(r+1)}\}, \tag{7}$$

i.e., if

$$\Delta_r^2 a_{n-1}^{(r-1)} \leqslant 0,$$

where

$$\Delta_r u_r \equiv u_r - u_{r+1}.$$

That this is the case is easily established by induction. If $n = 1$, we have

$$a_0^{(0)} = 0, \quad a_0^{(1)} = 1, \quad \ldots, \quad a_0^{(r-1)} = 1, \quad \ldots;$$

and if $n = 2$, we have

$$a_1^{(0)} = 0, \quad a_1^{(1)} = \tfrac{1}{2}, \quad \ldots, \quad a_1^{(r-1)} = 1 - (\tfrac{1}{2})^{r-1}, \quad \ldots;$$

and so

$$\Delta_r^2 a_0^{(r-1)} = 0, \qquad \Delta_r^2 a_1^{(r-1)} = -(\tfrac{1}{2})^{r+1} < 0.$$

Also

$$\begin{aligned}\Delta_r^2 a_{n-1}^{(r-1)} &= a_{n-1}^{(r-1)} - 2a_{n-1}^{(r)} + a_{n-1}^{(r+1)} \\ &\leqslant \tfrac{1}{2}[a_{n-2}^{(r-2)} + a_{n-2}^{(r)} - 2\{a_{n-2}^{(r-1)} + a_{n-2}^{(r+1)}\} + a_{n-2}^{(r)} + a_{n-2}^{(r+2)}] \\ &= \tfrac{1}{2}\{\Delta_r^2 a_{n-2}^{(r-2)} + \Delta_r^2 a_{n-2}^{(r)}\}.\end{aligned}$$

Hence (7) being true for $n = 1, 2$, is true for all values of n. Hence (6) is also true, and so, for any fixed value of r, $a_n^{(r)}$ tends to a limit $a_\infty^{(r)}$ as $n \to \infty$. Hence, from (5),

$$a_\infty^{(r)} \leqslant \tfrac{1}{2}\{a_\infty^{(r-1)} + a_\infty^{(r+1)}\},$$

or

$$a_\infty^{(r+1)} - a_\infty^{(r)} \geqslant a_\infty^{(r)} - a_\infty^{(r-1)}.$$

Therefore

$$a_\infty^{(r+1)} - a_\infty^{(r)} \geqslant a_\infty^{(s)} - a_\infty^{(s-1)} \quad (s \leqslant r).$$

Hence

$$a_\infty^{(r+1)} \geqslant a_\infty^{(s)} + (r - s + 1)[a_\infty^{(s)} - a_\infty^{(s-1)}]. \tag{8}$$

Now $0 \leqslant a_n^{(r)} \leqslant 1$, and so $0 \leqslant a_\infty^{(r)} \leqslant 1$. Hence, making $r \to \infty$ in (8), we see that we must have $a_\infty^{(s)} = a_\infty^{(s-1)}$; and therefore

$$a_\infty^{(s)} = a_\infty^{(0)} = \lim_{n \to \infty} a_n^{(0)} = \lim 0 = 0.$$

This is true for all values of s; for any fixed s, then, we can choose n so that $\alpha_n^{(s)} < \epsilon$, and our theorem is established.

The foregoing proof is the precise analytical expression of the following rough argument. If the order of $f^{(r)}$ increases as r increases, it does so by equal or increasing steps. The magnitude of the step must be less than any power of ψ; otherwise a finite number of steps would give an increase in order greater than ψ, which is contrary to the hypotheses of the theorem.

9. Theorem 1 may be generalised as follows:—

THEOREM 3.—*If ϕ and ψ are increasing functions, and $f^{(r)}$ is continuous, then if $0 < s < r$,*

(*a*) *if $f = O(\phi)$ and $f^{(r)} = O(\psi)$, then $f^{(s)} = O\{\phi^{(r-s)/r}\psi^{s/r}\}$;*

(*b*) *if $f = o(\phi)$, or if $\phi = 1$ and $f \to s$, and $f^{(r)} = O(\psi)$, then $f^{(s)} = o\{\phi^{(r-s)/r}\psi^{s/r}\}$;*

(*c*) *if $f = O(\phi)$ and $f^{(r)} = o(\psi)$, then $f^{(s)} = o\{\phi^{(r-s)/r}\psi^{s/r}\}$.*

Let
$$\chi_s(x) = \max_{y \leqslant x} |f^{(s)}(y)| / [\{\phi(y)\}^{(r-s)/r} \{\psi(y)\}^{s/r}],$$

and let
$$\chi(x) = \max_{0<s<r} \chi_s(x).$$

We shall first consider (*a*). The conclusion of (*a*) is certainly true if

$$\chi(x) = O(1).$$

Let us suppose that this last equation is false, so that χ, which is an increasing function, tends to infinity. We have

$$|f^{(s)}/\{\phi^{(r-s)/r}\psi^{s/r}\}| \leqslant \max_{y \leqslant x} |f^{(s)}(y)| / [\{\phi(y)\}^{(r-s)/r} \{\psi(y)\}^{s/r}] \leqslant \chi_s \leqslant \chi,$$

or
$$f^{(s)} = O\{\phi^{(r-s)/r}\psi^{s/r}\chi\}.$$

If now we apply Theorem 1*a* to f, using the relation $f = O(\phi)$, and that obtained by taking $s = 2$ in the formula just found, we obtain

$$f' = O\{\phi^{(r-1)/r}\psi^{1/r}\chi^{1/2}\}. \tag{1}$$

Repeating the argument with f', and using (1),

$$f'' = O\{\phi^{(r-2)/r}\psi^{2/r}\chi^{3/4}\}.$$

Similarly,
$$f''' = O\{\phi^{(r-3)/r}\psi^{3/r}\chi^{7/8}\},$$
... ...

(2) $$f^{(r-2)} = O\{\phi^{2/r}\psi^{(r-2)/r}\chi^{(2^{r-1}-1)/2^{r-1}}\}.$$

Now we have

(3) $$f^{(r)} = O(\psi) = O(\psi\chi).^*$$

Hence, applying Theorem 1*a* to $f^{(r-2)}$ and using (2) and (3),

(4) $$f^{(r-1)} = O\{\phi^{1/r}\psi^{(r-1)/r}\chi^{(2^r-1)/2^r}\}.$$

Consequently we have from (1), &c., (2), (3), and (4),
$$f^{(s)}/\{\phi^{(r-s)/r}\psi^{s/r}\} = O(\chi^a), \quad a = (2^r-1)/2^r,$$
for $0 < s < r$. Hence $\chi_s = O(\chi^a)$ for $0 < s < r$, and so
$$\chi = \max_{0<s<r} \chi_s = O(\chi^a).$$
Since $a < 1$, this is incompatible with $\chi \to \infty$. Hence
$$\chi = O(1),$$
and the proof of (*a*) is completed. To prove (*b*) and (*c*) we observe that in each case we can assume the result of (*a*). Hence, in the case of (*c*), we have
$$f^{(r-2)} = O\{\phi^{2/r}\psi^{(r-2)/r}\}, \quad f^{(r)} = o(\psi).$$
Applying Theorem 1*c*, we have
$$f^{(r-1)} = o\{\phi^{1/r}\psi^{(r-1)/r}\}.$$
Using this equation instead of $f^{(r)} = o(\psi)$, we now obtain the result of (*c*) for $s = r-2$, then for $s = r-3$, and so on. Similarly, starting at the other end of the series $f, f', \ldots, f^{(r)}$, we obtain (*b*).

A particularly interesting case of the theorem occurs when $\phi = 1$, $\psi = 1$, $f \to s$. We then have the theorem :—

THEOREM 3*a*.—*If f has derivates of all orders and if $f \to s$, then, if any derivate is bounded, all preceding derivates tend to zero, or, to state the matter in another way, if any derivate does not tend to zero, no subsequent derivate is bounded.*

This theorem contains in its turn as a particular case that quoted in § 1 of the introduction.†

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* This step is required to make the argument applicable in the proof of Theorem 6.

† Littlewood, *l.c.*, p. 437.

10. The corresponding extension of Theorem 2 is an immediate corollary of the last theorem.

THEOREM 4.—*If ϕ and ψ are increasing functions, and $\psi \to \infty$, and if*

$$f = O(\phi), \quad f^{(r)} = O(\phi\psi) \qquad (r \geqslant r_0),$$

then

$$f^{(r)} = O(\phi\psi^r),$$

for all values of r.

For, by Theorem 3, we have for $r < r_0$,

$$f^{(r)} = O(\phi\psi^{r/r_0}) = O(\phi\psi).$$

Hence

$$f^{(r)} = O(\phi\psi)$$

for all values of r, and the theorem follows by Theorem 2.

11. The foregoing theorems give no information concerning functions for which we are given relations of the form $f^{(r)} = O(\psi)$, where $\psi \to 0$, beyond, of course, what can be deduced from $f^{(r)} = o(1)$. Subject, however, to certain conditions, Theorems 3 and 4 remain true when $\psi \to 0$. The most important form of ψ is $x^{-\alpha}$, where $\alpha \leqslant r$. The case in which $\alpha > r$ is of less interest. Suppose, for example, $\alpha > r$, and $\phi = 1$. Then the result of Theorem 3 is trivial, since $f^{(s)}$, the $(r-s)$-th integral of $f^{(r)}$, is plainly of the form $O(x^{r-s-\alpha})$, and $r-s-\alpha$ is less than $-s\alpha/r$, which is the index assigned by the theorem.

On the other hand, if $\alpha \leqslant r$, $x^r\psi$ is an increasing function. This suggests that we shall find it convenient, instead of using relations of the type

$$f^{(r)} = O(\psi),$$

where ψ is decreasing, to use relations of the type

$$x^r f^{(r)} = O(\psi),$$

where ψ is increasing. The preceding remarks show that, in the cases of greatest interest, the two points of view are equivalent.*

12. We shall begin by establishing the result corresponding to Theorem 1, viz.,

THEOREM 5.—*If ϕ and ψ are increasing functions, and $\phi = O(\psi)$,*

* There are cases which are thus excluded, *e.g.*, that in which

$$f^{(r)} = O(1/x^r \log x).$$

But such cases do not seem to have applications of any particular interest.

and $f^{(r+2)}$ *is continuous, then*

(*a*) *if* $x^r f^{(r)} = O(\phi)$, $x^{r+2} f^{(r+2)} = O(\psi)$, *then* $x^{r+1} f^{(r+1)} = O\{\sqrt{(\phi\psi)}\}$;

(*b*) *if* $x^r f^{(r)} = o(\phi)$, $x^{r+2} f^{(r+2)} = O(\psi)$, *then* $x^{r+1} f^{(r+1)} = o\{\sqrt{(\phi\psi)}\}$.

Further, if $\phi = o(\psi)$, *then*

(*c*) *if* $x^r f^{(r)} = O(\phi)$, $x^{r+2} f^{(r+2)} = o(\psi)$, *then* $x^{r+1} f^{(r+1)} = o\{\sqrt{(\phi\psi)}\}$.

As in Theorem 1, we shall consider only the case (*c*) in detail. The argument is very similar to that of § 6.

We have $$|f^{(r+2)}(x)| < \epsilon\psi(x)/(2x)^{r+2}$$

for all values of x greater than some value ξ. Consider the curve $y = f^{(r+1)}(x)$. Let P or (x_0, y_0) be a point on it: we suppose $y_0 > 0$. Let N be the foot of the ordinate of P, M the mid-point of ON, C the point $(\xi, 0)$, and T the point to the left of N, such that

$$\tan\theta = \tan PTN = \epsilon\psi(x_0)/x_0^{r+2}.$$

We can suppose $x_0 > 2\xi$, so that M lies to the right of C.

When x_0 is sufficiently large, we have, for all values of x between M and N,

(1) $$|f^{(r)}(x)| < A\phi(x)/(2x)^r \leqslant A\phi(x_0)/x_0^r,$$

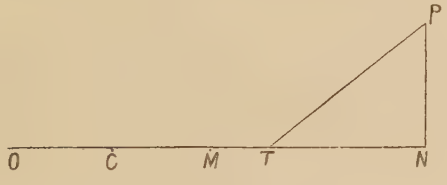

FIG. 3.

where A is a constant, and also

(2) $$\frac{dy}{dx} \leqslant |f^{(r+2)}(x)| < \epsilon\psi(x)/(2x)^{r+2} < \epsilon\psi(x_0)/x_0^{r+2} = \tan\theta.$$

Let x_1 be the abscissa of T. There are two cases to consider, according as T is to the right or to the left of M.

Case (i).—T is to the right of M. Then, by (2), the curve is above TP between T and N. Hence

$$2A\phi(x_0)/x_0^r \geqslant |f^{(r)}(x_0) - f^{(r)}(x_1)| = \left|\int_{x_1}^{x_0} y\,dx\right| > \text{area } PTN = \tfrac{1}{2}y_0^2\cot\theta.$$

Hence

$$y_0^2 < 4A\epsilon\phi(x_0)\,\psi(x_0)/x_0^{2r+2}. \tag{3}$$

Case (ii).—T is to the left of M. Then the curve is above PM between M and N, and so

$$2A\phi(x_0)/x_0^r > \text{area } PMN = \tfrac{1}{4}x_0y_0,$$

$$y_0 < 8A\phi(x_0)/x_0^{r+1},$$

$$y_0^2 < \frac{\phi(x_0)\,\psi(x_0)}{x_0^{2r+2}}\,\frac{64A^2\phi(x_0)}{\psi(x_0)}.$$

Since $\phi = o(\psi)$ it follows that, for sufficiently large values of x_0, we have

$$y_0^2 < \epsilon\phi(x_0)\,\psi(x_0)/x_0^{2r+2}, \tag{4}$$

and from (3) and (4) the result of the theorem follows.*

13. The analogues of Theorems 3 and 4 are as follows. That of Theorem 2 is, of course, included in that of Theorem 4.

Theorem 6.—*If ϕ and ψ are increasing functions, and $\phi = O(\psi)$, and $f^{(r)}$ is continuous, and $0 < s < r$, then*

(*a*) *if $f = O(\phi)$ and $x^rf^{(r)} = O(\psi)$, then*

$$x^sf^{(s)} = O\left\{\phi^{(r-s)/r}\psi^{s/r}\right\};$$

(*b*) *if $f = o(\phi)$, or if $\phi = 1$, $f \to s$, and $x^rf^{(r)} = O(\psi)$, then*

$$x^sf^{(s)} = o\left\{\phi^{(r-s)/r}\psi^{s/r}\right\}.$$

Further, if ϕ satisfies the more stringent condition $\phi = o(\psi)$, then

(*c*) *if $f = O(\phi)$, $x^rf^{(r)} = o(\psi)$, then $x^sf^{(s)} = o\left\{\phi^{(r-s)/r}\psi^{s/r}\right\}$.*

Theorem 7.—*If ϕ and ψ are increasing functions, and $\psi \to \infty$, and if*

$$f = O(\phi), \quad x^rf^{(r)} = O(\phi\psi),$$

for $r > r_0$, then $x^rf^{(r)} = O(\phi\psi^\epsilon)$ for $r > 0$ and for any position ϵ.

The analogue of Theorem 2 (*i.e.* Theorem 7 with $r_0 = 0$) follows from Theorem 5 precisely as Theorem 2 follows from Theorem 1; and Theorem 7

* The proof is simpler than that of Theorem 1, because we have an additional condition, $\phi = o(\psi)$ in the above argument, $\phi = O(\psi)$ in the corresponding proofs of (*a*) and (*b*). We are thus able to avoid the complications of Case (iii) of Theorem 1.

follows from Theorem 6 as Theorem 4 follows from Theorem 3. Theorem 6 itself may be proved by writing "$x^r f^{(r)}$" in place of "$f^{(r)}$," and "Theorem 5" in place of "Theorem 1" throughout § 9. The work of § 9 is so arranged that (as is easily verified) the additional condition

$$\phi = O(\psi) \quad or \quad \phi = o(\psi),$$

with which Theorem 5 is burdened, is satisfied whenever we have to appeal to the latter theorem.

14. There are theorems, similar to those we have been discussing, concerning the relative orders of the derivates of a function of x as x tends to a finite limit, which we may suppose to be zero. We shall, however, content ourselves by giving one pair of such theorems. The enunciations of these are the same, word for word, as those of Theorems 5 and 6, but it is to be understood that x tends to zero instead of to infinity, that ϕ and ψ increase steadily as x tends to zero, and that the hypothesis of continuity, where it occurs, refers to points other than the origin.

THEOREM 8.—*The result of Theorem* 6 *holds also when* x *tends to zero instead of to infinity.*

THEOREM 9.—*The result of Theorem* 7 *holds also with the same modification.*

The proofs of these theorems are almost exactly the same as those of Theorems 6 and 7. It is a sufficient indication of the necessary modifications to remark that the *rôle* of the point M, or $(\frac{1}{2}x_0, 0)$ in § 12, is now played by the point $(2x_0, 0)$.

15. We shall illustrate the range and limitations of Theorems 1–9 by a few simple applications.

(*a*) Let $f = x^n$. If $n \geqslant 2$, we may take $\phi = x^n$, $\psi = x^{n-2}$ in Theorem 1 and obtain

$$f' = O\left\{\surd(x^{2n-2})\right\} = O(x^{n-1}).$$

If $x \geqslant r$ we may take $\phi = x^n$, $\psi = x^{n-r}$ in Theorem 3, and obtain $f^{(s)} = O(x^k)$, where

$$k = \frac{r-s}{r}n + \frac{s}{r}(n-r) = n-s.$$

These results are evidently correct.

(*b*) Any number of illustrations may (as was pointed out, in the case of Theorem 1, in § 7) be obtained by considering functions of the form $\lambda \cos \mu$, where λ and μ are L-functions. Thus, if

$$f = x^a \cos(x^b) \quad (a > 0,\ b > 1),$$

the order of $f^{(r)}$ is $x^{a+r(b-1)}$: each differentiation increases the order by a factor x^{b-1}. This is

an illustration of the first group of theorems. A good illustration of the second is obtained by considering the function

$$f = x^a \cos x^b \quad (a > 0,\ 0 < b < 1).$$

Here the order of $x^r f^{(r)}(x)$ is x^{a+rb}.

(*c*) It was pointed out in § 7 that the result of Theorem 1 fails when $f = x \log x$. This illustrates the importance, in the first group of theorems, of the condition that ϕ and ψ are increasing functions. The same example may be used to show the necessity some such condition as $\phi = O(\psi)$ in the second group of theorems. Here we have

$$f \sim x \log x, \quad xf' \sim x \log x, \quad x^2 f'' \sim x,$$

and the result of Theorem 5 does not apply.

(*d*) The most subtle point which requires illustration is the necessity for the condition $\phi = o(\psi)$, in the third case of Theorem 5: we might well expect $\phi = O(\psi)$ to be a sufficient condition. But consider the function

$$f = x/(\log x).$$

Here

$$xf' \sim x/(\log x), \quad x^2 f'' \sim -x/(\log x)^2.$$

If $\phi = x/(\log x)$, $\psi = x/(\log x)$, we have $f = O(\phi)$, $x^2 f'' = o(\psi)$; but it is not true that

$$xf' = o\{\surd(\phi\psi)\}.$$

III.

Applications to the Theory of Dirichlet's Series.

16. THEOREM 10.—*Let r and k be integers and $0 < k < r$. Then* (*a*) *if* $\alpha \geqslant 1$ *and*

(i) $a_n = O(n^\alpha)$,

(ii) Σa_n *is summable* (*Cr*) *to sum* s,

then

$$s_n^k/n^k = o\{n^{(1+\alpha)(r-k)/(r+1)}\},$$

if $\alpha > -1$, *and*

$$s_n^k/n^k = s/k! + o(1),$$

if $\alpha = -1$. *Further* (*b*) *if* $\alpha > 1$ *and*

(i) $a_n = o(n^\alpha)$,

(ii) Σa_n *is finite* (*Cr*),

then

$$s_n^k/n^k = o\{n^{(1+\alpha)(r-k)/(r+1)}\}.$$

The first part of the theorem is a generalisation of the general Cesàro-Tauber theorem,* to which it reduces when $\alpha = -1$.

If $\alpha > r$, the theorem is trivial in either case in virtue of (ii), which gives

$$s_n^k = O(n^r) \quad (0 < k < r).$$

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 307.

For a similar reason the case $a = r$ is trivial in case (a). We may therefore suppose that $a < r$ in case (a), and $a \leqslant r$ in case (b).

Let $f(x)$ be a function whose $(r+1)$-th derivative is defined by the equation

$$f^{(r+1)}(x) = a_n + (x-n)(a_{n+1}-a_n) \quad (n \leqslant x \leqslant n+1),$$

so that $f^{(r+1)}(x)$ is continuous and assumes the value a_n for $x = n$. Then

$$\begin{aligned} f^{(r)}(x) &= C + \sum_0^{n-1} \int_\nu^{\nu+1} \{a_\nu + (t-\nu)(a_{\nu+1}-a_\nu)\}\, dt \\ &\qquad + \int_n^x \{a_n + (t-n)(a_{n+1}-a_n)\}\, dt \\ &= C + s_{n-1} + \tfrac{1}{2} \sum_0^{n-1} (a_{\nu+1}-a_\nu) \\ &\qquad + (x-n)\, a_n + \tfrac{1}{2}(x-n)^2\, (a_{n+1}-a_n) \\ &= A + s_{n-1} + (x-n+\tfrac{1}{2})\, a_n + \tfrac{1}{2}(x-n)^2\, (a_{n+1}-a_n) \\ &= A + s_n + O(a_n) + O(a_{n+1}-a_n). \end{aligned}$$

Here A and C are constants (independent of n and x), and $O(a_n)$ denotes a function of n and x the ratio of whose absolute value to that of a_n is less than a constant.*

It is easy to see that, if we repeat the process of integration, we obtain

$$f^{(r-1)}(x) = Ax + O(1) + s_n^1 + O(s_n) + O(a_n) + O(a_{n+1}-a_n),$$

$$f^{(r-2)}(x) = \frac{Ax^2}{2!} + O(x) + s_n^2 + O(s_n^1) + O(s_n) + O(a_n) + O(a_{n+1}-a_n),$$

and, generally,

$$(1) \quad f^{(r-k)}(x) = \frac{Ax^k}{k!} + O(x^{k-1}) + s_n^k + O(s_n^{k-1}) + \ldots + O(a_n) + O(a_{n+1}-a_n),$$

for $0 \leqslant k \leqslant r$.

So far the proofs of cases (a) and (b) of the theorem are identical;

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* Our use of O here involves an extension of our ordinary notation. Generally, in such a formula as $f = O(\phi)$, ϕ is an essentially positive function; but there is clearly no reason why we should not write, *e.g.*,

$$\tfrac{1}{2}\sin x = O(\sin x), \quad \sin^2 x = O(\sin x).$$

We may observe that the constants implied by the O's which occur in this proof may depend upon r and k, but are essentially independent of n and x.

From this point they differ slightly. We shall consider only case (*a*); the proof of case (*b*) may be left to the ingenuity of the reader.

The constant A is at our disposal: we take $A = -s$. Let us consider in particular the case $k = r$. Then, since Σa_n is summable (Cr) to sum s, we have

$$\frac{Ax^r}{r!} + s_n^r = -\frac{s}{r!}\{n^r + o(n^r)\} + \frac{sn^r}{r!} + o(n^r) = o(n^r) = o(x^r).$$

Also s_n^{r-1}, s_n^{r-2}, ..., s_n, a_n, and $a_{n+1} - a_n$ are all of the form $o(x^r)$. Thus, finally,

$$f(x) = o(x^r). \tag{2}$$

But
$$f^{(r+1)}(x) = O(a_n) + O(a_{n+1} - a_n) = O(x^\alpha),$$

$$x^{r+1} f^{(r+1)}(x) = O(x^{r+1+\alpha}). \tag{3}$$

Hence, by Theorem 6,* we have

$$x^{r-k} f^{(r-k)}(x) = o(x^{r+\rho_k}), \tag{4}$$

where
$$r + \rho_k = \{r(k+1) + (r-k)(r+1+\alpha)\}/(r+1),$$

or
$$\rho_k = \{(1+\alpha)(r-k)/(r+1)\}.$$

We can now combine equations (1) and (4) in such a way as to obtain the result of the theorem. Dividing (1) by x^k, and remembering that $A = -s$, we find

$$s_n^k/n^k - s/k! = o(x^{\rho_k}) + x^{-k}\{O(s_n^{k-1}) + \ldots + O(a_{n+1} - a_n)\}. \tag{5}$$

Suppose first $k = 0$. Then

$$\begin{aligned} s_n - s &= o(x^{\rho_0}) + O(a_n) + O(a_{n+1} - a_n) \\ &= o(x^{\rho_0}) + O(x^\alpha) = o(x^{\rho_0}), \end{aligned}$$

since
$$\alpha < \rho_0 = (1+\alpha)\, r/(r+1),$$

if
$$\alpha/(1+\alpha) < r/(r+1);$$

a condition which is satisfied, since $\alpha < r$.

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* It is important to observe that, as $r + 1 + \alpha \geqslant r$, the condition $\phi = O(\psi)$ is satisfied. It should also be observed that, at the corresponding point of the proof of case (*b*), we require the more stringent condition $\phi = o(\psi)$. This is satisfied if $\alpha > -1$, but not if $\alpha = -1$. As a matter of fact the result of case (*b*) is *not* true when $\alpha = -1$. It is not true, for example, that a series Σa_n, for which $a_n = o(1/n)$, cannot be finite $(C1)$ without being convergent. The series

$$\Sigma \frac{1}{n(\log n)^{1+\alpha i}} \quad (\alpha \neq 0),$$

for example, is finite $(C1)$, but not convergent, or even summable $(C1)$.

Next, suppose the result of the theorem established for $0, 1, \ldots, k-1$. Then, for $0 \leqslant p < k$, we have

$$x^{-k} O(s_n^p) = o(x^{\rho_p - k}) = o(x^{\rho_k}),$$

since $$\rho_p - \rho_k = (1+\alpha)(k-p)/(r+1) < k-p \leqslant k.$$

Also $$x^{-k}\{O(a_n) + O(a_{n+1} - a_n)\} = O(x^{\alpha-k}) = o(x^{\rho_k}),$$

since $$(\alpha-k)/(\alpha+1) < (r-k)/(r+1),$$

and so $\alpha - k < \rho_k$. Hence, finally,

$$s_n^k/n^k - s/k! = o(x^{\rho_k}), \tag{6}$$

and the theorem is established.

A simple and interesting case of Theorem 10 is that in which $\alpha = 0$, $r = 1$. The form of the theorem is then as follows.

THEOREM 11.—*If* $a_n = O(1)$, *and* Σa_n *is summable* $(C1)$, *then*

$$s_n = o(\sqrt{n}).^*$$

17. The result of Theorem 10 gains in interest when translated into the language of the theory of Dirichlet's series.

We shall, however, find it convenient, before we proceed to this application, to state, and in part prove, a number of theorems concerning Dirichlet's series which are logically independent of the previous results of the paper. These theorems are of considerable intrinsic interest, and will enable us to present our applications of the theorems of Section II in a more subtle and precise form than would otherwise be possible.

We shall first prove the following preliminary theorem. There is nothing new in the theorem, which is contained in Bohr's investigations on the summability of Dirichlet's series.† We have thought it well to include a proof (in a somewhat condensed form), partly on account of the language and comparative inaccessibility of Bohr's dissertation, and partly because we shall have occasion to refer later on to formulæ which occur in it.

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* If a_n is the Fourier's cosine coefficient of a function $f(\theta)$ continuous for $\theta = 0$, then the conditions are certainly satisfied. In this case, however, a good deal more can be said of s_n; for, by a theorem of Riesz and Chapman (Chapman, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 391; Riesz, *Comptes Rendus*, November 22, 1909), Σa_n is summable $(C\delta)$ for any positive δ, and so $s_n = o(n^\delta)$: and indeed it may easily be proved directly that $s_n = O(\log n)$.

This example might suggest that Theorem 10 does not give the best result of its kind that can be obtained concerning s_n. This, however, is untrue. We shall see later (cf. § 25) that for no $\beta < \frac{1}{2}$ is it true that $a_n = O(1)$, together with summability $(C1)$, implies $s_n = o(n^\beta)$.

† *Bidrag til de Dirichlet'ske Raekkers Theori*, Copenhagen, 1910.

THEOREM 12.—*If* $s_n^k = O(n^{k+\beta})$, *where* $\beta > 0$, *then* $\Sigma n^{-\gamma} a_n$ *is summable* $(C k)$ *for any* $\gamma > \beta$.

Let $$n^{-\gamma} a_n = b_n \ (n > 0), \quad a_0 = b_0 = 0,$$

and denote the Cesàro's sums, formed from b_n instead of a_n, by t_n, t_n^1, ... Then

$$t_n^k = \sum_{j=0}^{n} s_j^k \sum_{i=0}^{k} \binom{k+1}{i} \binom{n-j-i+k}{k-i} \Delta^{k+1-i} (j+i)^{-\gamma}. \tag{1}$$*

In this sum a finite number† of terms occur for which $j+i > n$. These terms are to be replaced by zero.‡ But it is easy to see that this convention may be abandoned without affecting the limit of t_n^k / A_n^k.§ For the terms in question are finite in number, and for all of them $n-j < k$. Further, we may write in all of them

$$s_j^k = O(n^{k+\beta}), \quad \Delta^{k+1-i} (j+i)^{-\gamma} = O(n^{-\gamma}),$$

and so their sum is of the form $O(n^{k+\beta-\gamma})$ or $o(n^k)$.

We now consider the terms of (1) for which $i = 0$, which we can write in the form

$${}_0t_n^k = \sum_{0}^{n} \binom{n-j+k}{k} s_j^k \Delta^{k+1} (j^{-\gamma}).$$

We have‖ $$-Kn^k < \binom{n-j+k}{k} - A_n^k < 0 \quad (0 \leqslant j \leqslant n),$$

$$-K\nu n^{k-1} < \binom{n-j+k}{k} - A_n^k < 0 \quad (0 \leqslant j \leqslant \nu < n).$$

Hence $${}_0t_n^k / A_n^k = \sum_{j=0}^{n} s_j^k \Delta^{k+1} j^{-\gamma} + R_0,$$

where $$R_0 = O\left(\frac{\nu}{n} \sum_{1}^{\nu} j^{k+\beta} \Delta^{k+1} j^{-\gamma}\right) + O\left(\sum_{\nu+1}^{n} j^{k+\beta} \Delta^{k+1} j^{-\gamma}\right).$$

The series $\Sigma j^{k+\beta} \Delta^{k+1} j^{-\gamma}$ converges (like $\Sigma j^{-1+\beta-\gamma}$). Choosing first ν and then n sufficiently large, we see that $R_0 \to 0$, and

$${}_0t_n^k / A_n^k \to \sum_{j=0}^{\infty} s_j^k \Delta^{k+1} j^{-\gamma}. \tag{2}$$

Consider next the terms in (1) for which i has a fixed value other than zero. Writing

$${}_it_n^k = \binom{k+1}{i} \sum_{j=0}^{n} \binom{n-j-i+k}{k-i} s_j^k \Delta^{k+1-i} (j+i)^{-\gamma},$$

and observing that $$\binom{n-j-i+k}{k-i} = O(n^{k-i}),$$

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* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 260, and Vol. 8, p. 279. In a footnote to the second paper a number of mistakes in the first are corrected. It should be observed that the notation of these papers differs slightly from that which we have since adopted, the series being there taken to be $a_0 + a_1 + \ldots$ instead of $a_1 + a_2 + \ldots$. As the argument which follows is very similar to that of the second paper quoted, we have thought it best to adhere to its notation for the purposes of this proof. The equation (1) contains one term for which $i = j = 0$, so that $(j+i)^{-\gamma}$ is meaningless, but as $s_0^k = 0$, this is immaterial.

† That is to say, a number independent of n.

‡ *E.g.*, if $i = k$, we must replace $\Delta (j+k)^{-\gamma}$ by $n^{-\gamma}$, if $j = n-k$, and by 0 if $j > n-k$.

§ A_n^k has the same meaning as in the papers cited: thus $A_n^k \sim n^k / k!$.

‖ *L.c.*, p. 279.

we find

$$ {}_it_n^k/A_n^k = O\left\{n^{-i}\sum_0^n j^{k+\beta}\Delta^{k+1-i}(j+i)^{-\gamma}\right\} = O\left\{\sum_0^n\left(\frac{j}{n}\right)^i j^{k-i+\beta}\Delta^{k+1-i}(j+i)^{-\gamma}\right\}. $$

We can now show that

$$ {}_it_n^k/A_n^k \to 0, \tag{3} $$

by what is practically a repetition of the final argument which led to (2). From (1) and (2) the theorem follows.

It is important to observe that $s_n^k = o(n^{k+\beta})$ does not (as might perhaps be expected) involve the summability (Ck) of $\Sigma n^{-\beta}a_n$. If, *e.g.*, $k = 0$, the equation

$$ \sum_1^n \nu^{-\beta}a_\nu = \sum_1^{n-1} s_\nu\Delta\nu^{-\beta} + n^{-\beta}s_n $$

shows that, when $s_n = o(n^\beta)$, the series

$$ \Sigma n^{-\beta}a_n, \qquad \Sigma n^{-\beta-1}s_n $$

converge together. But the second series is divergent if, *e.g.*, $s_n = n^\beta/(\log n)$.

It is evident that, in the enunciation of Theorem 12, we need only have supposed that

$$ s_n^k = O(n^{k+\beta+\epsilon}) $$

for all positive values of ϵ.

18. Our next theorem involves an argument of a more delicate character.

THEOREM 13.—*If* $s_n^k = o(n^{k+\beta})$, *where* $\beta > 0$, *then* $\Sigma n^{-\beta}a_n$ *is either summable* (Ck) *or summable by none of Cesàro's means. Further, the same is true of* $\Sigma n^{-\beta-ti}a_n$, *for any value of* t.*

Suppose that $\Sigma n^{-\beta}a_n$ is summable $(C,\ k+1)$. Then the necessary and sufficient condition that it should be summable (Ck) is that

$$ \tau_n^k = o(n^{1+k}), \tag{1} $$

where τ_n^k is the k-th Cesàro's sum formed from

$$ a_n = n^{1-\beta}a_n.\dagger $$

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* The arguments of this paragraph are in detail so similar to those of § 17 that we have not thought it worth while to give more than an exceedingly condensed account of the proof. Any attempt to extend the result of Theorem 12 to the case in which $\gamma = \beta$ would, of course, have been foredoomed to failure. If, in § 17, we take $\gamma = \beta$, we are led to the divergent series

$$ \Sigma j^{k+\beta}\Delta^{k+1}j^{-\beta}. $$

In this paragraph the fact that we already know our series to be summable $(C,\ k+1)$ enables us to use condition (1), and so argue in terms of sums τ_n^k formed from $n^{1-\beta}a_n$ instead of sums t_n^k formed from $n^{-\beta}a_n$. We thus avoid the logarithm associated with a series of the type $\Sigma(1/j)$.

† Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 304.

Suppose first that $k = 0$. Then

$$\tau_n = \sum_0^n \nu^{1-\beta} a_\nu = \sum_0^{n-1} s_\nu \Delta\nu^{1-\beta} + s_n n^{1-\beta} = O\left\{\sum_1^{n-1} o(\nu^\beta)\,\nu^{-\beta}\right\} + o(n) = o(n).$$

Thus (1) is true for $k = 0$.

When $k > 0$, we use the formula

$$\tau_n^k = \sum_{j=0}^n s_j^k \sum_{i=0}^k \binom{k+1}{i} \binom{n-j-i+k}{k-i} \Delta^{k+1-i}(j+i)^{1-\beta}, \tag{2}$$

analogous to (1) of § 17. In the first place, we have

$${}_0\tau_n^k = \sum_{j=0}^n s_j^k \binom{n-j+k}{k} \Delta^{k+1} j^{1-\beta} = A_n^k \left\{\sum_{j=0}^n s_j^k \Delta^{k+1} j^{1-\beta} + R_0\right\},$$

where R_0 differs from the R_0 of § 17 only in the replacing of $-\gamma$ by $1-\beta$. The first term on the right-hand side is

$$O\left\{n^k \sum_1^n o(j^{k+\beta})\, j^{-\beta-k}\right\} = O\left\{n^k \sum_1^n o(1)\right\} = o(n^{k+1}).$$

And that

$$R_0 = o(n)$$

is easily shown by an argument in essentials the same as that of § 17. Finally,

$$\begin{aligned} {}_i\tau_n^k &= \binom{k+1}{i} \sum_{j=0}^n \binom{n-j-i+k}{k-i} s_j^k \Delta^{k+1-i}(j+i)^{1-\beta} \\ &= O\left\{n^{k-i} \sum_{j=0}^n o(j^{k+\beta})\, j^{-\beta-k+i}\right\} \\ &= O\left\{n^{k-i} \sum_{j=0}^n o(j^i)\right\} = o(n^{k+1}). \end{aligned}$$

Combining these results, we see that (1) is satisfied, and the proof of Theorem 13, in so far as it is concerned with the point

$$s = \beta,$$

is completed.

The extension to complex values of s presents no difficulties. When, *e.g.*, $k = 0$, we have

$$\tau_n = \sum_0^n \nu^{1-\beta-ti} a_\nu = \sum_0^{n-1} s_\nu \Delta\nu^{1-\beta-ti} + s_n n^{1-\beta-ti} = O\left\{\sum_0^{n-1} \nu^{-\beta}\, o(\nu^\beta)\right\} + o(n) = o(n),$$

as before. Similarly in the general case.

The most interesting case is, of course, that in which $k = 0$.

With Theorem 13 may be associated the following theorem, the proof of which presents no difficulty.

THEOREM 14.—*If* $\Sigma n^{-\beta} a_n$ *is summable* (Ck), *and* $\beta > 0$, *then*

$$s_n^k = o(n^{k+\beta}).$$

The result of this theorem is a straightforward deduction from the formula

$$s_n^k = \sum_{j=0}^{n} t_j^k \sum_{i=0}^{k} \binom{k+1}{i} \binom{n-j-i+k}{k-i} \Delta^{k+1-i}(j+i)^{\beta},$$

analogous to (1) of § 17. The result, although useful, is much less subtle and interesting than that of the companion Theorem 13, and it seems hardly worth while to set out the details of the proof. The result obviously breaks down when $\beta = 0$.

19. Some very interesting results can be deduced from a mere combination of Theorems 13 and 14. Suppose that $\Sigma n^{-\beta} a_n$ is summable (Ck). Then, by Theorem 14, we have $s_n^k = o(n^{k+\beta})$. Hence, using Theorem 13, we obtain

THEOREM 15.—*If a Dirichlet's series is summable* (Ck) *for* $s = \beta$, *it is, at any point of the line* $s = \beta + ti$, *either summable* (Ck), *or summable by none of Cesàro's means.*

It will be observed that the condition $\beta > 0$, which is plainly irrelevant to the truth of the theorem, has now disappeared.

THEOREM 16.—*If a Dirichlet's series is summable* (Ck) *for* $s = \beta$, *and the function represented by the sum of the series can be continued, by the use of Cesàro's methods of summation, across the line* $s = \beta + ti$, *then it is summable* (Ck) *at all points of the line. Moreover, it is uniformly summable along any finite stretch of the line.*

This theorem is an immediate corollary of Theorem 15, except for the assertion of uniformity. The truth of this last assertion follows from the fact that the O's and o's which occur in the proof of Theorem 13 are uniform with respect to t.

Theorem 16 remains true if the words "by the use of Cesàro's methods of summation" are omitted. In fact, if the series is summable (Ck) for $s = \beta$, it is summable (Ck) at all points of the line at which it is regular. This is a fairly straightforward deduction from Riesz's analogue of Fatou's theorem,* viz., that *if* $a_n = o(1/n)$, *the series* $\Sigma a_n n^{-s}$ *is convergent at all regular points of the line* $\sigma = 0$.

* Riesz, *Comptes Rendus*, Nov. 22nd, 1909; see also Landau, *Prac Matematyczno-Fizycznych*, Vol. 21, p. 151.

In this connection it is interesting to give an example of a series, which represents an integral function, but is summable only in a half-plane. The series

$$\Sigma e^{i(\log n)^\alpha} n^{-s} \quad (\alpha \geqslant 2)$$

is such a series. Its lines of summability all coincide in the line $\sigma = 1$, at every point at which it is convergent.

20. We return now to Theorem 10. From Theorems 10 and 12 we can at once deduce

THEOREM 17.—*If* $0 \leqslant k \leqslant r$, *and* λ_k *is the abscissa of the line of summability* (Ck) *of the Dirichlet's series* $\Sigma a_n n^{-s}$, *and if*

(i) $a_n = O(n^{\alpha+\epsilon})$, *where* $\alpha > -1$, *for every positive* ϵ,

(ii) $\lambda_r \leqslant 0$,

then

$$\lambda_k \leqslant \frac{(\alpha+1)(r-k)}{r+1}.$$

We have, however, not yet extracted from Theorem 10 the maximum of information which it may be made to yield concerning the convergence and summability of a Dirichlet's series. This is contained in the following more precise theorem, which contains Theorem 17, and is an immediate corollary of Theorems 10, 13, and 16.

THEOREM 18.—*If* $a_n = O(n^\alpha)$, *where* $\alpha > -1$, *and* Σa_n *is summable* (Cr), *then* $\Sigma a_n n^{-s}$ *is summable* (Ck), *where* $0 \leqslant k < r$, *for*

$$s = \frac{(\alpha+1)(r-k)}{r+1}.$$

The same result follows from the assumptions that $a_n = o(n^\alpha)$ *and* Σa_n *is finite* (Cr). *Further, the series is, in either case, uniformly summable on any finite stretch of the straight line*

$$s = \frac{(\alpha+1)(r-k)}{r+1} + ti.$$

In fact we have, by Theorem 10,

$$s_n^k = o(n^{k+\beta}),$$

where

$$\beta = \frac{(\alpha+1)(r-k)}{r+1} > 0.$$

Hence, by Theorem 13, $\Sigma a_n n^{-s}$ is summable (Ck) or by none of Cesàro's means for $s = \beta + ti$. Therefore, as it is certainly summable (Cr), it is summable (Ck). Finally, the uniformity follows from Theorem 16.

We add a few remarks as to the relations between Theorems 10 and 18. Theorem 10 included two cases, both of which were trivial when $a > r$, while the first, but not the second, was also trivial when $a = r$. Theorem 18 includes two corresponding cases, but neither of these is trivial for $a = r$. Suppose, for simplicity, that $k = 0$. If Σa_n is summable (Cr), we must have $a_n = o(n^r)$: thus the first condition is included in the second, and the first case of the theorem reduces to a known theorem,* viz. that the summability (Cr) of Σa_n involves the convergence of $\Sigma a_n n^{-r}$. The second case gives us a new result, viz., that if the series Σa_n is finite (Cr), and $a_n = o(n^r)$—a condition no longer included in the first condition—then $\Sigma a_n n^{-r}$ is convergent.

It is possible to generalise Theorem 10 by adopting hypotheses of the type

$$a_n = O\{n^\alpha (\log n)^\beta (\log\log n)^\gamma \dots\},$$

instead of $a_n = O(n^\alpha)$. Such generalisations have no particular interest for our present purpose. The more special hypothesis is more than sufficient for the application of the theorem to the proof of Theorem 17, and is *necessary* in the proof of Theorem 18.

21. We shall now state and prove the second principal theorem of this part of the paper.

THEOREM 19.—*If* $\Sigma a_n n^{-s}$ *is summable* (Cr) *for* $s = 0$, *and summable* $(C, r-k)$ *for* $s = \beta$, *where* $0 < \beta \leqslant k \leqslant r$, *then it is summable* $(C, r-l)$, *where* $0 < l < k$, *for*

$$s = l\beta/k.$$

The conclusion still holds if either "summable" in the hypothesis is replaced by "finite": while, if both are so replaced, the "summable" in the conclusion must also be replaced by "finite."

We consider the case in which Σa_n is finite (Cr) and $\Sigma a_n n^{-\beta}$ summable $(C, r-k)$. The other cases may be settled similarly.

We define a function $f(x)$ as in § 16. As in § 16, we find

$$(1) \quad f^{(l)}(x) = O(x^{r-l-1}) + s_n^{r-l} + O(s_n^{r-l-1}) + \dots + O(s_n) + O(a_n) + O(\Delta a_n),$$

for $0 \leqslant l \leqslant r$. We have chosen zero as the value of the constant A of § 16. Taking first $l = 0$, we have

$$(2) \qquad f(x) = O(x^r).$$

* This result is a corollary of a theorem given by Riesz, *Comptes Rendus*, June 21st, 1909.

Now take $l = k$. Since $\Sigma a_n n^{-\beta}$ is summable $(C, r-k)$, we have, by Theorem 14,

$$s_n^{r-k} = o(n^{r-k+\beta});$$

and it is clear that every other term on the right-hand side of (1) is also of this form. Hence

$$f^{(k)}(x) = o(x^{r-k+\beta}),$$

$$x^k f^{(k)}(x) = o(x^{r+\beta}), \tag{3}$$

We now apply Theorem 5 to equations (2) and (3), observing that, as $\beta > 0$, the condition $\phi = o(\psi)$ is certainly satisfied. We obtain

$$x^l f^{(l)}(x) = o\{x^{r+(l\beta/k)}\}.$$

From this we can deduce

$$s_n^{r-l} = o\{n^{r-l+(l\beta/k)}\}, \tag{4}$$

for $0 \leqslant l \leqslant k$. For this is certainly true when $l = k$. Suppose it true for $l = k,\ k-1,\ \ldots,\ \lambda+1$; and consider equation (1), with $l = \lambda$. It is easy to see that every term of the equation, save $s_n^{r-\lambda}$, is of the form $o\{n^{r-\lambda+(\lambda\beta/k)}\}$. For example,

$$O(s_n^{r-\lambda-1}) = o\{n^{r-\lambda-1+[(\lambda+1)\beta/k]}\} = o\{n^{r-\lambda+(\lambda\beta/k)}\},$$

since (4) has *ex hypothesi* been proved for $l = \lambda+1$, and since $0 < \beta \leqslant k$. Hence $s_n^{r-\lambda}$ is of this form, and so the truth of (4) is established by induction.

Theorem 19 follows at once from (4) and Theorem 13, since $\Sigma a_n n^{-s}$ is certainly summable by *some* of Cesàro's means for any positive s.

Taking $k = 2$, $l = 1$ in Theorem 19, and effecting a suitable linear transformation on s, we deduce

THEOREM 20.—*If* $\Sigma a_n n^{-s}$ *is summable* (Cr) *for* $s = \alpha$, *and summable* $(C, r-2)$ *for* $s = \beta$, *it is summable* $(C, r-1)$ *for* $s = \frac{1}{2}(\alpha+\beta)$.

And from this we can at once deduce a very interesting theorem of Bohr,* viz.,

THEOREM 21.—*If* λ_r *is the abscissa of summability* (Cr) *of a Dirichlet's*

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* Bohr, *Nachrichten u.s.w. zu Göttingen*, 1909, p. 252; *Bidrag til de Dirichlet'ske Raekkers Theorie*, p. 101.

series $\Sigma a_n n^{-s}$, *then*

$$\lambda_{r-2}-\lambda_{r-1} \geqslant \lambda_{r-1}-\lambda_r,$$

or

$$\Delta^2\lambda_r \geqslant 0.$$

For, if we had $\lambda_{r-2}-\lambda_{r-1} < \lambda_{r-1}-\lambda_r$,

we could obviously choose values of α and β in such a way as to violate Theorem 20.

This theorem may be employed to simplify the proof of Theorem 17. For suppose this theorem established in the case $k = 0$. Then it follows from Theorem 21 that

$$(\lambda_0-\lambda_k)/k \geqslant (\lambda_k-\lambda_r)/(r-k),$$

$$\lambda_k \leqslant \frac{r-k}{r}\lambda_0 + \frac{k}{r}\lambda_r \leqslant \frac{(\alpha+1)(r-k)}{r+1}.$$

It should also be observed that, *unless*

$$\lambda_0-\lambda_1 = \lambda_1-\lambda_2 = \ldots = \lambda_{k-1}-\lambda_k,$$

we must have $\lambda_k < (\alpha+1)(r-k)/(r+1)$:

so that the series is summable (Ck) for $s = (\alpha+1)(r-k)/(r+1)$, and even for smaller values of s: it is only in the extreme case (which is, however, that of most frequent occurrence) that we require the more delicate Theorem 18.

22. Now let us suppose that $a_n = o(n^\delta)$,

for every posltive δ, so that the series is certainly absolutely convergent for $s > 1$. And let us suppose further that

$$\Lambda = \lim_{r\to\infty} \lambda_r < 1;$$

i.e., that the series is summable by some of Cesàro's means for some distance to the left of the line

$$\sigma = 1.$$

If η is positive and $s = \Lambda+\eta < 1$, the series is summable (Cr); here r is a function of η, which may tend to infinity as $\eta \to 0$. Also

$$b_n = a_n n^{-\Lambda-\eta} = o(n^{-\Lambda-\eta+\delta}).$$

Applying Theorem 17 to the series $\Sigma b_n n^{-s}$, with $k = 0$, we see that the abscissa of convergence of $\Sigma a_n n^{-s}$ is not greater than

$$\Lambda+\eta+\frac{r}{r+1}(1+\delta-\Lambda-\eta) = 1-\frac{1-\Lambda-\eta}{r+1}+\frac{r\delta}{r+1}.$$

This is true for all positive values of δ: hence the abscissa of convergence is not greater than

$$1-\frac{1-\Lambda-\eta}{r+1} < 1.$$

Hence we obtain

THEOREM 22.—*If* $a_n = o(n^\delta)$, *and the series* $\Sigma a_n n^{-s}$ *is summable by any of Cesàro's means for some points to the left of* $\sigma = 1$, *then it is convergent for some points to the left of* $\sigma = 1$.

The interesting case is, of course, that in which the abscissa $\bar{\lambda}$ of absolute convergence is 1. Let us consider this case more closely. The series is summable $(C1)$ for some points to

the left of $\sigma = 1$; let $\lambda_1 = 1-\zeta < 1$ be the abscissa of summability. The series

$$\Sigma b_n = \Sigma a_n n^{-1+\zeta-\rho}$$

is summable $(C1)$ for any positive ρ. Also

$$b_n = o(n^{-1+\zeta-\rho+\delta}).$$

Applying Theorem 17, we find $\lambda_0 \leqslant 1-\zeta+\rho+\frac{1}{2}(\zeta-\rho+\delta)$,

or $\lambda_0 \leqslant 1-\frac{1}{2}\zeta$,

since both δ and ρ are arbitrarily small. Our result is equivalent to the following theorem.

THEOREM 23.—*If a Dirichlet's series is such that* $a_n = o(n^{\bar{\lambda}+\delta-1})$, *where* $\bar{\lambda}$ *is the abscissa of absolute convergence, then*

$$\bar{\lambda}-\lambda_0 \geqslant \lambda_0-\lambda_1 \geqslant \lambda_1-\lambda_2 \geqslant \ldots .$$

The point of this theorem is to complete Bohr's theorem by showing that, when

$$a_n = o(n^{\bar{\lambda}+\delta-1}),$$

the breadth of the strip of conditional convergence cannot be less than that of any strip of Cesàro summability. The condition

$$a_n = o(n^{\bar{\lambda}+\delta-1})$$

is one satisfied by the most obvious series, such as

$$\Sigma n^{-s}, \quad \Sigma(-1)^n n^{-s}, \quad \Sigma e^{in^\alpha} n^{-s}.$$

If β is the lower limit of the values of s such that $a_n n^{-s} = o(1)$, it is evident that $\bar{\lambda} \leqslant \beta+1$. The series to which Theorem 23 is applicable are those for which $\bar{\lambda} = \beta+1$.

It is, of course, possible that $\bar{\lambda} < \beta+1$; indeed we may have $\bar{\lambda} = \beta$. This is the case, for example, if

$$a_n = (-1)^i (n = n_i), \quad a_n = 0 \ (n \neq n_i),$$

where (n_i) is such a sequence that Σn_i^{-s} is convergent for every positive s. A particularly interesting example of such a series has been given by Bohr.* This series is

$$1^{-s}-2^{-s}+4^{-s}-5^{-s}+27^{-s}-28^{-s}+\ldots,$$

where $a_n = 1$ if $n = m^m$, $a_n = -1$ if $n = m^m+1$, and $a_n = 0$ otherwise.

For this series $\bar{\lambda} = \lambda_0 = 0, \quad \lambda_1 = -1;$

there is a strip (of breadth 1) of summability $(C1)$, but no strip of conditional convergence. This example shows that no such result as that of Theorem 23 holds unless some restriction is placed upon a_n.

23. It may be deduced from Theorem 22, in conjunction with known theorems, that if the Riemann hypothesis as regards the roots of the ζ function is true (or indeed if Θ, the upper limit of their real parts, is less than 1), then the series

$$\frac{1}{\zeta(s)} = \Sigma \frac{\mu(n)}{n^s}$$

is convergent to the left of the line $\sigma = 1$. For it has been shown by Bohr that, for this function, $\Lambda = \Theta$. As a matter of fact, more than this is already known: it is known in fact

* *Nachrichten u.s.w., l.c.*, p. 261.

that the series is convergent for $s > \Theta$.* It should, moreover, be observed that the equation $\Lambda = \Theta$ is deduced by Bohr from theorems of the nature of the "Landau-Schnee" theorem mentioned in the next paragraph, and that the final result can be deduced from this theorem directly. Thus the result cannot be obtained as a *bona-fide* deduction from Theorem 22.†

24. The nature of the theorems which we have been discussing suggests that they may have some connection with an important group of theorems due to Landau and Schnee.‡ Suppose that the conditions of Theorem 17 are satisfied. Then $\Sigma a_n n^{-s}$ is summable (Cr) for $\sigma > 0$ and absolutely convergent for $\sigma > \alpha + 1$.

It follows first, by a theorem of Bohr and Riesz,§ that if $f(s)$ is the function represented by the sum of the series, then $f(s)$ is regular for $\sigma > 0$, and

$$f(s) = O(|t|^{r+1}),$$

for $\sigma \geqslant \delta > 0$.

Now Landau and Schnee have proved the following theorem :—

If $a_n = o(n^\lambda)$, *so that* $\Sigma a_n n^{-s}$ *is absolutely convergent for* $\sigma > 1$; *if further* $f(s)$ *is regular for* $\sigma > \eta$, *where* $\eta < 1$, *and*

$$f(s) = O(|t|^\lambda,$$

for $\sigma \geqslant \eta$, *then* $\Sigma a_n n^{-s}$ *is convergent for*

$$\sigma > (\eta + \lambda)/(1 + \lambda).$$

Write $n^{-\alpha} a_n = b_n$, and apply this theorem to $\Sigma b_n n^{-s}$, taking

$$\eta = -\alpha + \delta, \quad \lambda = r + 1.$$

We find at once that $\Sigma a_n n^{-s}$ is convergent for

$$\sigma > \alpha + \frac{r + 1 - \alpha}{r + 2} = \frac{(\alpha + 1)(r + 1)}{(r + 2)}.$$

This is a result very similar to that of Theorem 17, but, since

$$(r + 1)/(r + 2) > r/(r + 1),$$

not so good. Thus the greater depth of the considerations appealed to in this argument does not compensate for its lack of directness.

25. We conclude this section with a few miscellaneous remarks concerning the theorems we have proved. First, considering Theorem 17, we will show by an example that the information which it gives is the utmost which we can hope to obtain from the data.

The series $$\Sigma n^{-b} e^{in^\alpha} \quad (0 < a < 1)$$

(to which we have so frequently to appeal) is summable (Cr) if $(r + 1)\alpha + b > 1$.‖ In particular, if $b = 0$, the series is summable (Cr) if $(r + 1)\alpha > 1$ and finite (Cr) if $(r + 1)\alpha = 1$.

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* Littlewood, *Comptes Rendus*, January 29th, 1912; the best previous result in this direction was due to Landau, *Handbuch*, p. 871.

† That $\Lambda = \Theta$ can, however, be proved otherwise, by making use of a theorem of Riesz (*Comptes Rendus*, June 21st, 1909; cf. Bohr, *l.c.*, p. 259).

‡ Landau, *Handbuch*, pp. 853 *et seq.* See also Landau, *Rendiconti di Palermo*, Vol. 28, p. 113; Schnee, *Math. Annalen*, Vol. 66, p. 337.

§ Bohr, *Comptes Rendus*, January 11th, 1909; *Bidrag etc.*, p. 114; *Nachrichten u.s.w.*, p. 252; Riesz, *Comptes Rendus*, *l.c. supra*.

‖ Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 142.

Suppose then $$a = \frac{1}{r+1}.$$

Then the series $\Sigma n^{-s}a_n = \Sigma n^{-s}e^{in^\alpha}$ is summable (Cr) for $s > 0$, and $a_n = O(1)$. Hence Theorem 17 asserts that $\Sigma a_n n^{-s}$ is convergent for

$$s > r/(r+1).$$

The series is, in fact, convergent, if, and only if,

$$s > 1-a > r/(r+1),$$

which is the limit assigned by the theorem.

Similarly it may be shown that the number $(\alpha+1)(r-k)/(r+1)$ of the theorem cannot be improved upon.

26. Our other remarks have reference to Theorems 19 *et seq*. It follows from Theorems 19 and 20 that if a Dirichlet's series has three lines of summability λ_r, λ_{r+1}, λ_{r+2} at equal distances, the summability of the series at the foot of the two extreme lines involves its summability at the foot of the middle line: in fact, we need only assume summability $(C, r+2)$ for, *e.g.*, $s = \lambda_{r+2}$, and finitude (Cr) for $s = \lambda_r$. It should be observed that this result depends essentially on the equidistance of the lines of summability. Thus for the series

$$\Sigma \left\{ \frac{(-1)^n n}{\log n} + \log n\, e^{i\sqrt{n}} + \frac{e^{in^{1/4}}}{\sqrt{n}\log n} \right\} n^{-s},$$

we have $$\lambda_0 = 1, \quad \lambda_1 = 0, \quad \lambda_2 = -\tfrac{1}{4};$$

the series is summable $(C2)$ for $s = -\frac{1}{4}$ and convergent for $s = 1$; but is not summable $(C1)$, or even finite $(C1)$, for $s = 0$.

27. In conclusion we remark that all the theorems of this section have their analogues for integrals, which are, in fact, easier to deduce from the theorems of Section II. The reader will have no difficulty in stating these theorems for himself. We pass now to applications of a different character.

IV.

Additional Applications.

Theorems connected with the General Abel-Tauber Theorem.

28. THEOREM 24. — *If* $a_n = O(n^\alpha)$ *and* $f(x) = \Sigma a_n e^{-nx} = O(x^{-\beta})$ *as* $x \to 0$, *where* $0 < \beta < \alpha+1$, *then*

$$x^r f^{(r)} = O(x^{-\beta-\epsilon}).*$$

We have

$$|x^r f^{(r)}| < Kx^r \Sigma n^{\alpha+r} e^{-nx} = O(x^{-\alpha-1}) = O(x^{-\beta}\psi),$$

where $$\psi = x^{-(\alpha+1-\beta)} \to \infty.$$

* That $x^r f^{(r)} = O(x^{-\alpha-1})$ is trivial.

We may apply Theorem 9, obtaining

$$x^r f^{(r)} = O(x^{-\beta}\psi^\epsilon),$$

which gives the result of the theorem.

There is a similar result for the case $\alpha+1 = \beta = 0$, viz.,

THEOREM 25.—*If* $a_n = O(1/n)$, *and* $f(x) = \Sigma a_n e^{-nx} \to s$, *then*

$$x^r f^{(r)} \to 0.*$$

For we have $|x^{r+1} f^{(r+1)}| < K x^{r+1} \Sigma n^r e^{-nx} = O(1)$,

and our theorem follows from Theorem 9.

29. Theorem 10, which is an extension of the general Cesàro-Tauber theorem, suggests that there is a similar extension of the general Abel-Tauber theorem. Theorem 10 gives

$$s_n = o\left\{n^{(1+\alpha)r/(r+1)}\right\}, \tag{1}$$

as a consequence of $a_n = O(n^\alpha)$ and the summability (Cr) of Σa_n. If we make r tend to infinity in (1), we obtain $s_n = o(n^{1+\alpha})$, and this equation suggests itself as the corresponding consequence when summability (Cr) is replaced by the existence of Abel's limit.

That the theorem thus suggested is true is an immediate consequence of known results. That $s_n = O(n^{1+\alpha})$ is trivial; and it has already been shown† that the hypotheses involve the convergence of $\Sigma n^{-\alpha} a_n$, and therefore, by Theorem 14, the equation $s_n = o(n^{1+\alpha})$. But the result may be made more general by assuming less than the existence of Abel's limit. All, indeed, that it is necessary to assume is that

$$f(x) = \Sigma a_n e^{-nx} = o(x^{-1-\alpha}).$$

Thus the theorem we propose to prove is as follows.

THEOREM 26.—*If* $a_n = O(n^\alpha)$, *where* $\alpha \geqslant -1$, *and if*

$$f(x) = \Sigma a_n e^{-nx} = o(x^{-1-\alpha}) \ (\alpha > -1), \quad = s+o(1) \ (\alpha = -1),$$

then

$$s_n = o(n^{1+\alpha}) \ (\alpha > -1), \quad = s+o(1) \ (\alpha = -1).$$

The case $\alpha = -1$ is the general Abel-Tauber theorem; we shall therefore suppose $\alpha > -1$. The proof in this case, as also in the case $\alpha = -1$,‡ consists in dividing s_n (or $s_n - s$) into two parts, each of which

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* This theorem is proved by Littlewood, *l.c.*, p. 439.

† Littlewood, *l.c.*, p. 447.

‡ Cf. Littlewood, *l.c.*, p. 440.

contains an auxiliary constant r. In each case one of these parts is shown, by an application of the theorems of Section II, to be of the desired form, while we obtain for the other part an expression whose modulus is less than $\epsilon n^{1+\alpha}$, when r is sufficiently large. The detailed treatment of the second part we shall omit in the present case; it is not unlike the corresponding proof for the case $\alpha = -1$, and the proof of a similar result in Theorem 27. The former of these proofs is already published, and the latter is given in detail in §§ 35, 36.

We have $$f = o(x^{-1-\alpha}),$$

(2) $$x^r f^{(r)} = O(x^{-1-\alpha});$$

the first equation being one of our hypotheses and the second being an immediate consequence of $a_n = O(n^\alpha)$. Applying Theorem 8, with $\phi = \psi = x^{-1-\alpha}$, so that the condition $\phi = O(\psi)$ is satisfied, we see that (2) can be replaced by the better equation

(2′) $$x^r f^{(r)} = o(x^{-1-\alpha}).$$

Now
$$\begin{aligned}(-x)^r \Sigma s_n n^{r-1} e^{-nx} &= -x^r \left(\frac{d}{dx}\right)^{r-1} \Sigma s_n e^{-nx} \\ &= -x^r \left(\frac{d}{dx}\right)^{r-1} \left\{\frac{f(x)}{1-e^{-x}}\right\} \\ &= \Sigma A x^r f^{(p)}(x)(1-e^{-x})^{-q} e^{-sx},\end{aligned}$$

where the A's are constants, and the summation extends to a number of sets of values of p, q, s, such that $p+q \leqslant r$. Applying (2′) we obtain

$$\Sigma o(x^{r-p-q-1-\alpha}) = o(x^{-1-\alpha});$$

so that

(3) $$(-x)^r \Sigma s_n n^{r-1} e^{-nx} = o(x^{-1-\alpha}).$$

We choose $x = r/m$, where r is fixed but at our disposal, and make $m \to \infty$. From (3), we obtain

$$\Sigma s_n n^{r-1} e^{-nr/m} = o(m^{r+\alpha+1}),$$

and so

(4) $$s_m \Sigma n^{r-1} e^{-nr/m} = o(m^{r+\alpha+1}) + \Sigma (s_n - s_m) n^{r-1} e^{-nr/m}.$$

Now $$\Sigma n^{r-1} e^{-nx} \sim r!\, x^{-r},$$

so that $$\Sigma n^{r-1} e^{-nr/m} \sim r!\, r^{-r} m^r.$$

Hence, from (4),

(5) $$s_m = o(m^{1+\alpha}) + \{1+o(1)\} m^{-r} (r^r/r!) \Sigma (s_n - s_m) n^{r-1} e^{-nr/m}.$$

It is clear that our theorem is established if we can prove that

$$S \equiv m^{-r-1-\alpha}(r^r/r!) \sum_{n=1}^{\infty} |s_n - s_m| n^{r-1} e^{-nr/m} < \delta \tag{6}$$

when r is sufficiently large.

We divide the sum into three parts by writing

$$S = \sum_{1}^{(1-\lambda)m} + \sum_{(1-\lambda)m}^{(1+\lambda)m} + \sum_{(1-\lambda)m}^{\infty} = S_1 + S_2 + S_3,$$

where $0 < \lambda < 1$. It is easy to deduce from the equation $a_n = O(n^\alpha)$ that

$$\begin{aligned} |s_n - s_m| &< Km^{1+\alpha} && (in\ S_1), \\ &< Km^\alpha |n-m| && (in\ S_2), \\ &< Kn^{1+\alpha} && (in\ S_3), \end{aligned}$$

where the K's are independent of n, m, and r. If we substitute from these inequalities in the three sums it is not difficult to establish (6); as we have explained above, the kind of argument involved is sufficiently illustrated in § 35.

30. We pass now to a theorem of rather a different character. It is well known that the series

$$\Sigma \frac{1}{n^{1+\alpha i}}, \quad \Sigma \frac{1}{n(\log n)^{1+\alpha i}}, \quad \ldots,$$

are finitely oscillating (if $\alpha \neq 0$) and that the corresponding Abel limits do not exist, and it is easy to see that, at any rate in a number of special cases, the Abel limit corresponding to the series

$$\Sigma n^{-s} (\log n)^{-t} (\log \log n)^{-u} \ldots,$$

where s, t, u, ... are complex, does not exist unless the series is convergent. The most obvious common property of such series is expressed by the equation

$$\Delta a_n \equiv a_n - a_{n+1} = O(a_n/n);$$

this suggests that this property is a sufficient condition for the truth of the converse of Abel's theorem. It will be seen that we can prove a little more than this.

THEOREM 27. — *If* $\Delta a_n = O(a_n/n)$, *and if* $f(x) = \Sigma a_n e^{-nx} \to s$, *as* $x \to 0$, *then* $na_n \to 0$ *and* Σa_n *is convergent.*

We shall first establish independently the much easier theorem, deducible immediately from the above, with summability ($C1$) in the

hypothesis in place of the existence of Abel's limit. This seems to be worth while on account of the extreme difficulty of the proof of Theorem 27.

Theorem 28.—*If* $\Delta a_n = O(a_n/n)$, *and if* Σa_n *is summable* (*C*1), *then* $na_n \to 0$ *and* Σa_n *is convergent.*

Throughout the succeeding sections we shall assume that no a_n (or b_n) vanishes. If any one does, so do all which follow: thus the limitation is trivial.

31. It is convenient to begin by establishing five lemmas, of which the first four are deductions from the equation

$$\Delta b_n = O(b_n/n),$$

which is to be understood as their hypothesis.

Lemma 1.—*There is a constant* K *such that* $b_n = O(n^K)$.

Lemma 2.—*If* h *and* k *are any constants such that* $0<h<1<k<2$, *there is a constant* K *such that*

$$|b_n - b_m| < K|n-m||b_m|/m,$$

for $hm \leqslant n \leqslant km$.

Lemma 3.—*There exists a constant* c, *such that*

$$b_{m+r} = b_m(1+\eta_r), \qquad |\eta_r| < \tfrac{1}{2},$$

for

$$|r| \leqslant cm.$$

Lemma 4.—*We have*

$$|b_n/b_m| < K(n/m)^K \quad (n \geqslant m),$$

$$|b_n/b_m| < K(m/n)^K \quad (n \leqslant m),$$

where the K*'s are independent of both* m *and* n.

We have

$$\frac{b_{n+1}}{b_n} = \left(1 - \frac{\Delta b_n}{b_n}\right) = 1 + O\left(\frac{1}{n}\right) = \exp\left[O\left(\frac{1}{n}\right)\right].$$

Hence $$\left|\frac{b_{n+1}}{b_1}\right| = \exp\left[O\left(\frac{1}{1}\right)+O\left(\frac{1}{2}\right)+\ldots+O\left(\frac{1}{n}\right)\right],^*$$

$$|b_{n+1}| < K\exp[K(1+\tfrac{1}{2}+\ldots+1/n)] < Kn^K;$$

which is the result of Lemma 1.

Again, if $m \leqslant n \leqslant km$,

$$\left|\frac{b_n-b_m}{b_m}\right| = \left|\frac{b_{m+1}}{b_m}\frac{b_{m+2}}{b_{m+1}}\cdots\frac{b_n}{b_{n-1}}-1\right|$$

$$= \exp\left[O\left(\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n-1}\right)\right]-1$$

$$= \exp\left[O\left(\frac{n-m}{m}\right)\right]-1 = O\left(\frac{n-m}{m}\right),$$

since $$(n-m)/m = O(1).$$

A similar argument gives this same equation in the case $hm \leqslant n \leqslant m$; we have thus proved Lemma 2. Lemma 3 is an immediate corollary; we have only to take $c = k$ or $c = 1/(2K)$, whichever is least, where the K is that of Lemma 2. Finally, to prove Lemma 4, we have, when $n \geqslant m$,

$$|b_n/b_m| = \exp\left[O\left(\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n-1}\right)\right]$$

$$= \exp[O\{\log(n/m)\}+K]$$

$$< K(n/m)^K;$$

which is the first inequality of Lemma 4. The second inequality follows similarly.

The last lemma is of a different character.

LEMMA 5.—*If* $t < 1$, *then*

$$\sum_{-\infty}^{\infty} |\nu| e^{-t\nu^2} < K/t.$$

The maximum term of the series $\sum_1^{\infty} \nu e^{-t\nu^2}$ occurs at one of the integers p, $p+1$, separated by $\sqrt{(1/2t)}$. The terms increase up to this point

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* The constants implied in the O's in this paragraph are all independent of both m and n.

and then begin to decrease. Hence we have

$$\begin{aligned}\sum_1^\infty \nu e^{-t\nu^2} &< \int_0^p x e^{-tx^2} dx + 2\sqrt{(1/2t)}\, e^{-\frac{1}{2}} + \int_{p+1}^\infty x e^{-tx^2} dx \\ &< 2\int_0^\infty x e^{-tx^2} dx + 1/t \\ &< 2/t.\end{aligned}$$

Thus $K = 4$ satisfies the requirements of the lemma.

32. We proceed now to the proof of Theorem 28. We shall use η to denote a number, not always the same, whose modulus is less than $\frac{1}{2}$.

By Lemma 3, we have

$$a_{n+r} = a_n(1+\eta), \tag{1}$$

for $r \leqslant cn$. Hence

$$\begin{aligned}s_{n+r} - s_n &= a_{n+1} + a_{n+2} + \ldots + a_{n+r} \\ &= a_n(1+\eta) + a_n(1+\eta) + \ldots + a_n(1+\eta) \\ &= r a_n(1+\eta).\end{aligned} \tag{2}$$

If now

$$\sigma_n = s_1 + s_2 + \ldots + s_n,$$

$$\sigma_{n+r} - \sigma_n = \sum_{p=1}^{r} s_{n+p},$$

and so, by substitution from (2) in each s_{n+p},

$$\sigma_{n+r} - \sigma_n = r s_n + \tfrac{1}{2} r(r+1)\, a_n(1+\eta).$$

We shall suppose $r = \alpha n$, where $\frac{1}{8}c < \alpha \leqslant c$. Then

$$(\sigma_{n+r} - \sigma_n)/n = \alpha s_n + \tfrac{1}{2}\alpha(\alpha n + 1)\, a_n(1+\eta). \tag{3}$$

Now, since Σa_n is summable $(C1)$,

$$\sigma_{n+r} = (n+r)\{s + o(1)\} = (1+\alpha)n\{s + o(1)\},$$

and so

$$(\sigma_{n+r} - \sigma_n)/n = \alpha s + o(1).$$

Substituting in (3) and rearranging the terms, we obtain

$$s - s_n = o(1) + u n a_n, \tag{4}$$

where

$$u = \tfrac{1}{2}\{\alpha + 1/n\}(1+\eta).$$

We first choose $\alpha = \alpha_1$, where $\alpha_1 n$ is the greatest integer contained in cn. Then when n is large

$$w_1 = |u_1| > \tfrac{1}{4}(1 + [cn])/n > \tfrac{1}{4}c.$$

Similarly we choose $\alpha = \alpha_2$, where $\alpha_2 n+1$ is the greatest integer contained in $\frac{1}{4}cn$. Then

$$w_2 = |u_2| < \tfrac{3}{4}[\tfrac{1}{4}cn]/n < \tfrac{3}{16}c.$$

Now, from (4), we have

$$na_n(u_1-u_2) = o(1). \tag{5}$$

As $n \to \infty$, $\alpha_1 \to c$, $\alpha_2 \to \frac{1}{4}c$, and

$$|u_1-u_2| \geqslant w_1-w_2 > \tfrac{1}{4}c-\tfrac{3}{16}c = \tfrac{1}{16}c.$$

It follows from (5) that $na_n \to 0$, and the convergence of Σa_n follows from the special Cesàro-Tauber theorem.

33. There is another theorem closely allied to Theorem 28.

THEOREM 29.—*If* Σa_n *is summable* (*C*1), *and*

$$na_n = O\left(\frac{a_1+2a_2+\ldots+na_n}{n}\right),$$

then $na_n = o(1)$, *and* Σa_n *is convergent.*

We omit the proof of this theorem, which follows without serious difficulty from (1) some elementary lemmas analogous to those of § 31, (2) the line of argument used by Landau* to prove the general Cesàro-Tauber theorem. This argument is (as Landau points out) substantially the same as that used for a special purpose by de la Vallée-Poussin in his classical memoirs on the theory of prime numbers.†

34. We shall now attack the much more difficult Theorem 27. It will be convenient to adopt certain conventions as to notation. Throughout the proof K's denote constants independent of r as well as of m. O's and o's are relative to the tending to infinity of m, but they are not necessarily uniform with respect to r; *e.g.*, the K implied in O is not necessarily independent of r. Finally, we use ϵ_r to denote any number, *independent of* m, which tends to zero as $r \to \infty$.

35. We choose $x = (r-\theta)/m$, where r is fixed, but at our disposal,

* *Prac Matematyczno-Fizycnych*, Vol. 21, pp. 97 *et seq.*

† "Recherches Analytiques etc.," *Annales de la Société Scientifique de Bruxelles*, t. 20, pp. 247-250.

and $0 \leqslant \theta \leqslant 1$. Our first object is to establish the two following equations

$$ma_m = O\{x^r f^{(r)}(x)\}, \tag{1}$$

$$x^r f^{(r)}(x) = O(ma_m), \tag{2}$$

for all sufficiently great values of r, and uniformly in respect to θ. These equations do not require the hypothesis of the existence of Abel's limit.

We have
$$\begin{aligned}(-x)^r f^{(r)}(x) &= x^r \Sigma n^r a_n e^{-nx} \\ &= S_1+S_2+S_3+S_4,\end{aligned}$$

where

$$\left\{\begin{aligned} S_1 &= x^r \Sigma n^{r-1} m a_m e^{-nx}, \\ S_2 &= x^r \sum_{1}^{hm} n^{r-1}(na_n - ma_m) e^{-nx}, \\ S_3 &= x^r \sum_{hm}^{km} n^{r-1}(na_n - ma_m) e^{-nx}, \\ S_4 &= x^r \sum_{km}^{\infty} n^{r-1}(na_n - ma_m) e^{-nx}; \end{aligned}\right. \tag{3}$$

where h and k are independent of m and r, and $0 < h < 1 < k < 2$. It is convenient to suppose h and k irrational, so that hm and km cannot be integral.

We have

$$S_1 = ma_m r!\{1+o(1)\}. \tag{4}$$

We proceed to show that, for sufficiently large values of m,

$$|S_2| < \epsilon_r r! |ma_m|, \tag{5}$$

$$|S_4| < \epsilon_r r! |ma_m|. \tag{6}$$

Now $b_n = na_n$ is easily seen to satisfy $\Delta b_n = O(b_n/n)$, and so satisfies the results of Lemmas 1 and 4. Hence, to establish (5) and (6), it is clearly sufficient, to show that, if a is any number independent of m and r,

$$S_2' = x^r \sum_{1}^{hm} n^{r-1}(m/n)^a e^{-nx} < \epsilon_r r!, \tag{5'}$$

$$S_4' = x^r \sum_{km}^{\infty} n^{r-1}(n/m)^a e^{-nx} < \epsilon_r r!. \tag{6'}$$

The maximum value of $n^{r-1-a}e^{-nx}$, *qua* function of n, occurs when

$$n = (r-1-a)/x,$$

which is of the form $m(1+\epsilon_r)$.

Hence, when r is sufficiently large, this value of n lies outside the range of summation in (5′), and the summand is an increasing function of n. Similarly, in (6′), the summand is a decreasing function of n. Consequently

$$\begin{aligned} S_2' &< x^r hm(hm)^{r-1}(m/hm)^\alpha e^{-hmx} \\ &< h^{r-\alpha}(r-\theta)^r e^{-h(r-\theta)} < Kr!\, r^{-\frac{1}{2}} h^r e^{r(1-h)} * \\ &< Kr! \exp[r\{(1-h)+\log h\}] \\ &< \epsilon_r r!, \end{aligned}$$

since $1-h+\log h < 0$; and

$$\begin{aligned} S_4' &< x^r \int_0^\infty t^{r-1}(t/m)^\alpha e^{-tx}\, dt \\ &= (r-\theta)^r \int_0^\infty u^{r+\alpha-1} e^{-(r-1)u}\, du \\ &< Kr^{-\alpha}\Gamma(r+\alpha) \\ &< Kr^{-1}\Gamma(r+1) \\ &= \epsilon_r r!. \end{aligned}$$

We have thus proved (5) and (6).

There remains S_3 to be considered. We have in S_3 (applying Lemma 2 to $b_n = na_n$)

$$|na_n - ma_m| < K|n-m|\,|a_m|,$$

and so
$$\begin{aligned} |S_3| &< Kx^r |a_m| \sum_{hm}^{km} n^{r-1}|n-m|\, e^{-nx} \\ &< Kx^r |a_m| \sum_{\nu=(h-1)m}^{(k-1)m} (m+\nu)^{r-1}|\nu|\, e^{-(r-\theta)(m+\nu)/m}. \end{aligned}$$

Now $|\nu| < \kappa m$, where $0 < \kappa < 1$, and

$$u - \log(1+u) > Ku^2,$$

for $-\kappa < u < \kappa$. Hence

$$\begin{aligned} \left(\frac{m+\nu}{m}\right)^{r-1} e^{-(r-\theta)\nu/m} &= \exp\left[-(r-1)\left\{\frac{\nu}{m} - \log\left(1+\frac{\nu}{m}\right)\right\} - \frac{(1-\theta)\nu}{m}\right] \\ &< Ke^{-K(r-1)(\nu/m)^2}. \end{aligned}$$

* By Stirling's theorem.

Consequently

$$(7)\qquad \begin{aligned} |S_3| &< Kx^r\,|ma_m|\,e^{-r}m^{r-2}\sum_{-\infty}^{\infty}|\nu|\,e^{-K(\nu,m)^2(r-1)} \\ &< Kx^r\,|ma_m|\,e^{-r}m^r/(r-1)^* \\ &< K\,|ma_m|\,r^{r-1}e^{-r} \\ &< |ma_m|\,\epsilon_r r!, \end{aligned}$$

by Stirling's theorem.

From (3), (4), (5), (6), and (7), we have

$$|(-x)^r f^{(r)}(x) - r!\,\{1+o(1)\}\,ma_m| < \epsilon_r r!\,|ma_m|,$$

and from this both (1) and (2) follow immediately.

36. From (1) and (2) and the theorems of Section II it is not difficult to deduce our theorem. Let

$$g(y) = \max_{n\leqslant y}|na_n|,$$

so that $g(y)$ is an increasing function of y.

From Lemma 4, we have

$$(1)\qquad g(ay) = O\{g(y)\},$$

where a is any constant. Now, if r is sufficiently large,

$$x^r f^{(r)}(x) = O(ma_m),$$

by (2) of § 35, where $x = (r-\theta)/m$. Hence

$$(2)\qquad x^r f^{(r)}(x) = O\{g(m)\} = O[g\{(r-\theta)/x\}] = O\{g(1/x)\},$$

by (1). This equation is valid for all values of x as $x\to 0$, since the conditions $x = (r-\theta)/m$, $0\leqslant\theta\leqslant 1$, allow x to take all small values as m takes all large integral values. If now we suppose that $g(y)\to\infty$ as $y\to\infty$, we obtain by means of Theorem 9 (since $f\to s$), the equation

$$(3)\qquad x^r f^{(r)}(x) = O[\{g(1/x)\}^\epsilon] = o\{g(1/x)\},$$

which holds when $x\to 0$. But we have, by (1) of § 35,

$$(4)\qquad |ma_m| = O\{x^r f^{(r)}(x)\}.$$

Hence, from (3), we have

$$\begin{aligned} |ma_m| &= o[g\{m/(r-\theta)\}] = o\{g(m)\} \\ &< \epsilon g(m), \end{aligned}$$

* By Lemma 5.

for $m > m_0$, say. Then

$$g(m) = \max_{n \leqslant m} |na_n| \leqslant \max \{\max_{n \leqslant m_0} |na_n|, \epsilon g(m)\} \tag{5}$$

$$= \max \{O(1), \epsilon g(m)\},$$

so that

$$g(m) = o\{g(m)\} + O(1). \tag{5}$$

The equality (5) is incompatible with $g(y) \to \infty$; hence

$$g(y) = O(1).$$

It then follows from (2) of § 35 that

$$x^r f^{(r)}(x) = O(1),$$

as $x \to 0$, for sufficiently large values of r. But now, applying Theorem 8, we have

$$x^r f^{(r)}(x) = o(1),$$

for $r \geqslant 1$. Hence, by (1) of § 35,

$$ma_m = O\{x^r f^{(r)}(x)\} = o(1).$$

This is the first part of our theorem: the second part follows by Tauber's theorem.

37. In theorems of the "Tauberian" type, it usually happens that the "special" form, involving o in the hypothesis, is very much easier to prove than the "general" form involving O.

This is true in the present case. But the "special" theorem corresponding to Theorem 27 is trivial—it is true, but true only in that it asserts that two mutually incompatible propositions imply some other proposition; an assertion which is always true.* This is shown by

THEOREM 30.—*If* $\Delta a_n = o(a_n/n)$, *then* Σa_n *cannot converge, nor can Abel's limit exist, except in the trivial case when* $a_n = 0$ *from a certain point onwards. Also* $|na_n| \to \infty$, *and*

$$s_n \sim na_n, \quad s_n = n^{1+o(1)}.$$

We have

$$\frac{(n+1)a_{n+1}}{na_n} = 1 - \frac{(n+1)\Delta a_n - a_n}{na_n} = 1 + u_n,$$

* Any false proposition implies all other propositions, true or false: *vide* Whitehead and Russell, *Principia Mathematica*, Vol. 1, p. 108 (Prop. II, 21).

say, and $$na_n = a_1 \prod_1^{n-1} (1+u_\nu).$$

Now $$u_\nu = O(1/\nu),$$

so that $$\log(1+u_\nu) = u_\nu + O(1/\nu^2),$$

$$\log(na_n) = \sum_1^{n-1} u_\nu + O(1).$$

But $$u_\nu \sim 1/\nu,$$

and so $na_n \sim \log n$. It follows from Theorem 27 that Abel's limit cannot exist, and *a fortiori* that the series cannot converge.

Again, if $|a_n| = \alpha_n$, we have

$$s_n = \sum_1^n a_n = \sum_1^{n-1} \nu\Delta a_\nu + na_n,$$

$$\sigma_n = \sum_1^n \alpha_n = \sum_1^{n-1} \nu\Delta a_\nu + n\alpha_n.$$

Now σ_n is an increasing function of n, which tends to infinity with n. Also

$$\nu\Delta a_\nu \leqslant \nu\,|\,\Delta a_\nu\,| = o(\alpha_\nu),$$

$$\sum_1^{n-1} \nu\Delta a_\nu = o(\sigma_n),$$

and $\sigma_n \sim n\alpha_n$. Finally,

$$\sum_1^{n-1} \nu\Delta a_\nu = o\left(\sum_1^{n-1} \alpha_\nu\right) = o(\sigma_n) = o(n\alpha_n);$$

and therefore $s_n \sim na_n$. Also

$$na_n = e^{\log(n\alpha_n)} = e^{\log n + o(\log n)} = n^{1+o(1)}.$$

38. We proved recently* that a series Σa_n in which $a_n = o(1/\sqrt{n})$ cannot be summable by Borel's method unless it is convergent. The preceding results suggest that there should be a theorem related to this theorem as Theorem 27 is related to the general Abel-Tauber theorem.

This theorem is as follows.

THEOREM 31.—*If Σa_n is summable (B), and*

$$\Delta a_n = o(a_n/\sqrt{n}),$$

then $a_n = o(1/\sqrt{n})$ and the series is convergent.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 1.

From $$e^{-m}\Sigma s_n\frac{m^n}{n!}\to s,$$

it follows that

$$S=e^{-m}\Sigma(s_n-s_m)\frac{m^n}{n!}=s-s_m+o(1) \tag{1}$$

as $m\to\infty$.

We choose positive numbers H, α, such that $H<1$, and no one of Hm, $\alpha\sqrt{m}$ $(m=1,2,\ldots)$ is an integer; and we write

$$S=e^{-m}\left(\sum_{0}^{m-Hm}+\sum_{m-Hm}^{m-\alpha\sqrt{m}}+\sum_{m-\alpha\sqrt{m}}^{m+\alpha\sqrt{m}}+\sum_{m+\alpha\sqrt{m}}^{m+Hm}+\sum_{m+Hm}^{\infty}\right)=S_1+S_2+S_3+S_4+S_5.$$

In order to discuss these sums we shall require certain deductions from the hypothesis

$$\Delta a_n=o(a_n/\sqrt{n}).$$

As in § 30, we put aside from the beginning the trivial case in which $a_n=0$ from a certain point onwards.

39. This hypothesis involves the following consequences.

(*a*) We have $$a_n=e^{o(\sqrt{n})},\quad s_n=e^{o(\sqrt{n})}.*$$

We shall not require to use all that is implied in this assertion: it will be sufficient for our purposes that $|a_n|<Ke^{K\sqrt{n}}$ and

$$|s_n-s_m|<Ke^{K\sqrt{q}},$$

where q is the larger of m, n.

(*b*) We have $$|s_n-s_m|<K\sqrt{m}\,|a_m|\,e^{Kr/\sqrt{m}},$$

where $$r=|n-m|<Hm.$$

(*c*) We have $$s_n-s_m=(n-m)\,a_m(1+\epsilon_m),$$

if $$r=|n-m|<\alpha\sqrt{m}.$$

Here ϵ_m is a function of m which tends to zero when once α has been fixed.

We shall first prove these results, and then apply (*a*) to the discussion of S_1 and S_5, (*b*) to that of S_2 and S_4, and (*c*) to that of S_3.

40. (i) *Proof of* (*a*).—Given ϵ we can choose p so that

$$a_{n+1}=a_n-\Delta a_n=a_n\left(1-\frac{\eta^n}{\sqrt{n}}\right),$$

where $$|\eta_n|<\epsilon\quad(n\geqslant p).$$

Hence, if $n=p+r$, we have

$$a_{p+r}=\chi(p,r)\,a_p, \tag{1}$$

where $$|\chi|<\prod_{p}^{p+r-1}\left(1+\frac{\epsilon}{\sqrt{\nu}}\right)<\exp\left(\epsilon\sum_{1}^{n}\frac{1}{\sqrt{\nu}}\right)<e^{2\epsilon\sqrt{n}};$$

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* That is to say, given $\delta>0$ we can find n_0 so that

$$e^{-\delta\sqrt{n}}<|a^n|<e^{\delta\sqrt{n}}\quad(n>n_0).$$

and so $$|a_n| < e^{K\epsilon\sqrt{n}}\,|a_p|.$$

Similarly it may be shown that $$|a_n| > e^{-K\epsilon\sqrt{n}}\,|a_p|.$$*

Thus $a_n = e^{o(\sqrt{n})}$. That $s_n = e^{o(\sqrt{n})}$ is an immediate corollary.

(ii) *Proof of* (*b*).—Suppose, *e.g.*, $n > m$. It follows from (1) above that

$$|a_n| = |a_{m+r}| = \chi(m, r)\,|a_m|,$$

where $$|\chi| < \prod_{m}^{m+r-1}\left(1+\frac{K}{\sqrt{\nu}}\right) < \exp\left(\frac{Kr}{\sqrt{m}}\right).$$

Hence $$|s_n - s_m| < |a_{m+1}| + \ldots + |a_n| < |a_m|\,\frac{e^{K(r+1)/\sqrt{m}}}{e^{K/\sqrt{m}}-1} < K\sqrt{m}\,|a_m|\,e^{Kr/\sqrt{m}}.$$

The case in which $n < m$ may be discussed similarly.

(iii) *Proof of* (*c*).—Suppose, *e.g.*, $0 < r < \alpha\sqrt{m}$. We observe first that

$$|a_n/a_m| < e^{Kr/\sqrt{m}} < K,$$

$$a_n = O(a_m).$$

Also $$a_{n+1} = a_n + o(a_n/\sqrt{n}) = a_n + o(a_m/\sqrt{n}),$$

$$a_{m+r} = a_m\left\{1 + o\left(\sum_{m}^{m+r-1}\frac{1}{\sqrt{\nu}}\right)\right\} = a_m\left\{1 + o(r/\sqrt{m})\right\} = a_m\left\{1 + o(1)\right\},$$

$$s_n - s_m = a_{m+1} + \ldots + a_{m+r} = ra_m\left\{1 + o(1)\right\};$$

which establishes the result.†

41. We are now in a position to obtain upper limits for S_1, S_2, ...

In S_1, we write $$|s_n - s_m| < Ke^{K\sqrt{m}},$$

and in S_5, $$|s_n - s_m| < Ke^{K\sqrt{n}};$$

and there is no serious difficulty in establishing the inequalities

(1) $$|S_1| < e^{-Km}, \quad |S_5| < e^{-Km}.$$‡

We next consider S_2 and S_4; let us select the latter. In this sum we use the inequality

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* We can use the inequality $1/(1-u) < e^{2u}$, which holds when u is small and positive, instead of $1 + u < e^u$.

† The K's, and the additional K's and ϵ's implied in the O's and o's, in this proof, of course, depend upon α.

‡ See *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 6. The introduction of the factors $e^{K\sqrt{m}}$ or $e^{K\sqrt{n}}$, instead of m^K, n^K, will be found to make no material difference.

(b), which gives

$$S_4 = e^{-m}\sqrt{m}\, a_m\, O\left\{\sum_{\alpha\sqrt{m}}^{Hm} \frac{m^{m+r}}{(m+r)!}\, e^{Kr/\sqrt{m}}\right\}$$

$$= a_m O\left\{\sum_{\alpha\sqrt{m}}^{Hm} \exp\left(\frac{Kr}{\sqrt{m}} - \frac{r^2}{4m}\right)\right\} *$$

$$= a_m O\left\{\int_{\alpha\sqrt{m}}^{\infty} \exp\left(\frac{Kr}{\sqrt{m}} - \frac{r^2}{4m}\right) dr\right\}$$

$$= a_m \sqrt{m}\, O\left(\int_{\alpha}^{\infty} e^{Ku - \frac{1}{4}u^2}\, du\right).$$

It is essential to observe that the K which figures in this formula, and the further constant implied in the O, *are independent of* α. We may then suppose $\alpha > 8K > 1$, in which case

$$S_4 = O\left(|a_m|\sqrt{m}\int_{\alpha}^{\infty} e^{-\frac{1}{8}u^2}\, du\right) = O\left(e^{-\frac{1}{8}\alpha^2}|a_m|\sqrt{m}\right); \tag{2}$$

where the constant implied by O is still independent of α. In the same way we can express S_2 in the same form.

There remains S_3. To this we apply the result (c). We thus obtain

$$S_3 = e^{-m} a_m \sum_{-\alpha\sqrt{m}}^{\alpha\sqrt{m}} r\, \frac{m^{m+r}}{(m+r)!} + e^{-m} a_m o(1) \sum_{-\alpha\sqrt{m}}^{\alpha\sqrt{m}} |r|\, \frac{m^{m+r}}{(m+r)!} = S_3' + S_3'',$$

say. The work of our previous paper† shows that

$$S_3'' = o(|a_m|\sqrt{m}). \tag{3}$$

The series which occurs in S_3' is

$$\sum_{1}^{\alpha\sqrt{m}} r\left\{\frac{m^{m+r}}{(m+r)!} - \frac{m^{m-r}}{(m-r)!}\right\} = \frac{m^m}{m!}\sum_{1}^{\alpha\sqrt{m}} r\left\{\frac{1}{\left(1+\frac{1}{m}\right)\ldots\left(1+\frac{r}{m}\right)} - \left(1-\frac{1}{m}\right)\ldots\left(1-\frac{r-1}{m}\right)\right\}$$

$$< \frac{m^m}{m!}\sum_{1}^{\alpha\sqrt{m}} r\left\{\frac{1-\left(1-\frac{1^2}{m^2}\right)\left(1-\frac{2^2}{m^2}\right)\ldots\left(1-\frac{r^2}{m^2}\right)}{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\ldots\left(1+\frac{r}{m}\right)}\right\}.$$

Now, throughout the range of values considered,

$$\left(1+\frac{1}{m}\right)\ldots\left(1+\frac{r}{m}\right) > \exp\left(\frac{1}{2m}+\ldots+\frac{r}{2m}\right) > e^{r^2/4m},$$

$$1-\left(1-\frac{1^2}{m^2}\right)\ldots\left(1-\frac{r^2}{m^2}\right) < \frac{1^2}{m^2}+\ldots+\frac{r^2}{m^2} < \frac{r^3}{m^2}.$$

Hence

$$S_3' = |a_m| O\left(\frac{m^{m-2}e^{-m}}{m!}\sum_{1}^{\alpha\sqrt{m}} r^4 e^{-r^2/4m}\right) = |a_m| O\left(m^{-5/2}\int_0^{\infty} r^4 e^{-r^2/4m}\, dr\right)$$

$$= O(|a_m|) = o(|a_m|\sqrt{m}). \tag{4}$$

From (3) and (4) it follows that

$$S_3 = o(|a_m|\sqrt{m}). \tag{5}$$

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* Since

$$\frac{m^{m+r}}{(m+r)!} < \frac{K}{\sqrt{m}} \exp\left(m - \frac{r^2}{4m}\right)$$

(cf. *Proc. London Math. Soc.*, Ser. 2, Vol., 11, p. 7).

† *L.c.*, p. 7.

From (1), (2), (5), and (1) of § 38, we deduce

$$s - s_m + o(1) = O(e^{-\frac{1}{2}\alpha^2} |\, a_m \,|\, \sqrt{m}) + o(|\, a_m \,|\, \sqrt{m}), \tag{6}$$

where the constant of the O is independent of α, and O, o refer to the tending of m to infinity after α is fixed.

As $e^{-\frac{1}{2}\alpha^2} \to 0$, as $\alpha \to \infty$, it follows that, given any positive δ, we can choose first a sufficiently large value of α and then a number m_0, so that

$$|\, s - s_m + o(1) \,| < \delta \,|\, a_m \,|\, \sqrt{m}, \tag{7}$$

for $m > m_0$.

Let n be the integer nearest to $m + \sqrt{m}$. Then $|\, a_n \,| / |\, a_m \,|$ is less than a constant.* Hence we may suppose m_0 so chosen that (7) is also satisfied when we write n for m in its left-hand member. Hence

$$|\, s_n - s_m + o(1) \,| < 2\delta \,|\, a_m \,|\, \sqrt{m}.$$

But we have already seen that

$$s_n - s_m = (n-m)\{1 + o(1)\}\, a^m = \sqrt{m}\,\{1 + o(1)\}\, a_m.$$

And the inequality $$|\, \sqrt{m}\,\{1 + o(1)\}\, a_m + o(1) \,| < 2\delta \,|\, a_m \,|\, \sqrt{m}$$

is plainly contradictory, when δ is small, unless

$$a_m \sqrt{m} = o(1).$$

But then, in virtue of the Borel-Tauber theorem, Σa_n is convergent. Thus Theorem 31 is proved.†

The standard type of series for which the condition is satisfied is

$$\Sigma\, \phi(n)\, e^{i\psi(n)},$$

where $\phi(n)$ and $\psi(n)$ are L-functions, and

$$e^{-\delta\sqrt{n}} \prec \phi(n) \prec e^{\delta\sqrt{n}}, \quad 1 \prec \psi(n) \prec \sqrt{n}.$$

No such series can be summable (B) unless it is convergent; and

$$f(x) = \Sigma\, \phi(n)\, e^{i\psi(n)}\, x^n$$

cannot be regular for $x = 1$ unless the series is convergent for $x = 1$. In particular

$$\Sigma\, n^{-b}\, e^{in^a} \quad (0 < a < \tfrac{1}{2})$$

is summable (B) if, and only if, it is convergent. This we proved in our former paper, but by a special investigation.

42. Theorems 27 and 31 at once suggest a corresponding analogue of Fatou's theorem,‡ viz., *if* $\Delta a_n = o(a_n)$, *and* $f(x) = \Sigma a_n x^n$ *is regular for* $x = 1$, *then* $a_n = o(1)$ *and* Σa_n *is convergent.* This theorem is true, but trivial in the sense explained in § 37—the hypotheses are incompatible. In fact the conditions on a_n used in Theorems 27 and 31, and here, may be

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* See § 40, (*c*).

† The "general" form of Theorem 31 is no doubt true. But it would be futile to look for a proof until the general Borel-Tauber theorem is established.

‡ On the subject of this theorem see Hardy and Littlewood, *l.c. supra*, pp. 9–11, where references are given.

exhibited in the form

$$\frac{a_{n+1}}{a_n} = 1 + O\left(\frac{1}{n}\right), \quad \frac{a_{n+1}}{a_n} = 1 + o\left(\frac{1}{\sqrt{n}}\right), \quad \frac{a_{n+1}}{a_n} = 1 + o(1),$$

and the last equation asserts simply that $a_{n+1}/a_n \to 1$. The point $x = 1$ is therefore a singular point of $f(x)$.*

43. While we are on the subject of these theorems involving Δa_n, it seems worth while to state the following two, which are easy deductions from known results.

THEOREM 32.—*If, for some integral* r, $\Delta^r a_n = o(1)$, *and* $f(x)$ *is regular for* $x = 1$, *then* $a_n = o(1)$ *and the series* Σa^n *is convergent.*

THEOREM 33.—*If* $\Delta a_n = O(1/n)$, *and* $(1-x)f(x) = o(1)$, *as* $x \to 1$, *then* $a_n = o(1)$.

To prove Theorem 32, we suppose first $r = 1$, and apply Fatou's theorem to

$$\Sigma \Delta a_n x^n = \{a_0 - (1-x) f(x)\}/x.$$

A repetition of this argument leads to the result. Theorem 33 may be deduced similarly from the general Abel-Tauber theorem.

It follows, *e.g.*, from Theorem 32, that a series such as

$$\Sigma n^p e^{Ai(\log n)^q} x^n,$$

for which $\Delta^r a_n = o(1)$, for sufficiently large values of r, has certainly a singular point at $x = 1$.

Finally, the following theorem is interesting because of its relations to the general Abel-Tauber theorem.

THEOREM 34.—*If* $\Delta a_n = O(1/n^2)$, *and* $f(x) \to s$ *as* $x \to 1$, *then* Σa_n *is convergent.*

This is evidently only a special case of the Abel-Tauber theorem; and it is somewhat remarkable that the latter can be deduced from it in an elementary way. Suppose that Σa_n is a series for which $a_n = O(1/n)$. Then we have

$$f(x) = \Sigma a_n x^n = \Sigma \frac{t_n}{n(n+1)} x^{n+1} + (1-x) \Sigma \frac{t_n}{n} x^n,$$

where $$t_n = a_1 + 2a_2 + \ldots + na_n ;$$

or, say, $$f(x) = \Phi(x) + (1-x)\Phi'(x).$$

It can easily be shown† that $f(x) \to s$ involves

$$\Phi(x) \to s, \quad (1-x)\Phi'(x) \to 0.$$

Moreover, it will easily be verified that, if $a_n = O(1/n)$, then

$$\Delta \frac{t_n}{n(n+1)} = O\left(\frac{1}{n^2}\right), \quad \Delta \frac{t^n}{n} = O\left(\frac{1}{n}\right).$$

Applying Theorems 33 and 34, we see that $\Sigma \{t_n/n(n+1)\}$ is convergent, and $t_n/n = o(1)$. From this the convergence of Σa_n follows at once.

The form of Theorem 34 suggests that it might be easier to prove than the Abel-Tauber theorem (from the nature of the case it could not be more difficult). If so it would become of great interest. But our attempts in this direction have not been successful.

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* See § 3 of the Introduction.

† Cf. Hardy, *Quarterly Journal*, Vol. 43, p. 143.

V.

Miscellaneous Theorems.

The Multiplication of Series.

44. One of the most interesting of the properties of series whose n-th term is of order $1/n$ is their *multiplicative* character: if

$$(A) \quad a_1+a_2+\ldots, \qquad (B) \quad b_1+b_2+\ldots,$$

are two such series, and

$$(C) \quad c_1+c_2+\ldots,$$

is their product formed in accordance with Cauchy's rule, then the convergence of A and B is enough to ensure the convergence of C and the equality $AB = C$.*

The form of the theorem at once suggests that it must be connected with the Cesàro-Tauber theorem. If, indeed, it were true that the hypotheses

$$a_n = O(1/n), \qquad b_n = O(1/n),$$

implied

$$c_n = a_1 b_n+\ldots+a_n b_1 = O(1/n),$$

then, of course, the multiplication theorem would be an immediate corollary of the Cesàro-Tauber theorem—for the product of two convergent series is necessarily summable ($C1$). But this is not the case: in fact, if

$$\Sigma a_n = \Sigma b_n = 1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots,$$

we find

$$(-1)^{n-1} c_n = \frac{2}{n+1}\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n}\right) \sim \frac{2\log n}{n}.$$

Thus the connection must lie somewhat deeper.

45. The clue to the connection is to be found in the negative indices of summability introduced by Chapman.†

In order to assimilate our notation to that of Cesàro and Chapman we shall, throughout this part of our work, take our series in the form $a_0+a_1+a_2+\ldots$. We shall write

$$(1-x)^{-r-1}\Sigma a_n x^n = \Sigma A_n^r x^n, \qquad (1-x)^{-r-1} = \Sigma M_n^r x^n,$$

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* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410, and *ibid.*, Vol. 10, p. 396. In the second of these papers the corresponding theorem is proved for the general form of Dirichlet's multiplication.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 369.

so that Σa_n is summable (Cr) to sum s, if

$$A_n^r/M_n^r \to s,$$

or

$$\Gamma(r+1)\, A_n^r/n^r \to s.$$

Here r is any number greater than -1.*

46. Many of the characteristic properties of Cesàro's means still subsist when $-1 < r < 0$. In particular we may mention (1) the condition of consistency,† (2) the Cesàro multiplication theorem,‡ and (3) the theorem that, for a series summable (Cr), we must have

$$a_n = o(n^r).$$§

But one must not assume too hastily that *all* interesting properties of summable series are capable of this generalisation. As an example let us take the following theorem, which is relevant to our present purpose.

THEOREM 35.—*If A is absolutely convergent, and B summable (Cr), where $r \geqslant 0$, then C is summable (Cr).*

This is a generalisation of Mertens' classical theorem, to which it reduces when $r = 0$.‖ The proof is very simple. We may, without real loss of generality, suppose $B = 0$, so that

$$B_n^r = o(n^r).$$

Choose n_0 so that

$$|B_\nu^r| < \epsilon \nu^r \quad (\nu \geqslant \tfrac{1}{2} n_0).$$

Then

$$C_n^r = a_0 B_n^r + a_1 B_{n-1}^r + \ldots + a_n B_0^r,$$

$$|n^{-r} C_n^r| < \epsilon \sum_{\nu < \frac{1}{2}n} |a_\nu| + K \sum_{\frac{1}{2}n \leqslant \nu \leqslant n} |a_\nu| \quad (n \geqslant n_0),$$

which is less than $K\epsilon$, if $n \geqslant n_1 \geqslant n_0$, and n_1 is large enough.

This proof fails if $r < 0$, and the result fails too. Let us assume that (as will be proved in a moment) the series $1 - \frac{1}{2} + \frac{1}{3} - \ldots$ is summable $(C, -1+\delta)$ for every positive δ. Then we have

THEOREM 36.—*It is possible to find an absolutely convergent series A and a series B summable $(C, -1+\delta)$, such that their product is not summable $(C, -\delta)$.*¶

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* It is obvious that all our results will have analogues for Riesz's methods of summation and Dirichlet's multiplication. We do not consider these here: there are certain preliminary complications which arise when we try to extend Riesz's results to negative indices, and the length of this paper makes it impossible that we should consider the matter further at present.

† Chapman, *l.c.*, p. 377.

‡ Chapman, *l.c.*, p. 378.

§ Chapman, *l.c.*, p. 379.

‖ The corresponding theorem for Dirichlet's multiplication is given by Hardy and Riesz, "The General Theory of Dirichlet's Series," one of the *Cambridge Tracts*, shortly to be published.

¶ Here, of course, δ is an arbitrarily small positive number.

We take $$A = 0 + \frac{1}{1^2} + 0 + \frac{1}{2^2} + 0 + 0 + 0 + \frac{1}{3^2} + 0 + \dots$$

(where $a_n = 1/\nu^2$, if $n = 2^\nu - 1$, $\nu > 0$, and $a_n = 0$ otherwise), and $B = 1 - \frac{1}{2} + \frac{1}{3} - \dots$. We find

$$c_3 = 1\,\frac{1}{2^2} + \frac{1}{3}\,\frac{1}{1^2} > \frac{1}{2^2}, \qquad c_7 = 1\,\frac{1}{3^2} + \frac{1}{5}\,\frac{1}{2^2} + \frac{1}{7}\,\frac{1}{1^2} > \frac{1}{3^2},$$

$$c_{15} > \frac{1}{4^2}, \dots, c_{2^\nu - 1} > \frac{1}{n^2}, \dots;$$

and plainly it is not true that $c_n = o(n^{-\delta})$ for any positive value of δ. Using the third theorem of Chapman referred to above, the theorem follows.

47. THEOREM 37.—*If $a_n = O(1/n)$ and Σa_n is summable (Cr), then Σa_n is summable $(C, -1+\delta)$ for any positive value of δ.*

By the Cesàro-Tauber theorem the series is convergent, and therefore, by Chapman's generalised condition of consistency, it is summable (C, δ). Hence the necessary and sufficient condition that it should be summable $(C, -1+\delta)$, is

$$t_n^k = o(n^{k+1}), \tag{1}$$

where $k = -1+\delta$, and t_n^k is the Cesàro's sum of order $-1+\delta$ formed with $b_n = na_n$: that is to say,

$$t_n^k = \sum_{p=0}^{n-1} \frac{(k+1)(k+2)\dots(k+p)}{p!}\, b_{n-p}^{*} \tag{1'}$$

$$= \sum_{0}^{\nu-1} + \sum_{\nu}^{n-1} = S_1 + S_2,$$

say. It will make the proof clearer if we explain at once that we are going to take $\nu \sim \theta n$, where θ is a number a sufficiently small value of which will be fixed before n is made to tend to infinity.

In the first place, as $b_n = O(1)$, we have

$$S_1 = O\left\{\sum_{0}^{\nu-1} \frac{(k+1)\dots(k+p)}{p!}\right\} = O\left\{\frac{(k+2)\dots(k+\nu)}{(\nu-1)!}\right\} = O(\nu^{k+1}). \tag{2}$$

The constant of the O is independent of θ.

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 304. In this paper only integral orders of summability are considered, but, so far as the condition quoted is concerned, the work is independent of this restriction.

Again,

$$(3)\quad \Gamma(k+1)\,S_2$$

$$= \sum_1^{n-\nu} \frac{\Gamma(k+n-q+1)}{\Gamma(n-q+1)} b_q$$

$$= \sum_1^{n-\nu-1} B_q \left\{ \frac{\Gamma(k+n-q+1)}{\Gamma(n-q+1)} - \frac{\Gamma(k+n-q)}{\Gamma(n-q)} \right\} + \frac{\Gamma(k+\nu+1)}{\Gamma(\nu+1)} B_{n-\nu}$$

$$= S_2' + S_2'',$$

say. Now $B_q = o(q)$, since Σa_n is convergent. Hence, first,

$$(4)\qquad S_2'' = o(n\nu^k),$$

uniformly with respect to θ. Finally,

$$S_2' = O\left\{ \sum_1^{n-\nu-1} o(q) \frac{\Gamma(k+n-q)}{\Gamma(n-q+1)} \right\} = o\left\{ \sum_1^{n-\nu-1} q(n-q)^{k-1} \right\}.$$

But

$$\sum_1^{n-\nu-1} q(n-q)^{k-1} = n^k \sum_1^{n-\nu-1} \frac{q}{n}\left(1-\frac{q}{n}\right)^{k-1}$$

$$= O\left\{ n^{k+1} \int_0^{1-\theta} u(1-u)^{k-1}\, du \right\},$$

and so

$$(5)\qquad S_2' = o\{n^{k+1} K(\theta)\},$$

where $K(\theta)$ is a number dependent on θ alone and the o is uniform in respect to θ.*

From (1′)–(5) it follows that

$$(6)\qquad t_n^k = O(\nu^{k+1}) + o(n\nu^k) + o\{n^{k+1} K(\theta)\}$$

$$= n^{k+1}\{\theta^{k+1} O(1) + o(\theta^k) + o(K(\theta))\}.$$

First choose θ so that $\theta^{k+1} O(1) < \epsilon$. When θ is fixed we can choose n_0 so that

$$|o(\theta^k)| < \epsilon, \qquad |o(K(\theta))| < \epsilon,$$

$$|t_n^k| < 3\epsilon n^{k+1},$$

for $n > n_0$. Thus (1) is established and the theorem proved.

---

* $K(\theta) \to \infty$ as $\theta \to 0$. So does θ^k in (6) below. No difficulty is caused by this, but the uniformity of the O's and o's in respect to θ is essential to the proof.

48. It appears, then, that all Cesàro's means, of index greater than -1, are equivalent in regard to series whose general term is of order $1/n$.*

As a corollary we deduce

THEOREM 38.—*The product of any number of convergent series for which* $a_n = O(1/n)$ *is convergent.*

This follows at once from the fact that the index of summability of the product of two such series is $-1-1+1 = -1$.† We might say, more generally, "the product of any number of such series and any number of absolutely convergent series"—for we can multiply the absolutely convergent and the non-absolutely convergent series separately, and then form the product of their products, the final result being independent of the order of multiplication.‡

49. The definition of Cesàro summability breaks down when r is a negative integer, and the concept does not seem to be a useful one for any value of r less than -1.§ The question is, however, suggested whether it is not possible to frame a satisfactory definition in the case $r = -1$. A series summable $(C, -1)$ might be expected to have the property of giving a convergent product when multiplied by *any* convergent series.

It is, however, certain that series whose general term is $O(1/n)$ do not necessarily possess this property. In fact, it is easy to see that the general term of the product of

$$1 - \frac{1}{2} + \frac{1}{3} - \ldots, \qquad \frac{1}{\sqrt{\log 3}} - \frac{1}{\sqrt{\log 4}} + \frac{1}{\sqrt{\log 5}} - \ldots$$

does not tend to zero.

In this connection the following result is interesting.

THEOREM 39.—*Given any numbers* α, β, *such that* $\alpha < 1$, $\beta < 1$, *it is possible to find two convergent series* Σa_n, Σb_n, *such that*

$$a_n = o(n^{-\alpha}), \qquad b_n = o(n^{-\beta}),$$

and the product series is not convergent.‖

The example required is to be found by considering the square of the series

$$\Sigma n^{-b} e^{in^a},$$

where $0 < a < 1$, $b < 1$. The series itself is summable (Ck) if $(k+1)a + b > 1$.

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* That a convergent series of positive decreasing terms is summable $(C, -1+\delta)$ was proved by Chapman (*l.c.*, p. 405). It was this result which suggested Theorem 37. The general term of such a series is, of course, of the form $o(1/n)$.

† Hardy (*Proc. London Math. Soc.*, Ser. 2. Vol. 6, p. 414) proved only that the product of *two* such series was convergent; it did not follow from his argument that, *e.g.*, the series formed by cubing the series $1-\frac{1}{2}+\frac{1}{3}-\ldots$ was convergent: and as the property expressed by $a_n = O(1/n)$ is lost in the process of multiplication, his argument could not be repeated.

‡ This was pointed out to us by Mr. Chapman.

§ Cf. Chapman, *l.c.*, p. 376.

‖ This theorem answers a question raised by Hardy, *l.c.*, p. 419.

The behaviour of the general term of the squared series, viz.,

$$c_n = \sum_1^n \frac{e^{i\{\nu^a + (n+1-\nu)^a\}}}{\{\nu(n+1-\nu)\}^b},$$

may be shown to be similar to that of the integral

$$\int_0^x \frac{e^{i\{t^a + (x-t)^a\}}}{\{t(x-t)\}^b}\, dt = x^{1-2b} F(x^a),$$

where $x = n$, and

$$F(z) = \int_0^1 \frac{e^{zi\{u^a + (1-u)^a\}}}{\{u(1-u)\}^b}\, du.$$

The behaviour of this function for large positive values of z may be studied by methods depending on Cauchy's theorem.* It can be proved that

$$F(z) = Kz^{-\frac{1}{2}} e^{2^{1-a} zi}(1 + o(1)) + Kz^{-(1-b)/a} e^{zi}(1 + o(1)).\dagger$$

From this it is easy to deduce that the squared series is convergent or summable if, and only if, *both* of the series

$$(1) \qquad \Sigma n^{1-\frac{1}{2}a-2b} e^{2^{1-a} in^a}, \qquad \Sigma n^{-b} e^{in^a},$$

of which the second is the same as the original series, are convergent or summable. We can now distinguish two cases.

(i) If $b \geqslant 1-\frac{1}{2}a$,‡ the convergence or summability of the second of the series (1) carries with it that of the first. In this case *the index of summability of the squared series is the same as that of the original series.* Now the index k of the original series is given by

$$(k+1)a + b = 1.$$

Thus this state of affairs occurs if, and only if, $k \leqslant -\frac{1}{2}$.

(ii) If $b < 1-\frac{1}{2}a$, the first of the series (1) is the more important. Thus the index of summability l of the squared series is given by

$$(l+1)a + \tfrac{1}{2}a + 2b - 1 = 1,$$

so that

$$l = 2k + \tfrac{1}{2}.$$

The index of summability of the squared series has a value less by $\frac{1}{2}$ than the value $2k+1$ assigned by Cesàro's multiplication theorem.

In particular, *the squared series is convergent if* $b > 1-\frac{3}{4}a$. If

$$1-a < b < 1-\tfrac{3}{4}a,$$

---

* We are not professing to do more than sketch the general lines of the proof of Theorem 39.

† Each $o(1)$ is in reality an asymptotic series of descending powers of z. The actual values of the K's are

$$2^{2(b-1)} \sqrt{\left\{\frac{2^a \pi}{ia(1-a)}\right\}}, \qquad \frac{2}{a}\Gamma\left(\frac{1-b}{a}\right) i^{1+a-b, a}.$$

where the powers of i have their principal values.

‡ This condition is the same as the condition that $f(x) = \Sigma n^{-b} e^{in^a} x^n$ should be bounded in the unit circle.

the original series is convergent, but not the squared series. In order to obtain an example such as is required by the theorem, we have only to take a sufficiently small.

50. Before leaving the subject of multiplication we may mention one or two results whose proof is easy, and which possess no particular theoretical interest, but which are sometimes useful in practice. It is sufficient for the convergence of C that

$$a_n = O(n^{-\alpha}), \quad B_n - B = o(n^{-\beta}),$$

where $0 < \alpha \leqslant 1$, $0 < \beta \leqslant 1$, $\alpha + \beta \geqslant 1$. The O and the o may be interchanged. The condition $B_n - B = o(n^{-\beta})$ is certainly satisfied if $\Sigma n^{\beta} b_n$ is convergent; and, if $\beta < 1$, either of these conditions involves the summability $(C, -\beta)$ of the series B.

Tauber's Theorem for Double Series.

51. In this sub-section we propose to investigate the analogue of Tauber's theorem for double series. We shall also prove the analogue of the generalised form of the theorem, given by Tauber himself and Pringsheim :* the theorem that

$$a_1 + 2a_2 + \ldots + na_n = o(n),$$

together with the existence of Abel's limit, involves the convergence of Σa_n. We must begin by a few preliminary explanations.

When we say that a function of two variables, $a_{m,n}$ or $f(x, y)$, tends to a limit, we mean always a Pringsheim double limit; we shall not be concerned with *repeated* limits. The existence of a double limit for, *e.g.*, $a_{m,n}$ does not imply the existence of a simple limit when m or n tends separately to infinity. Nor does it imply the inequality

$$|a_{m,n}| < K$$

for all values of m and n.†

This last inequality, which plays an important part in all these questions, we call the *condition of finitude*.‡ When the double limit exists, the condition of finitude may or may not be satisfied. It is important to distinguish these two cases in our notation. We shall write the condition of finitude in the form

$$a_{m,n} = O(1).$$

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* See Pringsheim, *Münchener Sitzungsberichte*, Vol. 30, p. 37, and Vol. 31, p. 507.

† For the elements of the theory of double limits and series, see Pringsheim, *Munchener Sitzungsberichte*, Vol. 27, p. 101, and *Math. Ann.*, Vol. 53, p. 289; London, *Math. Ann.*, Vol. 53, p. 322; Bromwich, *Infinite Series*, Chap. 5.

‡ Cf. Bromwich and Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 163.

If the condition of finitude is satisfied, and $a_{m,n} \to 0$, we shall write

$$a_{m,n} = o(1).$$

Generally, by $a_{m,n} = o(\phi)$, we imply (i) that $a_{m,n} = O(\phi)$, and (ii) that $a_{m,n}/\phi \to 0$. It is plain that similar definitions may be given for functions of two continuous variables which tend to limiting values.

We shall find it convenient to assume throughout that $a_{m,n} = 0$, if either m or n is zero.

The reader will find no difficulty in verifying the following lemmas.

LEMMA 1.—*If* $a_{m,n} = o(1)$, *then*

$$\sum_1^m a_{\mu,n} = o(m), \qquad \sum_1^n a_{m,\nu} = o(n), \qquad \sum_1^m \sum_1^n a_{\mu,\nu} = o(mn).$$

LEMMA 2.—*If* $a_{m,n} = o(1)$, *then*

$$(1-x)(1-y)\Sigma a_{m,n} x^m y^n = o(1).$$

In the second lemma it is supposed that $0 < x < 1$, $0 < y < 1$, and $x \to 1$, $y \to 1$. Neither lemma is true if we substitute $a_{m,n} \to 0$ for $a_{m,n} = o(1)$.*

We shall write

$$\sum_1^n a_{m,\nu} = p_{m,n}, \qquad \sum_1^m a_{\mu,n} = q_{m,n}, \qquad \sum_1^m \sum_1^n a_{\mu,\nu} = s_{m,n},$$

$$\sum_1^m \mu p_{\mu,n} = \sum_1^m \sum_1^n \mu a_{\mu,\nu} = \rho_{m,n}, \qquad \sum_1^n \nu q_{m,\nu} = \sum_1^m \sum_1^n \nu a_{\mu,\nu} = \sigma_{m,n},$$

$$\sum_1^m \sum_1^n \mu\nu a_{\mu,\nu} = t_{m,n}.$$

It is obvious from Lemma 1 that

(1) $mp_{m,n} = o(1)$, (2) $nq_{m,n} = o(1)$, (3) $mna_{m,n} = o(1)$,

imply respectively

(1′) $\rho_{m,n} = o(m)$, (2′) $\sigma_{m,n} = o(n)$, (3′) $t_{m,n} = o(mn)$.

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* Thus, if $a_{m,1} = m$, $a_{m,n} = 0\ (n > 1)$, we find

$$\frac{1}{mn} \sum_1^m \sum_1^n a_{\mu,\nu} = \frac{m+1}{2n}.$$

The condition $a_{m,n} \to 0$ does not even involve the convergence of the power series for $x < 1$, $y < 1$ (cf. Bromwich and Hardy, *l.c. supra* p. 166).

Again, it is easily verified that

$$t_{m,n} = (n+1)\rho_{m,n} - \sum_1^n \rho_{m,\nu} = (m+1)\sigma_{m,n} - \sum_1^m \sigma_{\mu,n},$$

$$\sum_1^n \nu a_{m,\nu} = (n+1)p_{m,n} - \sum_1^n p_{m,\nu}, \qquad \sum_1^m \mu a_{\mu,n} = (m+1)q_{m,n} - \sum_1^m q_{\mu,n}.$$

Hence it follows that either of (1′) or (2′) implies (3′), and *a fortiori* either of (1) or (2) implies (3′), and that (1) and (2) imply respectively

$$(1'') \quad m\sum_1^n \nu a_{m,\nu} = o(n), \qquad (2'') \quad n\sum_1^m \mu a_{\mu,n} = o(m).$$

52. We can now prove

THEOREM 40.—*If* (1) $mp_{m,n} = o(1)$, $nq_{m,n} = o(1)$, *and*

$$(2) \quad f(x, y) = \Sigma a_{m,n} x^m y^n \to A,$$

as $x \to 1$, $y \to 1$, *then* $\Sigma a_{m,n}$ *is convergent to sum* A.

We shall first show that the conclusion holds if conditions (1), (2), and (3) of § 51, are all satisfied. We shall then show that condition (3) may be dispensed with.

We suppose $\qquad x = 1 - \frac{1}{\mu}, \qquad y = 1 - \frac{1}{\nu},$

and write
$$f(x, y) = \left(\sum_1^\mu \sum_1^\nu + \sum_1^\mu \sum_{\nu+1}^\infty + \sum_{\mu+1}^\infty \sum_1^\nu + \sum_{\mu+1}^\infty \sum_{\nu+1}^\infty\right) a_{r,s} x^r y^s$$
$$= \phi_1 + \phi_2 + \phi_3 + \phi_4,$$

say. Choose μ and ν so that $|rsa_{r,s}| < \epsilon$ for $r \geqslant \mu$, $s \geqslant \nu$. Then

$$|\phi_4| < \frac{\epsilon}{\mu\nu(1-x)(1-y)} = \epsilon,$$

so that $\phi_4 \to 0$. Again,

$$\phi_2 = \sum_{\nu+1}^\infty y^s \sum_1^\mu a_{r,s} x^r = (1-x)\sum_{\nu+1}^\infty y^s \sum_1^{\mu-1} q_{r,s} x^r + x^\mu \sum_{\nu+1}^\infty q_{\mu,s} y^s$$
$$= \phi_2' + \phi_2'',$$

* This theorem seems tc be the appropriate form of the analogue of Tauber's theorem. The introduction of the conditions $mp_{m,n} = o(1)$, $nq_{m,n} = o(1)$ is due to Dr. W. H. Young, who was the first to show that a double series which satisfies them cannot be summable by Cesàro's means without being convergent (see Young, "On Multiple Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 167).

say. Choosing μ and ν so that $|sq_{r,s}| < \epsilon$, for $r \geqslant \mu$, $s \geqslant \nu$, we see, by an argument similar to that used above for ϕ_4, that $\phi_2' \to 0$. And

$$|\phi_2'| < (1-x)\sum_{\nu+1}^{\infty} y^s \sum_{1}^{\lambda-1} |q_{r,s}| + (1-x)\sum_{\nu+1}^{\infty} y^s \sum_{\lambda}^{\mu-1} |q_{r,s}|\, x^r.$$

Now $|sq_{r,s}| < K$ for all values of r and s, and we can choose λ so that $|sq_{r,s}| < \epsilon$ for $r \geqslant \lambda$, $s \geqslant \nu$. Then

$$|\phi_2'| < \frac{K(1-x)}{\nu(1-y)} + \frac{\epsilon}{\nu(1-y)} = \frac{K}{\mu} + \epsilon,$$

and so $\phi_2' \to 0$. Thus $\phi_2 \to 0$, and similarly $\phi_3 \to 0$. Hence

$$\phi_1 \to A. \tag{1}$$

But $$\sum_{1}^{\mu}\sum_{1}^{\nu} a_{r,s} - \phi_1 = \sum_{1}^{\mu}\sum_{1}^{\nu} a_{r,s}\{(1-x^r)+(1-y^s)-(1-x^r)(1-y^s)\}$$

$$= \chi_1 + \chi_2 + \chi_3,$$

say. In the first place

$$|\chi_3| < (1-x)(1-y)\sum_{1}^{\mu}\sum_{1}^{\nu} rs\,|a_{r,s}| = \frac{1}{\mu\nu}\sum_{1}^{\mu}\sum_{1}^{\nu} rs\,|a_{r,s}|,$$

and so $\chi_3 \to 0$. In the second place

$$|\chi_1| = \left|\sum_{1}^{\mu} p_{r,\nu}(1-x^r)\right| < (1-x)\sum_{1}^{\mu} r\,|p_{r,\nu}| = \frac{1}{\mu}\sum_{1}^{\mu} r\,|p_{r,\nu}|,$$

and so $\chi_1 \to 0$. Similarly $\chi_2 \to 0$. Hence

$$s_{\mu,\nu} \to A,$$

and the result of Theorem 40 is established if conditions (1), (2), and (3) of § 51 are all satisfied.

53. Now $$f(x,y) = \sum_{1}^{\infty} y^n \sum_{1}^{\infty} m a_{m,n}(x^m/m)$$

$$= \sum_{1}^{\infty} y^n \sum_{1}^{\infty} (a_{1,n} + 2a_{2,n} + \ldots + ma_{m,n})\,\Delta\frac{x^m}{m} = S_1 + S_2,$$

where $$S_1 = \Sigma\frac{g_{m,n}}{m(m+1)}x^m y^n, \qquad S_2 = (1-x)\,\Sigma\frac{g_{m,n}}{m+1}\,x^m y^n,$$

$$g_{m,n} = a_{1,n} + 2a_{2,n} + \ldots + ma_{m,n}.$$

In the first place $$S_2 = (1-x)(1-y)\,\Sigma\,\gamma_{m,n}x^m y^n,$$

where $$\gamma_{m,n} = \frac{1}{m+1} \sum_1^m \sum_1^n r a_{r,s} = \frac{\rho_{m,n}}{m+1} = o(1).$$

Hence, by Lemma 2 of § 51, $S_2 \to 0$, and so $S_1 \to A$. We shall now show that, if $mp_{m,n} = o(1)$, $nq_{m,n} = o(1)$, the coefficients

$$a_{m,n} = \frac{g_{m,n}}{m(m+1)}$$

of S_1 satisfy the *three* conditions corresponding to (1), (2), (3) of § 51. We can then, after what was established in the last paragraph, infer the convergence of S_1 for $x = 1$, $y = 1$.

Denoting by $\varpi_{m,n}$, $\kappa_{m,n}$ the sums formed from $a_{m,n}$ as $p_{m,n}$, $q_{m,n}$ were from $a_{m,n}$, we find first that

$$m\varpi_{m,n} = \frac{1}{m+1} \sum_1^n g_{m,s} = \frac{1}{m+1} \sum_1^m \sum_1^n r a_{r,s} = \frac{\rho_{m,n}}{m+1} = o(1),$$

by (1′) of § 51. Secondly,

$$\begin{aligned} n\kappa_{m,n} &= n \sum_1^m \left(\frac{1}{\mu} - \frac{1}{\mu+1}\right)(a_{1,n} + 2a_{2,n} + \ldots + \mu a_{\mu,n}) \\ &= n\left\{\left(1 - \frac{1}{m+1}\right) a_{1,n} + \left(\frac{1}{2} - \frac{1}{m+1}\right) 2a_{2,n} + \ldots + \left(\frac{1}{m} - \frac{1}{m+1}\right) m a_{m,n}\right\} \\ &= nq_{m,n} - \frac{n}{m+1} \sum_1^m r a_{r,n} = o(1), \end{aligned}$$

by (2″) of § 51. Finally,

$$mna_{m,n} = \frac{n}{m+1} \sum_1^m r a_{r,n} = o(1).$$

Hence S_1 converges for $x = 1$, $y = 1$. But it will be found, by a transformation similar to that used immediately above, that

$$\sum_1^m \sum_1^n \frac{a_{1,s} + 2a_{2,s} + \ldots + r a_{r,s}}{r(r+1)} = \sum_1^m \sum_1^n a_{r,s} - \frac{\rho_{m,n}}{m+1}.$$

Therefore $\Sigma a_{m,n}$ is convergent, and Theorem 40 is proved.

It will be observed that (as in the ordinary proof of Tauber's theorem) we have in reality proved a good deal more than the convergence of the series. We have proved, *e.g.*, that if $f(x, y) \to A$, when $x, y \to 1$ through a particular sequence of values $1 - 1/m$, $1 - 1/n$, then $s_{m,n} \to A$ when m and n tend to infinity through the corresponding sequence.

54. We can easily establish a more general theorem.

THEOREM 41.—*If* (1) $\rho_{m,n} = o(m)$, $\sigma_{m,n} = o(n)$, *and* (2) $f(x, y) \to A$, *then* $\Sigma a_{m,n}$ *converges to the sum* A.

If we refer back to the proof of Theorem 40, it will be seen that all that was actually assumed in § 53 is that

$$\rho_{m,n} = o(m), \qquad nq_{m,n} = o(1).$$

These conditions are therefore sufficient, and so, of course, are

$$mp_{m,n} = o(1), \qquad \sigma_{m,n} = o(n).$$

We shall now prove that these conditions are satisfied by the series S_1 of § 53 whenever the conditions of the present theorem are satisfied by the original series. It will then follow that S_1 is convergent for $x = 1$, $y = 1$, and the proof may be completed as before.

The verification of the first test is precisely the same as before. Hence all that we have to show is that

$$\sum_1^m \sum_1^n \frac{s(a_{1,s} + 2a_{2,s} + \ldots + ra_{r,s})}{r(r+1)} = o(n).$$

But it will easily be verified that this expression is equal to

$$\sigma_{m,n} - t_{m,n}/(m+1) = o(n),$$

by (3′) of § 51.

THEOREM 42. — *If* $(m+n)^2 a_{m,n} = o(1)$, *and* $f(x, y) \to A$, *then* $\Sigma a_{m,n}$ *converges to the sum* A.

This theorem is merely a special case of Theorem 40.* It is interesting on account of its simplicity, and also because, when this special condition is satisfied, we can prove rather more. Suppose that we are given only that

$$f(x, x) \to A.$$

Then $$f(x, x) = \Sigma c_n x^{n+1},$$

where $$c_n = a_{n,1} + a_{n-1,2} + \ldots + a_{1,n} = o(1/n).$$†

Hence, by Tauber's theorem, the *diagonal series* of $\Sigma a_{m,n}$ converges. That the double series is convergent follows from another general theorem, whose proof we may leave to the reader.

THEOREM 43.—*If* $(m+n)^2 a_{m,n} = o(1)$, *the convergence of the diagonal series involves that of the double series, and conversely.*‡

* It is the form of the theorem that we had adopted before we saw Dr. Young's conditions.

† The proof of this is almost obvious.

‡ As all the rows and columns converge, the repeated series also converge to the same sum.

Two final remarks may be added before we leave this subject. It is not unnatural to suppose that the condition

$$mna_{m,n} = o(1) \tag{1}$$

might prove to be sufficient for the truth of the analogue of Tauber's theorem. It is, however, possible to construct such a series, for which $f(x, x) \to A$ while the diagonal series oscillates: and it can be shown that, in virtue of (1), the double series also must oscillate. Thus the truth of the suggestion is made to appear exceedingly improbable.

Lastly, as regards the analogue for double series of the Cesàro-Tauber theorem, Dr. Young has shown that the conditions (1) and (2) of § 51, together with the summability $C(k, l)$ of the series, involve its convergence. It is easy to prove directly that the more general conditions (1′), (2′) are also sufficient when $k = l = 1$; and there is no doubt of the correctness of the extension of this result to series summable by more general Cesàro's means, though we have not troubled to work this out. No doubt also (as is suggested by Dr. Young) the corresponding theorems with O instead of o can be established with comparatively little trouble. The extension of Theorems 40 and 41 to this case, on the other hand, would in all probability be exceedingly troublesome.

It would be natural to try to exhibit the analogues of the Cesàro-Tauber theorem as corollaries of Theorems 40 and 41, using the extensions of the theorems of Abel, Frobenius, and Hölder, given by Bromwich and Hardy in these *Proceedings*.* But difficulties arise owing to the *rôle* played by the "condition of finitude" in these theorems, and the relations between them are less evident than in the case of simple series.

Additional Extensions of the Ordinary Form of Tauber's Theorem.

55. The argument by which the usual form of Tauber's theorem is established may be adapted to prove a more general theorem.

THEOREM 44.—*Suppose that ϕ is a real or complex function of its argument, and that*

$$f(x) = \Sigma a_n x^n \sim \phi\left(\frac{1}{1-x}\right),$$

as $x \to 1$ by real values. Suppose further that it is possible to find a function ψ which satisfies the conditions

(i) *$\psi(x)$ is positive and increasing, and ψ/x decreasing,*

(ii) $\psi = O(\phi)$,

(iii) $na_n = o\{\psi(n)\}$.

Then

$$a_0 + a_1 + \ldots + a_n \sim \phi(n),$$

as $n \to \infty$. In conditions (ii) *and* (iii) *the O and the o may be interchanged.*

When $\psi = \phi = 1$, this is equivalent to Tauber's theorem.

We have, when $x = 1 - 1/n$, and we take the first form of the conditions,

$$\sum_0^n a_\nu - f\left(1-\frac{1}{n}\right) = \sum_0^n a_\nu(1-x^\nu) - \sum_{n+1}^\infty a_\nu x^\nu = (1-x)\sum_0^n o\{\psi(\nu)\} + \frac{o\{\psi(n)\}}{n(1-x)}$$

$$= (1-x)\, o\{n\psi(n)\} + o\{\psi(n)\} = o\{\psi(n)\};$$

* Bromwich and Hardy, *l.c., supra;* see also Bromwich, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 67.

and the result follows immediately. Similarly when the second form of the conditions is taken. Thus, if

$$f(x) = \sum_1^\infty \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right),$$

we could take $\psi = 1$, and deduce that

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{n} \sim \log n.$$

On the other hand, if $a_n = n^{-s}$, where $0 < s < 1$, we have $\phi(n) = \Gamma(1-s)\, n^{1-s}$, and it is impossible to choose ψ so as to satisfy the conditions. In this case

$$1 + \frac{1}{2^s} + \ldots + \frac{1}{n^s} \sim \frac{n^{1-s}}{1-s},$$

so that the result of the theorem is untrue.

In any particular case the equation

$$s_n = f\left(1 - \frac{1}{n}\right) + o\{\psi(n)\}$$

(or the corresponding equation with an O) would convey, as a rule, more precise information than is stated in the conclusion of the theorem.

From Theorem 44 we can easily deduce a more general theorem.

THEOREM 45.—*The result of Theorem* 44 *is still true if condition* (iii) *is replaced by the more general condition*

$$\text{(iii')} \quad a_1 + 2a_2 + \ldots + na_n = o(n\psi).$$

A similar modification may be made in the second form of the theorem.

For, if

$$a_1 + 2a_2 + \ldots + na_n = t_n,$$

we have

$$f(x) - a_0 = \sum_1^\infty a_n x^n = \sum_1^\infty t_n \Delta \frac{x^n}{n} = \Sigma \frac{t_n x^n}{n(n+1)} + (1-x)\Sigma \frac{t_n x^n}{n} = f_1 + f_2,$$

say. Now

$$f_2 = o\{(1-x)\Sigma\psi(\nu)x^\nu\} = o\left\{(1-x)\sum_1^n \psi(\nu) + (1-x)\frac{\psi(n)}{n}\sum_{n+1}^\infty \nu x_\nu\right\}$$

$$= o\left\{(1-x)n\psi(n) + \frac{\psi(n)}{(1-x)n}\right\} = o\{\psi(n)\},$$

if $x = 1 - 1/n$. And so, as $x \to 1$ through these values,

$$f_1(x) = \Sigma \frac{t_n x^n}{n(n+1)} \sim \phi\left(\frac{1}{1-x}\right).$$

But this series satisfies the conditions of Theorem 44. Hence

$$\sum_1^n \frac{t_\nu}{\nu(\nu+1)} = \phi(n) + o\{\psi(n)\},$$

$$\sum_0^n a_\nu = a_0 - a_{n+1} + \sum_1^{n+1} a_\nu = \sum_1^n \frac{t_\nu}{\nu(\nu+1)} - \frac{t_{n+1}}{n+1} + a_0 - a_{n+1}$$

$$= \phi(n) + o\{\psi(n)\} \sim \phi(n).$$

Similarly we can prove the alternative form of the theorem.

56. Theorems 44 and 45 have, of course, analogues for the most general form of Dirichlet's series. We content ourselves with stating the analogue of Theorem 45 for ordinary Dirichlet's series.

THEOREM 46.—*Suppose that*

$$f(s) = \Sigma a_n n^{-s} \sim \phi(1/s),$$

as $s \to 0$ *by real values. Suppose also that we can find a function* ψ *such that*

(i) ψ *increases and* $\psi/\log n$ *decreases,*

(ii) $\psi(n) = O\{\phi(\log n)\}$,

(iii) $n \log n\, a_n = o(\psi)$,

or, more generally,

$$\text{(iii')} \qquad \sum_1^n \log \nu\, a_\nu = o(\log n\, \psi).$$

Then

$$\sum_1^n a_\nu \sim \phi(\log n).$$

In conditions (ii) *and* (iii), *or* (iii'), *the O and o can be interchanged.*

Suppose, for example, that $a_n = 1/n$, if n is a prime p, and $a_n = 0$ otherwise. Then

$$f(s) = \Sigma \frac{1}{p^{1+s}} \sim \log\left(\frac{1}{s}\right),$$

$$\sum_1^n \log \nu\, a_\nu = \sum_{p \leqslant n} \frac{\log p}{p} = O(\log n).$$

We take $\psi = 1$, using conditions (ii) (with o) and (iii') (with O), and deduce

$$\sum_{p \leqslant n} \frac{1}{p} \sim \log\log n.$$

A more precise discussion would lead to the well known result

$$\sum_{p \leqslant n} \frac{1}{p} = \log\log n + O(1).$$

By a similar argument we can deduce the more accurate equation*

$$\sum_{p \leqslant n} \frac{1}{p} = \log\log n + A + o(1),$$

---

* Landau, *Handbuch*, p. 102. The result can also be deduced in a very elegant manner from Landau's form of Tauber's theorem for Dirichlet's series (*Monatshefte für Math. u. Phys.*, Vol. 18, p. 12), to which Theorem 46 reduces for $\psi = \phi = 1$. We take

$$a_n = \frac{1}{n} - \frac{1}{nln} \quad (n = v), \qquad a^n = -\frac{1}{nln} \quad (\textit{otherwise}),$$

so that

$$\frac{l1\, a_1 + l2\, a_2 + \ldots + ln\, a_n}{ln} = \frac{1}{ln} \sum_{p < n} \frac{lp}{p} - \frac{1}{ln} \sum_1^n \frac{1}{\nu} = o(1).$$

Also

$$\Sigma a^n n^{-s} = \Sigma \frac{1}{p^{1+s}} - \Sigma \frac{1}{n^{1+s} ln} \to A,$$

as $s \to 0$; whence the result.

where A is a constant, from the equations

$$f(s) = \log\left(\frac{1}{s}\right) + A + o(1), \qquad \sum_{p \leqslant n} \frac{\log p}{p} \sim \log n.$$

57. We shall conclude this paper with some remarks relating to the extension of Abel's and Tauber's theorems to the case in which $x \to 1$ along a path which *touches* the unit circle.

Let r, θ be polar coordinates. We shall call a curve C, passing from the inside of the circle to the point 1, *a regular path of order* α, if (i) r, regarded as a function of θ, has a continuous first derivative,*

$$\text{(ii)} \quad 1-r \sim H\theta^{\alpha},$$

as $r \to 1$, $\theta \to 0$. Thus a "Stolz-path," *i.e.*, a path such as is contemplated in Stolz's extension of Abel's theorem, is of order 1. The circle described on (0, 1) as diameter is of order 2, and so on. A path such as $r = 1-e^{-1/\theta}$ or $r = 1-e^{-(1/\theta)^2}$ may be described as *of infinite order*.

So long as we are concerned with paths of zero order (*i.e.*, paths which do not touch the circle) the consideration of complex paths introduces no new difficulty of a serious nature.† But as soon as we consider paths tangential to the circle, Abel's theorem ceases to hold. In this connection we have proved the following theorems.

THEOREM 47.—*It is possible to find a convergent series* Σa_n *such that* $f(x) = \Sigma a_n x^n$ *does not tend to a limit when* $x \to 1$ *along any circle interior to and touching the unit circle.*

An example is afforded by the series $\Sigma n^{-b} e^{in^a}$, where $0 < a < \frac{1}{2}$. In this case $f(x)$ tends to a limit when x approaches 1 along the upper half of the smaller circle, but oscillates when it approaches along the lower half.

THEOREM 48.—*If* $s_n = A + o(n^{-\beta})$, *then Abel's limit exists along any regular path of order* α, *where* $\alpha \leqslant 1/(1-\beta)$. *In particular it exists if*

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* This condition is, of course, quite unnecessarily restrictive. But the adoption of a more general definition would confuse rather than help to elucidate the results we wish to discuss.

† For proofs of the generalised forms of Abel's and Tauber's theorems see Bromwich, *Infinite Series*, p. 211 and p. 251. We have proved that the summability of Σa_n by Borel's method is enough to ensure the existence of Abel's limit along any "Stolz-path."

*$\Sigma n^\beta a_n$ is convergent. If $s_n = A + o(1/n)$, and in particular if $\Sigma n a_n$ is convergent, it exists along any regular path of finite order.**

We do not propose to write out proofs of these theorems. That of Theorem 47 would carry us too far from our subject, while that of Theorem 48 should be easily supplied by the reader. We mention them because they indicate a state of affairs in view of which it is rather remarkable that Tauber's theorem should hold in the form in which we propose now to prove it.

We suppose now that C is any regular path of finite or infinite order. That is to say, we suppose only that it is defined by a relation between r and θ, in virtue of which r has a continuous derivative, and we place no limitation on the closeness of its contact with the unit circle. We suppose, however, that it has only the one point $x = 1$ in common with the circle.

The curve C is a rectifiable curve of finite length, and the integral

$$\int_x^1 f(t)\,dt,$$

taken along the curve, has certainly a definite meaning, whenever f is a function continuous along the curve. And if $f(x) \to A$, when $x \to 1$ along C, it is easy to see that

$$\frac{1}{1-x}\int_x^1 f(t)\,dt \to A. \tag{1}$$

THEOREM 49. — *Suppose that $na_n = o(1)$ and $f(x) = \Sigma a_n x^n \to A$ as $x \to 1$ along C. Then Σa_n converges to the sum A.*

Obviously we may suppose $A = 0$. If $1-\xi$ is any point on C, other than 1, $\Sigma a_n t^n$ is uniformly convergent on the arc of C which joins x and $1-\xi$, and

$$\int_x^{1-\xi} f(t)\,dt = \Sigma \frac{a_n}{n+1}\{(1-\xi)^{n+1} - x^{n+1}\}.$$

Since $\Sigma \dfrac{a_n x^n}{n+1}$ is uniformly convergent for $|x| \leqslant 1$, it follows that we may replace ξ in this equation by 0. Hence, using (1), we see that

$$\Sigma \frac{a_n}{n+1}(1-x^{n+1}) = o(1-x), \tag{2}$$

as $x \to 1$ along C.

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* The closeness of the contact may be as great as that of the path $r = 1 - e^{-1/\theta}$. It is not difficult to prove the more general result in which we start from $k!\, s_n^k/n^k = A + o(n^{-\beta})$.

Let $$\nu = \left[\frac{1}{|1-x|}\right]$$

and $$\Sigma \frac{a_n}{n+1}(1-x^{n+1}) = \sum_0^{\nu} + \sum_{\nu+1}^{\infty} = \phi_1+\phi_2.$$

Then $$\phi_2 = o\left(\sum_{\nu+1}^{\infty} \frac{1}{n^2}\right) = o(1/\nu).$$

Also $$1-x^{n+1} = (n+1)(1-x)+O\{n^2(1-x)^2\},$$

and so $$\phi_1 = (1-x)\sum_0^{\nu} a_n + O\left\{(1-x)^2 \sum_0^{\nu} n|a_n|\right\}$$

$$= (1-x)\, s_n + o\{\nu(1-x)^2\}.$$

Substituting these expressions for ϕ_1 and ϕ_2 in (2), and dividing by $1-x$, we obtain

$$s_n = o\{\nu(1-x)\} + o\left\{\frac{1}{\nu(1-x)}\right\} = o(1),$$

and the theorem is established.

58. The result of Theorem 49 may be extended to include cases in which the path C has points (other than $x = 1$) in common with the unit circle.* In particular it holds *when C is the unit circle itself.* It is to observed that the condition $f(x) \to A$ was only used as a special hypothesis which ensures the truth of the relation (1); a more general theorem is obtained by taking that relation as a hypothesis. When stated in this form, and when C is the circle itself, Theorem 49 reduces to a known theorem in the theory of Fourier series, and it appears that our argument does not differ in principle from an argument used by Riemann in his classical memoir on trigonometrical series.†

Theorem 49 is a "Tauberian" theorem; it is natural to try to state a corresponding "Abelian" theorem. This theorem is

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* It has not seemed to us worth while at the present moment to go deeply into the question as to the most general hypotheses possible in regard to the nature of C. The essence of the matter lies in the fact that the restrictions which have to be imposed on C *have nothing to do with the closeness of its contact with the circle*: they are conditions relating to continuity, rectifiability, and so forth.

† See Riemann, *Gesammelte Werke*, p. 213; Fatou, *Thèse* (Stockholm, 1906), and *Acta Mathematica*, Vol. 30.

THEOREM 50.—*If* $a_n = o(1/n)$, *and* Σa_n *is convergent to sum* A, *then*

$$\frac{1}{1-x}\int_x^1 f(t)\,dt \to A,$$

as $x \to 1$ *along any regular path* C.

If C is of order 1 (a Stolz-path) this conclusion can be replaced by $f(x) \to A$; but that this is so in general appears to us to be improbable.

[*Added November 2nd*, 1912.—We add a few remarks in reference to the questions mentioned at the end of § 4. We understand from Dr. Marcel Riesz, to whom we communicated the proof-sheets of this paper, that he has succeeded in proving the analogue of Theorem 19 for the general Dirichlet's series $\Sigma a_n e^{-\lambda_n s}$ and for all orders of summation integral or non-integral, and in effecting similar generalisations of some of the theorems of Section II.

The supposition that the theorems of § 2, and some, at any rate, of the "Tauberian" theorems, may be generalised by the use of right- or left-handed conditions, proves to be well founded. As an example, we may quote the theorem : *if* $f(x) = \Sigma a_n x^n \to s$, *and* $a_n = O_L(1/n)$, *then* Σa_n *is convergent to sum* s. This is the generalisation of the general Abel-Tauber theorem.]

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